Computer Algebra for Lattice Path Combinatorics

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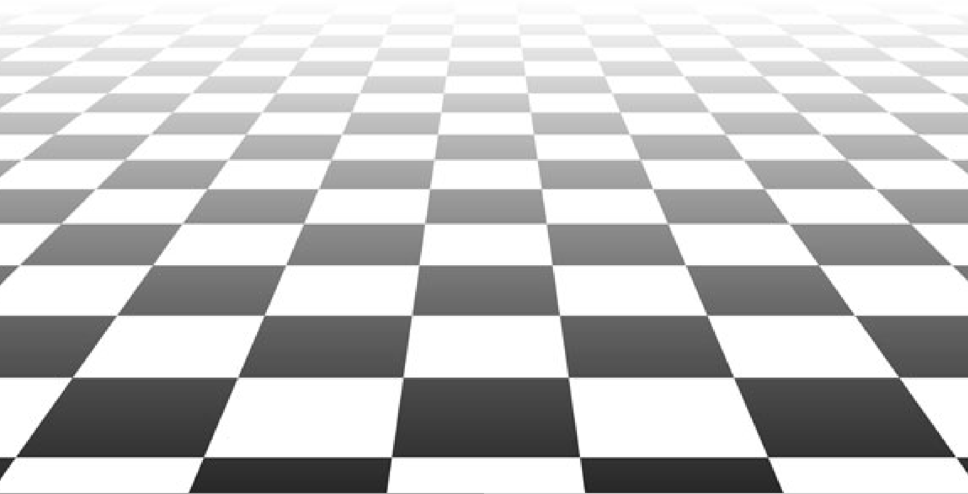
informatics mathematics Inria

The 74th Séminaire Lotharingien de Combinatoire
Ellwangen, March 23–25, 2015

Overview

1. Monday: General presentation
2. Tuesday: Guess'n'Prove
3. Wednesday: Creative telescoping

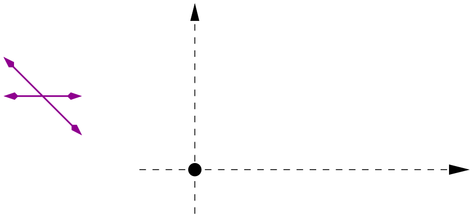
Part I: General presentation



General context: lattice paths confined to cones

Let $\mathfrak{S}$ be a subset of $\mathbb{Z}^d$ (step set, or model) and $p_0 \in \mathbb{Z}^d$ (starting point).

Example: $\mathfrak{S} = \{(1,0),(-1,0),(1,-1),(-1,1)\}$, $p_0 = (0,0)$



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A path (walk) of length n starting at p_0 is a sequence $(p_0, p_1, \ldots, p_n)$ of elements in $\mathbb{Z}^d$ such that $p_{i+1} - p_i \in \mathfrak{S}$ for all i.

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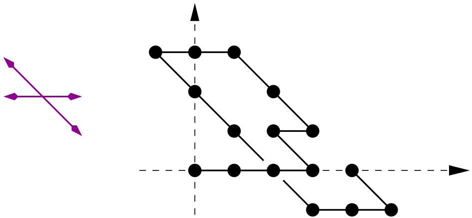

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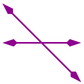
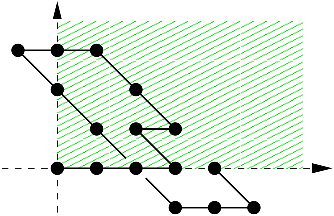
Let C be a cone of $\mathbb{R}^d$ (if $x \in C$ and $r \geq 0$ then $r \cdot x \in C$).

Example: $\mathfrak{S} = \{(1,0), (-1,0), (1,-1), (-1,1)\}$, $p_0 = (0,0)$ and $C = \mathbb{R}^2_+$





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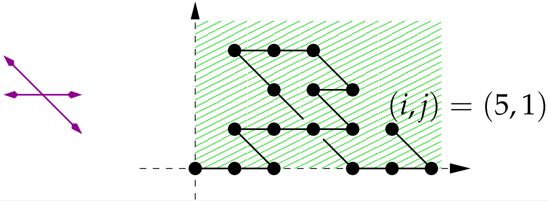

Questions

- What is the number $a(n)$ of n-step walks contained in C?
- For $\boldsymbol{i} \in C$, what is the number $a(n;\boldsymbol{i})$ of such walks that end at $\boldsymbol{i}$?
- What about their generating series $A(t)$, resp. $A(t;\boldsymbol{x})$?

Why count walks in cones?

Many discrete objects can be encoded in that way:

- discrete mathematics (permutations, trees, words, urns, ...)
- statistical physics (Ising model, ...)
- probability theory (branching processes, games of chance, ...)
- operations research (queueing theory, ...)

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A history and a survey of lattice path enumeration

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ABSTRACT

In celebration of the Sixth International Conference on Lattice Path Counting and Applications, it is befitting to review the history of lattice path enumeration and to survey how the topic has progressed thus far.

We start the history with early games of chance specifically the ruin problem which later appears as the ballot problem. We discuss André's Reflection Principle and its misnomer, its relation with the method of images and possible origins from physics and Brownian motion, and the earliest evidence of lattice path techniques and solutions.

In the survey, we give representative articles on lattice path enumeration found in the literature in the last 35 years by the lattice, step set, boundary, characteristics counted, and solution method. Some of this work appears in the author's 2005 dissertation.

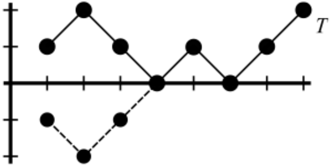
An old topic: The ballot problem and the reflection principle

Ballot problem [Bertrand, 1887]

> Suppose that candidates A and B are running in an election. If a votes are cast for A and b votes are cast for B, where $a > b$, then the probability that A stays ahead of B throughout the counting of the ballots is $(a-b)/(a+b)$.

Lattice path reformulation: given positive integers a, b with $a > b$, find the number of Dyck paths starting at the origin and consisting of a upsteps $\nearrow$ and b downsteps $\searrow$ such that no step ends on the x-axis.

Reflection principle: Dyck paths from $(1,1)$ to $T(a+b, a-b)$ that touch the x-axis $\equiv$ Dyck paths from $(1,-1)$ to T



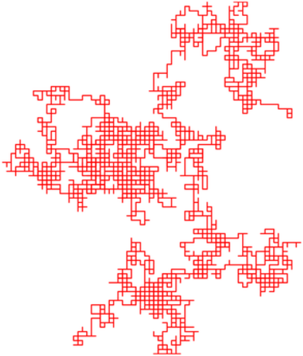
Answer: good paths = paths from $(1,1)$ to T that never touch the x-axis

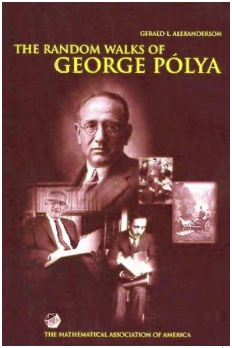
$$\binom{a+b-1}{a-1} - \binom{a+b-1}{b-1} = \frac{a-b}{a+b}\binom{a+b}{a}$$

Motto: Drunkard: "Will I ever, ever get home again?"
Polya (1921): "You can't miss; just keep going and stay out of 3D!" (Adam and Delbruck, 1968)

[Pólya, 1921] The simple random walk on $\mathbb{Z}^d$ is recurrent in dimensions $d = 1, 2$ ("Alle Wege fuehren nach Rom"), and transient in dimension $d \geq 3$

Über eine Aufgabe der Wahrscheinlichkeitsrechnung betreffend die Irrfahrt im Straßennetz.





