Computer Algebra for Lattice Path Combinatorics

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Let us consider the following questions arising in enumerative combinatorics:

(1) A chess rook can move any number of squares horizontally or vertically on an $n \times n$ chessboard. Assuming that the rook moves right or up at every step, how many different paths can the rook travel in moving from the bottom left corner to the top right corner? Same question for a 3-dimensional chessboard.

(2) A flea starts at $(0, 0)$ on the infinite two-dimensional integer lattice and executes a biased random walk: at each step it hops north or south with probability $\frac{1}{4}$, west with probability $\frac{1}{4} - \varepsilon$ and east with probability $\frac{1}{4} + \varepsilon$. What is the probability that the flea returns to $(0, 0)$ sometime during its wanderings?

(3) An exotic king rambles on an $n \times n$ chessboard. Contrary to classical kings, he can only move to an adjacent square in one of the four directions: west, south-west, east, or north-east. What is the total number of different paths of length n that this exotic king can take starting from the bottom left corner?

In this course we will demonstrate that such questions can be tackled in a systematic way, using an experimental mathematics approach combined with modern computer algebra algorithms. The aim is to give an overview of several techniques and tools leading to the computer-driven *discovery* and *proof* of the following answers, expressed in terms of generating functions of the corresponding sequences:

$$\frac{1}{2}\left(1+\frac{1-t}{\sqrt{(1-t)(1-9t)}}\right) \quad \text{and} \quad 1+6\cdot\int_0^t \frac{{}_2F_1\left(\begin{matrix}\frac{1}{3} & \frac{2}{3}\\ & 2\end{matrix}\,\middle|\,\frac{27x(2-3x)}{(1-4x)^3}\right)}{(1-4x)(1-64x)}\,dx \quad \text{for (1)},$$

$$1-\sqrt{\frac{A}{2}}\cdot{}_2F_1\left(\begin{matrix}\frac{1}{2},\frac{1}{2}\\ 1\end{matrix}\,\middle|\,\frac{2\sqrt{1-16\varepsilon^2}}{A}\right)^{-1} \quad \text{with } A=1+8\varepsilon^2+\sqrt{1-16\varepsilon^2} \quad \text{for (2)},$$

$$\text{and} \quad \frac{1}{2t}\cdot{}_2F_1\left(\begin{matrix}-\frac{1}{12} & \frac{1}{4}\\ & \frac{2}{3}\end{matrix}\,\middle|\,-\frac{64t(1+4t)^2}{(1-4t)^4}\right)-\frac{1}{2t} \quad \text{for (3)},$$

where ${}_2F_1\left(\begin{matrix}a & b\\ & c\end{matrix}\,\middle|\,t\right)$ denotes the Gaussian hypergeometric series defined by

$${}_2F_1\left(\begin{matrix}a & b\\ & c\end{matrix}\,\middle|\,t\right)=\sum_{n=0}^{\infty}\frac{t^n}{n!}\prod_{k=0}^{n-1}\frac{(a+k)(b+k)}{c+k}.$$

Lattice paths in question (3) are called *Gessel walks*. Let $g(n, i, j)$ denote the number of such walks using n steps and ending at the point (i, j). Among other things, we will describe a computer-assisted proof of the following structural result: the trivariate generating function of the sequence $g(n, i, j)$ is an algebraic power series.

The main outcome of the approach described and promoted in these lectures is a complete classification of lattice walks with small steps in the quarter plane. We will also report on ongoing work for extensions to higher dimensions and more general sets of allowed steps.