Still a topical issue

Many recent contributions:

Adan, Banderier, Bernardi, Bostan, Bousquet-Mélou, Chyzak, Cori, Denisov, Duchon, Dulucq, Fayolle, Fisher, Flajolet, Garbit, Gessel, Guttmann, Guy, Gouyou-Beauchamps, van Hoeij, Janse van Rensburg, Johnson, Kauers, Koutschan, Krattenthaler, Kreweras, Kurkova, van Leeuwarden, MacMahon, Melczer, Mishna, Niederhausen, Pech, Petkovšek, Prellberg, Raschel, Rechnitzer, Sagan, Salvy, Viennot, Wachtel, Wilf, Yeats, Zeilberger...

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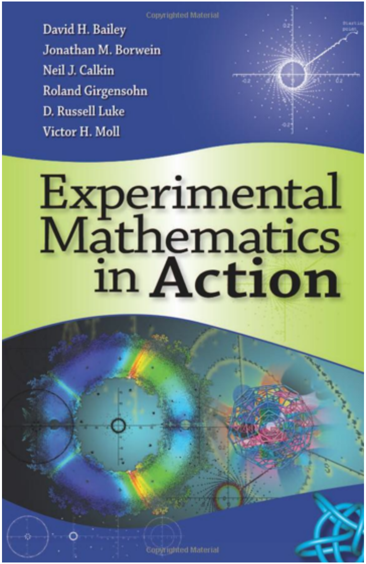
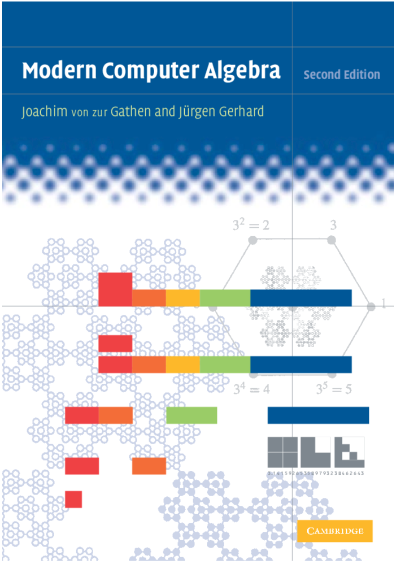
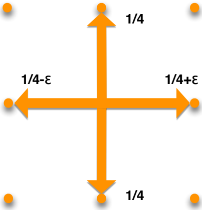
~~Specific question~~
~~Ad hoc solution~~ $\longrightarrow$ **Systematic approach**

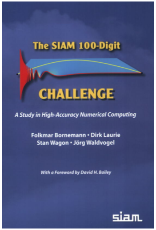
Personal bias: Experimental Mathematics using Computer Algebra





Example: From the SIAM 100-Digit Challenge [Trefethen 2002]





Problem 6

A flea starts at $(0,0)$ on the infinite two-dimensional integer lattice and executes a biased random walk: At each step it hops north or south with probability $1/4$, east with probability $1/4+\epsilon$, and west with probability $1/4-\epsilon$. The probability that the flea returns to $(0,0)$ sometime during its wanderings is $1/2$. What is ϵ?

- Computer algebra conjectures and proves

$$p(\epsilon) = 1 - \sqrt{\frac{A}{2}} \cdot {}_2F_1\left(\begin{matrix} \frac{1}{2}, \frac{1}{2} \\ 1 \end{matrix} \,\middle|\, \frac{2\sqrt{1-16\epsilon^2}}{A} \right)^{-1}, \quad \text{with } A = 1 + 8\epsilon^2 + \sqrt{1-16\epsilon^2}.$$

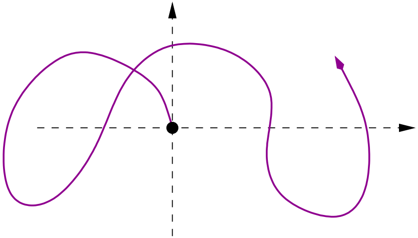
Rational series

If $\mathfrak{S} \subset \mathbb{Z}^d$ is finite and $C = \mathbb{R}^d$, then $A(t;\boldsymbol{x})$ is rational:

$$a(n) = |\mathfrak{S}|^n \quad \Leftrightarrow \quad A(t) = \sum_{n\geq 0} a(n)t^n = \frac{1}{1-|\mathfrak{S}|\,t}$$

More generally:

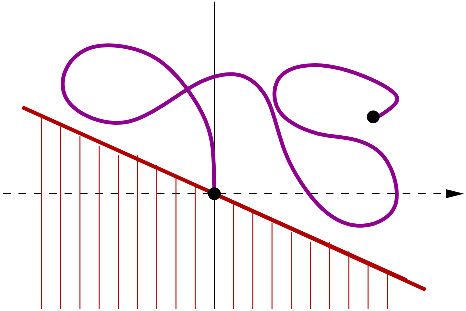
$$A(t;\boldsymbol{x}) = \frac{1}{1 - t\sum_{s\in\mathfrak{S}} \boldsymbol{x}^s}.$$



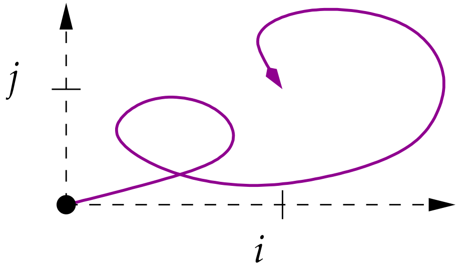
Also well-known: a (rational) half-space

Algebraic series [Bousquet-Mélou-Petkovšek 00]

If $\mathfrak{S} \subset \mathbb{Z}^d$ is finite and C is a rational half-space, then $A(t; \boldsymbol{x})$ is algebraic, given by an explicit system of polynomial equations.



The "next" case: intersection of two half-spaces



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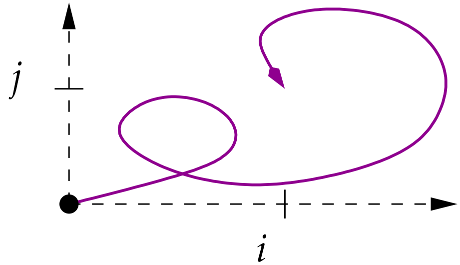
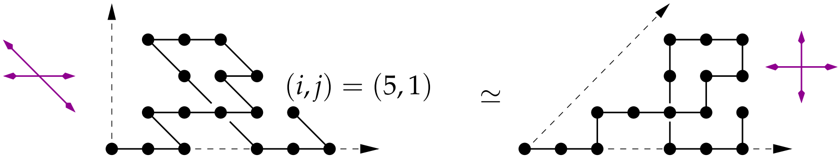
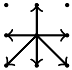


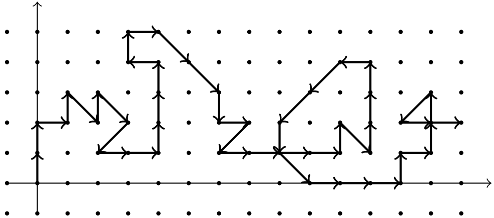


▶ From now on: we focus on nearest-neighbor walks in the quarter plane, i.e. walks in $\mathbb{N}^2$ starting at $(0,0)$ and using steps in a prefixed subset $\mathfrak{S}$ of

$$\{\swarrow, \leftarrow, \nwarrow, \uparrow, \nearrow, \rightarrow, \searrow, \downarrow\}.$$

▶ Example with $n = 45$, $i = 14$, $j = 2$ for:

$\mathfrak{S} =$ 



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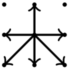
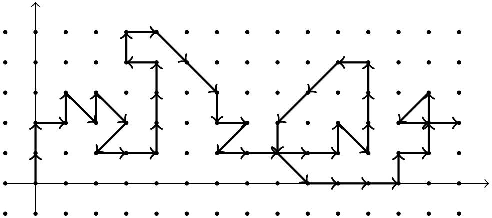
$\mathfrak{S} =$ 



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Lattice walks with small steps in the quarter plane

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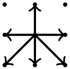
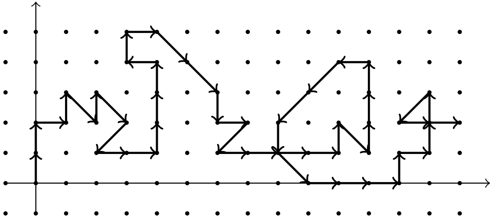
$\mathfrak{S} =$ 



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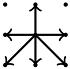
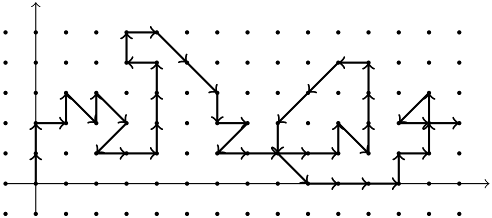
$\mathfrak{S} =$ 



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- $f_n = \sum_{i,j\geq 0} f_{n;i,j}$ = number of total walks with length n.

- Complete generating series:

$$F(t;x,y) = \sum_{n=0}^{\infty} \left(\sum_{i,j=0}^{\infty} f_{n;i,j} x^i y^j \right) t^n \in \mathbf{Q}[x,y][[t]].$$

Generating series and combinatorial problems

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- Special, combinatorially meaningful specializations:
  - $F(t;0,0)$ counts excursions;
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Combinatorial questions: Given $\mathfrak{S}$, what can be said about $F(t;x,y)$, resp. $f_{n;i,j}$, and their variants?

- Properties of F: algebraic? transcendental? D-finite?
- Explicit form: of F? of f?
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Our goal: Use computer algebra to give computational answers.

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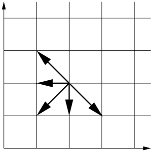

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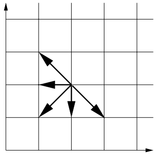

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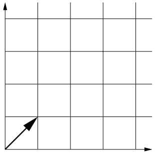

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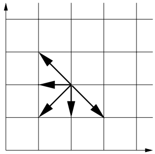

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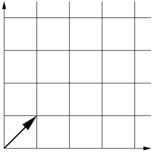

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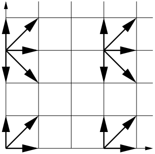

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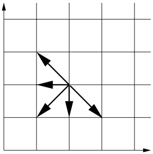

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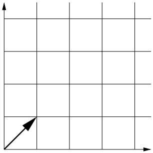

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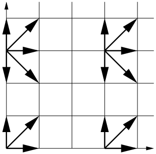

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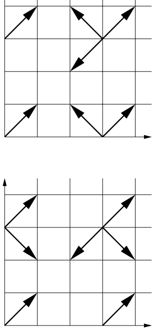

symmetrical.

Small-step walks of interest

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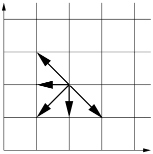

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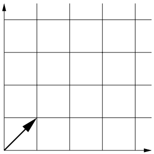
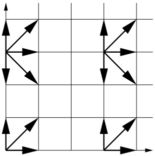
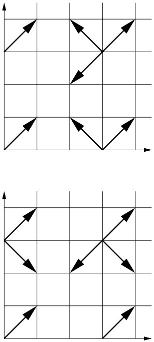


simple,

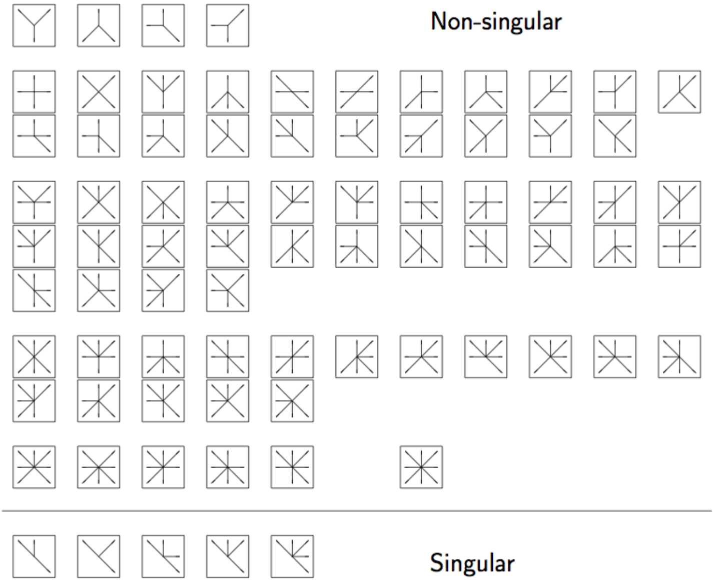


intrinsic to the half plane,

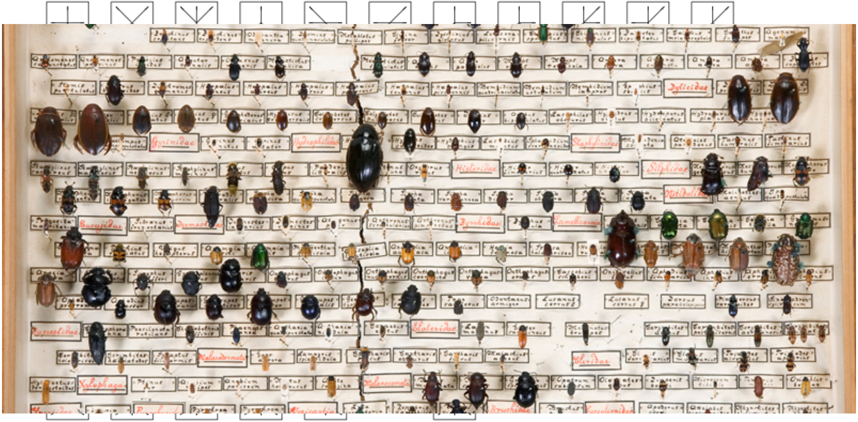


symmetrical.

One is left with 79 interesting distinct models.


Non-singular
Singular

Non-singular

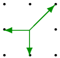


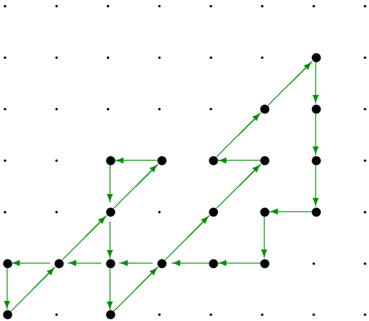
Singular

Two important models: Kreweras and Gessel walks

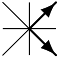
$\mathfrak{S} = \{\downarrow, \leftarrow, \nearrow\}$ $\qquad F_{\mathfrak{S}}(t;x,y) \equiv K(t;x,y)$

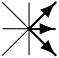
$\mathfrak{S} = \{\nearrow, \swarrow, \leftarrow, \rightarrow\}$ $\qquad F_{\mathfrak{S}}(t;x,y) \equiv G(t;x,y)$





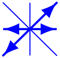
Example: A Kreweras excursion.

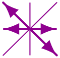
Dyck: 

Motzkin: 

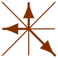
Pólya: 

Kreweras: 

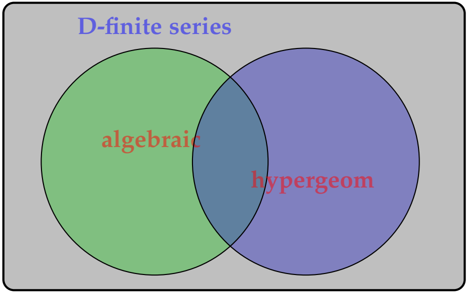
Gessel: 

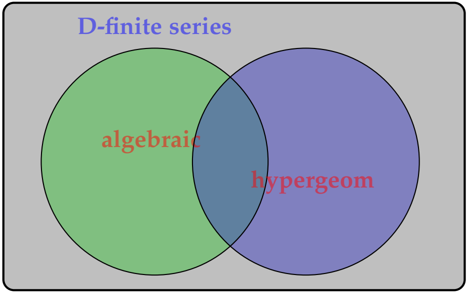
Gouyou-Beauchamps: 

King: 

Exercise: 

Important classes of univariate power series





Algebraic: $S(t) \in \mathbf{Q}[[t]]$ root of a polynomial $P \in \mathbf{Q}[t, T]$, i.e., $P\big(t, S(t)\big) = 0$.

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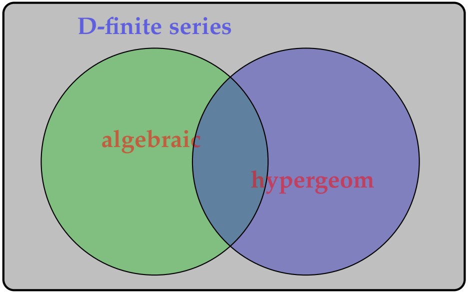

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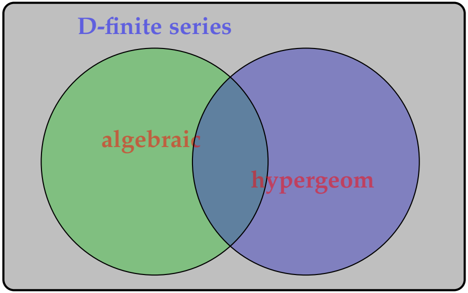

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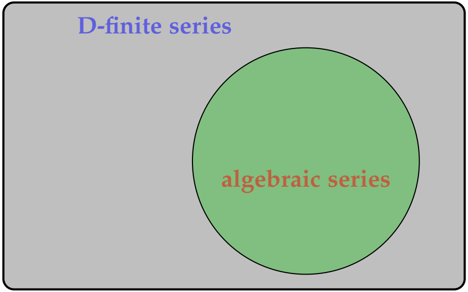
Hypergeometric: $S(t) = \sum_{n=0}^{\infty} s_n t^n$ such that $\frac{s_{n+1}}{s_n} \in \mathbf{Q}(n)$. E.g.,

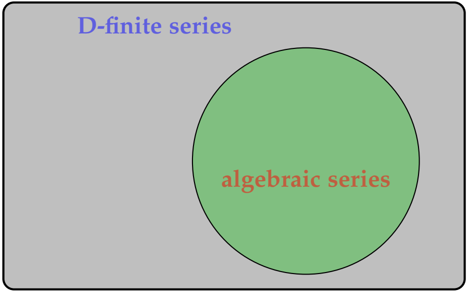
$$ {}_2F_1\left(\begin{matrix} a\ b \\ c \end{matrix} \,\middle|\, t\right) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{t^n}{n!}, \quad (a)_n = a(a+1)\cdots(a+n-1). $$

Important classes of multivariate power series



$S \in \mathbb{Q}[[x, y, t]]$ is algebraic if it is the root of a $P \in \mathbb{Q}[x, y, t, T]$.

Important classes of multivariate power series



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$S \in \mathbb{Q}[[x, y, t]]$ is D-finite if the set of all partial derivatives of S spans a finite-dimensional vector space over $\mathbb{Q}(x, y, t)$.

Main results (I): algebraicity of Gessel walks

Theorem [Kreweras 1965; 100 pages combinatorial proof!]

$$K(t;0,0) = {}_3F_2\left(\left.\begin{matrix}1/3 & 2/3 & 1\\ & 3/2 & 2\end{matrix}\,\right|\, 27\,t^3\right) = \sum_{n=0}^{\infty} \frac{4^n\binom{3n}{n}}{(n+1)(2n+1)} t^{3n}.$$

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- Fresh news: recent human proof [B., Kurkova & Raschel 2015].

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- ▶ Guess'n'Prove method, using Hermite-Padé approximants $\longrightarrow$ Tuesday

Main results (II): Explicit form for $G(t;x,y)$

Theorem [B., Kauers & van Hoeij 2010]

Let $V = 1 + 4t^2 + 36t^4 + 396t^6 + \cdots$ be a root of

$$(V-1)(1+3/V)^3 = (16t)^2,$$

let $U = 1 + 2t^2 + 16t^4 + 2xt^5 + 2(x^2+83)t^6 + \cdots$ be a root of

$$x(V-1)(V+1)U^3 - 2V(3x+5xV-8Vt)U^2$$
$$-xV(V^2-24V-9)U + 2V^2(xV-9x-8Vt) = 0,$$

let $W = t^2 + (y+8)t^4 + 2(y^2+8y+41)t^6 + \cdots$ be a root of

$$y(1-V)W^3 + y(V+3)W^2 - (V+3)W + V - 1 = 0.$$

Then $G(t;x,y)$ is equal to

$$\frac{\frac{64(U(V+1)-2V)V^{3/2}}{x(U^2-V(U^2-8U+9-V))^2} - \frac{y(W-1)^4(1-Wy)V^{-3/2}}{t(y+1)(1-W)(W^2y+1)^2}}{(1+y+x^2y+x^2y^2)t-xy} - \frac{1}{tx(y+1)}.$$

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- Computer-driven discovery and proof; no human proof yet.
- Proof uses guessed minimal polynomials for $G(t;x,0)$ & $G(t;0,y)$

Main results (III): Conjectured D-Finite $F(t;1,1)$ [B. & Kauers 2009]

	OEIS	$\mathfrak{S}$	Pol size	ODE size
1	A005566		—	3, 4
2	A018224		—	3, 5
3	A151312		—	3, 8
4	A151331		—	3, 6
5	A151266		—	5, 16
6	A151307		—	5, 20
7	A151291		—	5, 15
8	A151326		—	5, 18
9	A151302		—	5, 24
10	A151329		—	5, 24
11	A151261		—	4, 15
12	A151297		—	5, 18
13	A151275		—	5, 24
14	A151314		—	5, 24
15	A151255		—	4, 16
16	A151287		—	5, 19
17	A001006		2, 2	2, 3
18	A129400		2, 2	2, 3
19	A005558		—	3, 5
20	A151265		6, 8	4, 9
21	A151278		6, 8	4, 12
22	A151323		4, 4	2, 3
23	A060900		8, 9	3, 5

Equation sizes = {order, degree}@(algeq, diffeq)

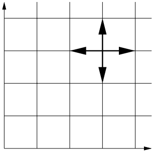
- Computerized discovery by enumeration + Hermite–Padé
- 1–22: Confirmed by human proofs in [Bousquet-Mélou & Mishna 2010]
- 23: Confirmed by a human proof in [B., Kurkova & Raschel 2015]

Main results (III): Conjectured D-Finite $F(t;1,1)$ [B. & Kauers 2009]

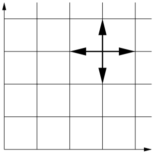
	OEIS	$\mathfrak{S}$	alg	asympt		OEIS	$\mathfrak{S}$	alg	asympt
1	A005566		N	$\frac{4}{\pi}\frac{4^n}{n}$	13	A151275		N	$\frac{12\sqrt{30}}{\pi}\frac{(2\sqrt{6})^n}{n^2}$
2	A018224		N	$\frac{2}{\pi}\frac{4^n}{n}$	14	A151314		N	$\frac{\sqrt{6}\lambda\mu C^{5/2}}{5\pi}\frac{(2C)^n}{n^2}$
3	A151312		N	$\frac{\sqrt{6}}{\pi}\frac{6^n}{n}$	15	A151255		N	$\frac{24\sqrt{2}}{\pi}\frac{(2\sqrt{2})^n}{n^2}$
4	A151331		N	$\frac{8}{3\pi}\frac{8^n}{n}$	16	A151287		N	$\frac{2\sqrt{2}A^{7/2}}{\pi}\frac{(2A)^n}{n^2}$
5	A151266		N	$\frac{1}{2}\sqrt{\frac{3}{\pi}}\frac{3^n}{n^{1/2}}$	17	A001006		Y	$\frac{3}{2}\sqrt{\frac{3}{\pi}}\frac{3^n}{n^{3/2}}$
6	A151307		N	$\frac{1}{2}\sqrt{\frac{5}{2\pi}}\frac{5^n}{n^{1/2}}$	18	A129400		Y	$\frac{3}{2}\sqrt{\frac{3}{\pi}}\frac{6^n}{n^{3/2}}$
7	A151291		N	$\frac{4}{3\sqrt{\pi}}\frac{4^n}{n^{1/2}}$	19	A005558		N	$\frac{8}{\pi}\frac{4^n}{n^2}$
8	A151326		N	$\frac{2}{\sqrt{3\pi}}\frac{6^n}{n^{1/2}}$					
9	A151302		N	$\frac{1}{3}\sqrt{\frac{5}{2\pi}}\frac{5^n}{n^{1/2}}$	20	A151265		Y	$\frac{2\sqrt{2}}{\Gamma(1/4)}\frac{3^n}{n^{3/4}}$
10	A151329		N	$\frac{1}{3}\sqrt{\frac{7}{3\pi}}\frac{7^n}{n^{1/2}}$	21	A151278		Y	$\frac{3\sqrt{3}}{\sqrt{2}\Gamma(1/4)}\frac{3^n}{n^{3/4}}$
11	A151261		N	$\frac{12\sqrt{3}}{\pi}\frac{(2\sqrt{3})^n}{n^2}$	22	A151323		Y	$\frac{\sqrt{2}3^{3/4}}{\Gamma(1/4)}\frac{6^n}{n^{3/4}}$
12	A151297		N	$\frac{\sqrt{3}B^{7/2}}{2\pi}\frac{(2B)^n}{n^2}$	23	A060900		Y	$\frac{4\sqrt{3}}{3\Gamma(1/3)}\frac{4^n}{n^{2/3}}$

$A = 1+\sqrt{2},\ B = 1+\sqrt{3},\ C = 1+\sqrt{6},\ \lambda = 7+3\sqrt{6},\ \mu = \sqrt{\frac{4\sqrt{6}-1}{19}}$

- Computerized discovery by enumeration + Hermite–Padé + LLL/PSLQ.

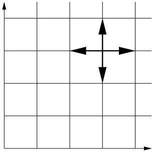


The characteristic polynomial $\chi_{\mathfrak{S}} := x + \frac{1}{x} + y + \frac{1}{y}$



The characteristic polynomial $\chi_{\mathfrak{S}} := x + \frac{1}{x} + y + \frac{1}{y}$ is left invariant under

$$\psi(x,y) = \left(x, \frac{1}{y}\right), \quad \phi(x,y) = \left(\frac{1}{x}, y\right),$$



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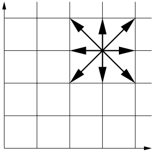
$$\psi(x,y) = \left(x, \frac{1}{y}\right), \quad \phi(x,y) = \left(\frac{1}{x}, y\right),$$

and thus under any element of the group

$$\langle \psi, \phi \rangle = \left\{ (x,y), \left(x, \frac{1}{y}\right), \left(\frac{1}{x}, \frac{1}{y}\right), \left(\frac{1}{x}, y\right) \right\}.$$

The group of a model: the general case



The polynomial $\chi_{\mathfrak{S}} := \sum_{(i,j)\in\mathfrak{S}} x^i y^j = \sum_{i=-1}^{1} B_i(y) x^i = \sum_{j=-1}^{1} A_j(x) y^j$

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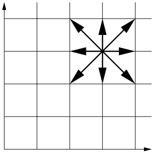

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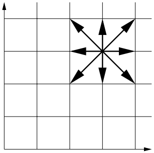

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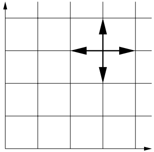
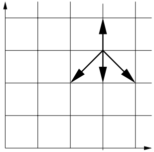
and thus under any element of the group

$$\mathcal{G}_{\mathfrak{S}} := \langle \psi, \phi \rangle.$$

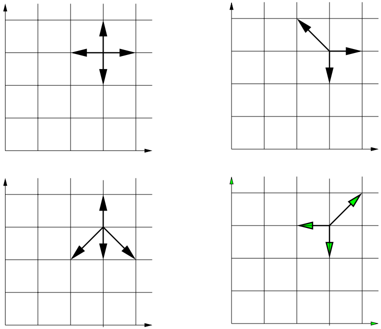
Examples of groups





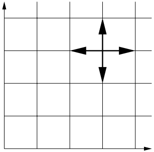
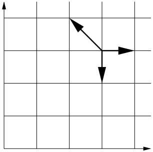
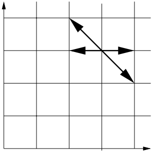
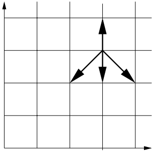
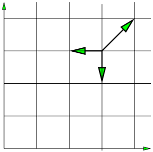
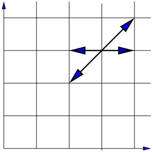
Order 4,

Examples of groups

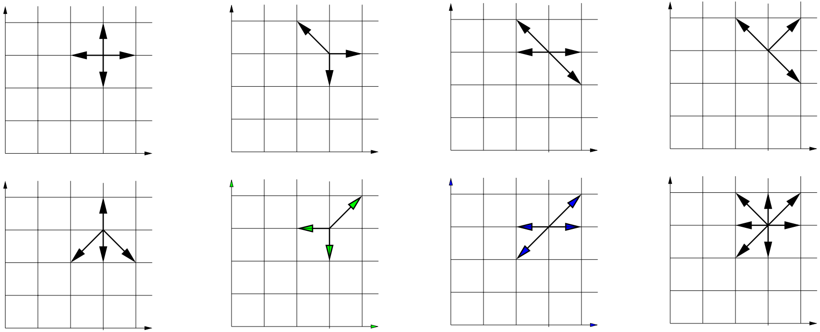


Order 4, order 6,

Examples of groups

Order 4, order 6, order 8,

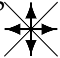


Order 4, order 6, order 8, order ∞.

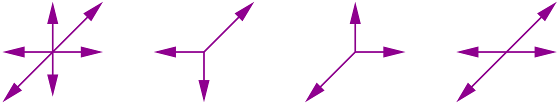
The orbit sum of a model $\mathfrak{S}$ is the following polynomial in $\mathbb{Q}[x, x^{-1}, y, y^{-1}]$:

$$\mathrm{OrbitSum}(\mathfrak{S}) := \sum_{\theta \in \mathcal{G}_{\mathfrak{S}}} (-1)^{\theta} \theta(xy)$$

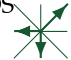
- E.g., for the simple walk:

OS  $= x \cdot y - \frac{1}{x} \cdot y + \frac{1}{x} \cdot \frac{1}{y} - x \cdot \frac{1}{y}$

- For 4 models, the orbit sum is zero:



E.g. for the Kreweras model:

OS  $= x \cdot y - \frac{1}{xy} \cdot y + \frac{1}{xy} \cdot x - y \cdot x + y \cdot \frac{1}{xy} - x \cdot \frac{1}{xy} \quad = 0$

The 79 models: finite and infinite groups

79 models

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79 models

23 admit a finite group

[Mishna'07]

56 have an infinite group

[Bousquet-Mélou & Mishna'10]

all $F(t;x,y)$ D-finite

79 models

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19 transcendental
($\mathrm{OS} \neq 0$)
[Gessel & Zeilberger'92]
[Bousquet-Mélou'02]

4 algebraic ($\mathrm{OS} = 0$)
(3 Kreweras-type + Gessel)
[BMM'10] + [B. & Kauers'10]

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56 have an infinite group

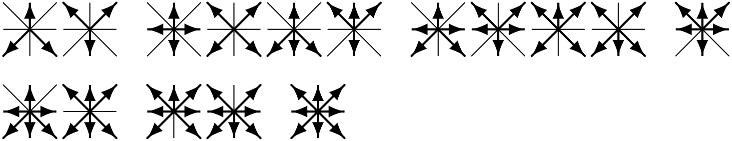
[Bousquet-Mélou & Mishna'10]

$\longrightarrow$ all non-D-finite

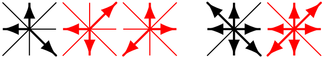
- [Mishna & Rechnitzer'07] and [Melczer & Mishna'13] for 5 singular models
- [Kurkova & Raschel'13] and [B., Raschel & Salvy'13] for all others

The 23 models with a finite group

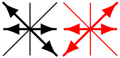
(i) 16 with a vertical symmetry, and group isomorphic to D_2



(ii) 5 with a diagonal or anti-diagonal symmetry, and group isomorphic to D_3



(iii) 2 with group isomorphic to D_4



(i): vertical symmetry; (ii)+(iii): zero drift $\sum_{s \in \mathfrak{S}} s$

In red, models with OS $= 0$ and algebraic GF

Main results (IV): explicit expressions for the 19 D-finite transcendental models

Theorem [B., Chyzak, van Hoeij, Kauers & Pech 2015]

Let $\mathfrak{S}$ be one of the 19 models with finite group $\mathcal{G}_{\mathfrak{S}}$, and non-zero orbit sum. Then F is expressible using iterated integrals of ${}_2F_1$ expressions.

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Example (King walks in the quarter plane, A025595)

F  $$(t;1,1) = \frac{1}{t}\int_0^t \frac{1}{(1+4x)^3} \cdot {}_2F_1\left(\begin{matrix}\frac{3}{2} & \frac{3}{2}\\ & 2\end{matrix}\,\middle|\, \frac{16x(1+x)}{(1+4x)^2}\right) dx$$

$$= 1 + 3t + 18t^2 + 105t^3 + 684t^4 + 4550t^5 + 31340t^6 + 219555t^7 + \cdots$$

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- Computer-driven discovery and proof; no human proof yet.
- Proof uses creative telescoping, ODE factorization, ODE solving. $\longrightarrow$ Wednesday

Hypergeometric Series Occurring in Explicit Expressions for $F(t;1,1)$

	hyp$_1$	hyp$_2$	w		hyp$_1$	hyp$_2$	w
1	${}_2F_1\left(\begin{array}{c}\frac{1}{2}\;\frac{1}{2}\\ 1\end{array}\middle\vert\, w\right)$	${}_2F_1\left(\begin{array}{c}\frac{1}{2}\;\frac{3}{2}\\ 2\end{array}\middle\vert\, w\right)$	$16t^2$	10	${}_2F_1\left(\begin{array}{c}\frac{7}{4}\;\frac{9}{4}\\ 2\end{array}\middle\vert\, w\right)$	${}_2F_1\left(\begin{array}{c}\frac{9}{4}\;\frac{11}{4}\\ 3\end{array}\middle\vert\, w\right)$	$\frac{64(t^2+t+1)t^2}{(12t^2+1)^2}$
2	${}_2F_1\left(\begin{array}{c}\frac{1}{2}\;\frac{1}{2}\\ 1\end{array}\middle\vert\, w\right)$		$16t^2$	11	${}_2F_1\left(\begin{array}{c}\frac{1}{2}\;\frac{3}{2}\\ 2\end{array}\middle\vert\, w\right)$	${}_2F_1\left(\begin{array}{c}\frac{1}{2}\;\frac{5}{2}\\ 3\end{array}\middle\vert\, w\right)$	$\frac{16t^2}{4t^2+1}$
3	${}_2F_1\left(\begin{array}{c}\frac{3}{2}\;\frac{3}{2}\\ 2\end{array}\middle\vert\, w\right)$		$\frac{16t}{(2t+1)(6t+1)}$	12	${}_2F_1\left(\begin{array}{c}\frac{5}{4}\;\frac{7}{4}\\ 1\end{array}\middle\vert\, w\right)$	${}_2F_1\left(\begin{array}{c}\frac{5}{4}\;\frac{7}{4}\\ 2\end{array}\middle\vert\, w\right)$	$\frac{64t^3(2t+1)}{(8t^2-1)^2}$
4	${}_2F_1\left(\begin{array}{c}\frac{3}{2}\;\frac{3}{2}\\ 2\end{array}\middle\vert\, w\right)$		$\frac{16t(1+t)}{(1+4t)^2}$	13	${}_2F_1\left(\begin{array}{c}\frac{7}{4}\;\frac{9}{4}\\ 2\end{array}\middle\vert\, w\right)$	${}_2F_1\left(\begin{array}{c}\frac{7}{4}\;\frac{9}{4}\\ 3\end{array}\middle\vert\, w\right)$	$\frac{64t^2(t^2+1)}{(16t^2+1)^2}$
5	${}_2F_1\left(\begin{array}{c}\frac{3}{4}\;\frac{5}{4}\\ 1\end{array}\middle\vert\, w\right)$	${}_2F_1\left(\begin{array}{c}\frac{5}{4}\;\frac{7}{4}\\ 2\end{array}\middle\vert\, w\right)$	$64t^4$	14	${}_2F_1\left(\begin{array}{c}\frac{7}{4}\;\frac{9}{4}\\ 2\end{array}\middle\vert\, w\right)$	${}_2F_1\left(\begin{array}{c}\frac{9}{4}\;\frac{11}{4}\\ 3\end{array}\middle\vert\, w\right)$	$\frac{64(t^2+t+1)t^2}{(12t^2+1)^2}$
6	${}_2F_1\left(\begin{array}{c}\frac{7}{4}\;\frac{9}{4}\\ 2\end{array}\middle\vert\, w\right)$	${}_2F_1\left(\begin{array}{c}\frac{7}{4}\;\frac{9}{4}\\ 3\end{array}\middle\vert\, w\right)$	$\frac{64t^3(1+t)}{(1-4t^2)^2}$	15	${}_2F_1\left(\begin{array}{c}\frac{1}{4}\;\frac{3}{4}\\ 1\end{array}\middle\vert\, w\right)$	${}_2F_1\left(\begin{array}{c}\frac{3}{4}\;\frac{5}{4}\\ 2\end{array}\middle\vert\, w\right)$	$64t^4$
7	${}_2F_1\left(\begin{array}{c}\frac{1}{2}\;\frac{1}{2}\\ 1\end{array}\middle\vert\, w\right)$	${}_2F_1\left(\begin{array}{c}\frac{1}{2}\;\frac{3}{2}\\ 1\end{array}\middle\vert\, w\right)$	$\frac{16t^2}{4t^2+1}$	16	${}_2F_1\left(\begin{array}{c}\frac{7}{4}\;\frac{9}{4}\\ 2\end{array}\middle\vert\, w\right)$	${}_2F_1\left(\begin{array}{c}\frac{9}{4}\;\frac{11}{4}\\ 3\end{array}\middle\vert\, w\right)$	$\frac{64t^3(1+t)}{(1-4t^2)^2}$
8	${}_2F_1\left(\begin{array}{c}\frac{5}{4}\;\frac{7}{4}\\ 2\end{array}\middle\vert\, w\right)$	${}_2F_1\left(\begin{array}{c}\frac{7}{4}\;\frac{9}{4}\\ 2\end{array}\middle\vert\, w\right)$	$\frac{64t^3(2t+1)}{(8t^2-1)^2}$				
9	${}_2F_1\left(\begin{array}{c}\frac{7}{4}\;\frac{9}{4}\\ 2\end{array}\middle\vert\, w\right)$	${}_2F_1\left(\begin{array}{c}\frac{7}{4}\;\frac{9}{4}\\ 3\end{array}\middle\vert\, w\right)$	$\frac{64t^2(t^2+1)}{(16t^2+1)^2}$	19	${}_2F_1\left(\begin{array}{c}-\frac{1}{2}\;\frac{1}{2}\\ 1\end{array}\middle\vert\, w\right)$	${}_2F_1\left(\begin{array}{c}\frac{1}{2}\;\frac{1}{2}\\ 2\end{array}\middle\vert\, w\right)$	$16t^2$

Main results (V): non-D-finiteness in models with an infinite group

Theorem [B., Rachel & Salvy 2013]

Let $\mathfrak{S}$ be one of the 51 non-singular models with infinite group $\mathcal{G}_{\mathfrak{S}}$.
Then $F_{\mathfrak{S}}(t;0,0)$, and in particular $F_{\mathfrak{S}}(t;x,y)$, are non-D-finite.

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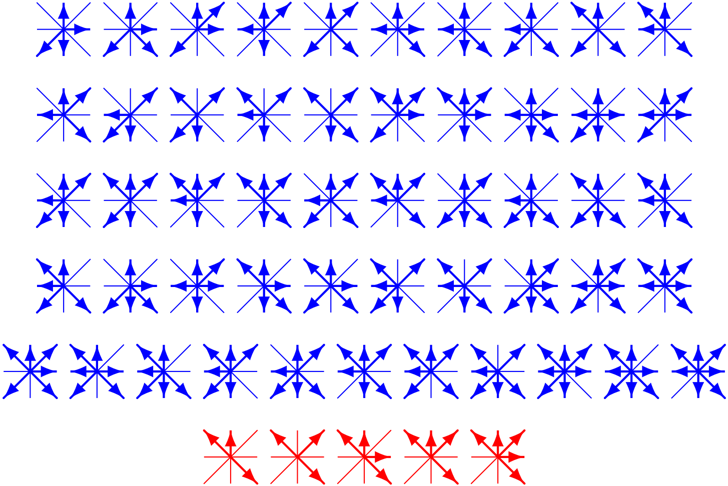
- Algorithmic proof. Uses Gröbner basis computations, polynomial factorization, cyclotomy testing.
- Based on two ingredients: asymptotics + irrationality.

- [Kurkova & Raschel 2013] Human proof that $F_{\mathfrak{S}}(t;x,y)$ is non-D-finite.
- No human proof yet for $F_{\mathfrak{S}}(t;0,0)$ non-D-finite.

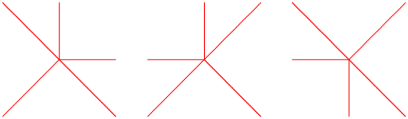
The 56 models with infinite group



In blue, non-singular models, solved by [B., Raschel & Salvy 2013]
In red, singular models, solved by [Melczer & Mishna 2013]

Example: the scarecrows

[B., Raschel & Salvy 2013]: $F_{\mathfrak{S}}(t;0,0)$ is not D-finite for the models



For the 1st and the 3rd, the excursions sequence $[t^n]\, F_{\mathfrak{S}}(t;0,0)$

$$1, 0, 0, 2, 4, 8, 28, 108, 372, \dots$$

is $\sim K \cdot 5^n \cdot n^{-\alpha}$, with $\alpha = 1 + \pi / \arccos(1/4) = 3.383396\dots$

The irrationality of α prevents $F_{\mathfrak{S}}(t;0,0)$ from being D-finite.

The Main Theorem   Let $\mathfrak{S}$ be one of the 74 non-singular models. The following assertions are equivalent:

(1) The full generating series $F_{\mathfrak{S}}(t;x,y)$ is D-finite

(2) the excursions generating series $F_{\mathfrak{S}}(t;0,0)$ is D-finite

(3) the excursions sequence $[t^n]\,F_{\mathfrak{S}}(t;0,0)$ is $\sim K \cdot \rho^n \cdot n^\alpha$, with $\alpha \in \mathbb{Q}$

(4) the group $\mathcal{G}_{\mathfrak{S}}$ is finite (and $|\mathcal{G}_{\mathfrak{S}}| = 2 \cdot \min\{\ell \in \mathbb{N}^\star \mid \frac{\ell}{\alpha+1} \in \mathbb{Z}\}$)

(5) the step set $\mathfrak{S}$ has either an axial symmetry, or zero drift and cardinal different from 5.

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Moreover, under (1)–(5), $F_{\mathfrak{S}}(t;x,y)$ is algebraic if and only if the model $\mathfrak{S}$ has positive covariance $\sum_{(i,j)\in\mathfrak{S}} ij - \sum_{(i,j)\in\mathfrak{S}} i \cdot \sum_{(i,j)\in\mathfrak{S}} j > 0$, and iff it has $\mathrm{OS} = 0$.

Summary: Classification of 2D non-singular walks

The Main Theorem Let $\mathfrak{S}$ be one of the 74 non-singular models. The following assertions are equivalent:

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In this case, $F_{\mathfrak{S}}(t;x,y)$ is expressible using nested radicals.
If not, $F_{\mathfrak{S}}(t;x,y)$ is expressible using iterated integrals of ${}_2F_1$ expressions.

Main methods

(1) for proving algebraicity / D-finiteness
- (1a) Guess'n'Prove
- (1b) Creative telescoping

(2) for proving non-D-finiteness
- (2a) Infinite number of singularities, or lacunary
- (2b) Asymptotics

Main methods

(1) for proving algebraicity / D-finiteness

(1a) Guess'n'Prove — Hermite-Padé approximants

(1b) Creative telescoping — Diagonals of rational functions

(2) for proving non-D-finiteness

(2a) Infinite number of singularities, or lacunary

(2b) Asymptotics

▶ All methods are algorithmic.

Summary: Walks with unit steps in $\mathbb{N}^2$

quadrant models: 79

$|G| < \infty$: 23

$|G| = \infty$: 56

nonzero orbit sum: 19

zero orbit sum: 4

asymptotics + GB

Kernel method + CT

Guess'n'Prove

not D-finite

D-finite

algebraic

11 074 225 distinct interesting models

3D octant models with ≤ 6 steps: 20804

$|G| < \infty$: 170

$|G| = \infty$?: 20634

orbit sum $\neq 0$: 108

orbit sum $= 0$: 62

not D-finite?

kernel method

2D-reducible: 43

not 2D-reducible: 19

D-finite

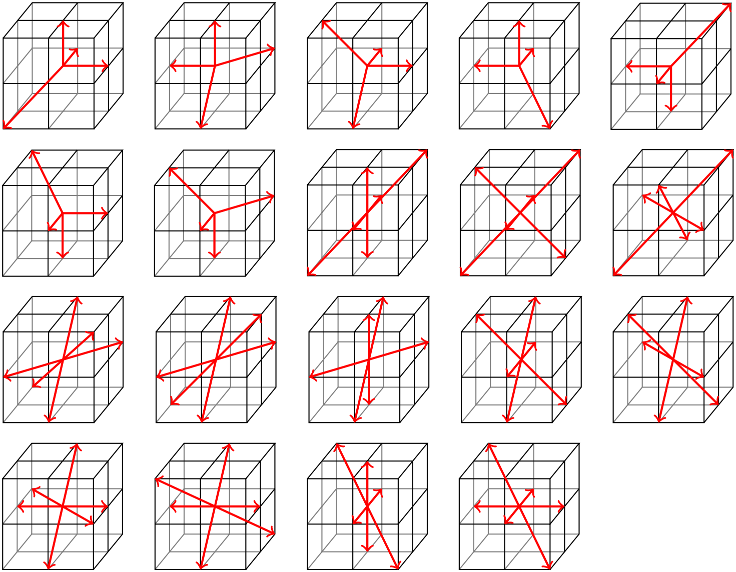
D-finite

not D-finite?

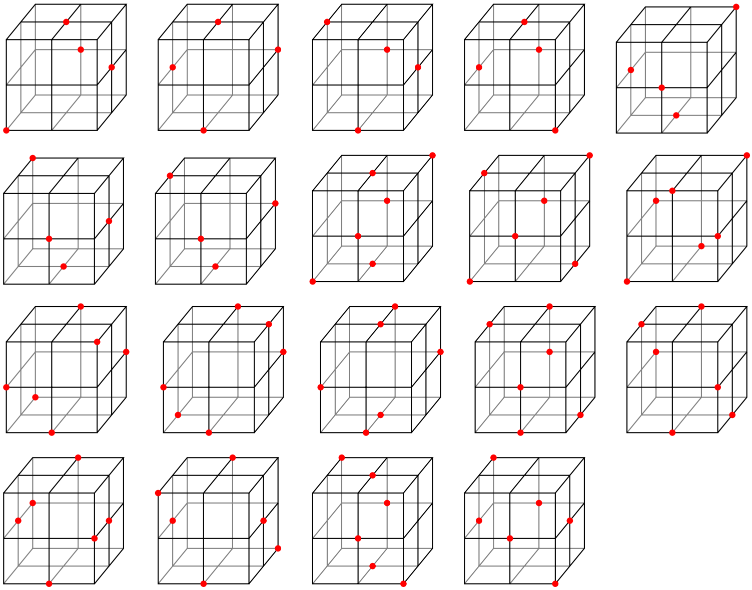
[B., Bousquet-Mélou, Kauers, Melczer 2015]

- Open question: some non-D-finite models with a finite group?

The 19 mysterious 3D-models



The 19 mysterious 3D-models

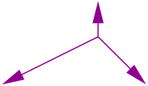


Extensions: Walks in $\mathbb{N}^2$ with longer steps

- Define (and use) a group $\mathcal{G}$ for models with larger steps?

- Example: When $\mathfrak{S} = \{(0,1), (1,-1), (-2,-1)\}$, there is an underlying group that is finite and

$$xyF(t;x,y) = [x^{>0}y^{>0}] \frac{(x-2x^{-2})(y-(x-x^{-2})y^{-1})}{1-t(xy^{-1}+y+x^{-2}y^{-1})}$$



[B., Bousquet-Mélou & Melczer, in progress]

▶ Current status:

- 680 models with one large step, 643 proved non D-finite, 32 of 37 have differential equations guessed.
- 5910 models with two large steps, 5754 proved non D-finite, 69 of 156 have differential equations guessed.

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- Explicit formula for the generating series of diagonal 3D Rook paths, with F. Chyzak, M. van Hoeij and L. Pech. Séminaire Lotharingien de Combinatoire, 2011.
- Non-D-finite excursions in the quarter plane, with K. Raschel and B. Salvy. Journal of Combinatorial Theory A, 2013.
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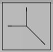
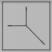
- A human proof of Gessel's lattice path conjecture, with I. Kurkova, K. Raschel, 2015.
- Explicit Differentiably Finite Generating Functions of Walks with Small Steps in the Quarter Plane, with F. Chyzak, M. van Hoeij, M. Kauers and L. Pech, 2015.

An exercise involving the model 

Let $\mathfrak{S} = \{\mathsf{N}, \mathsf{W}, \mathsf{SE}\}$. A $\mathfrak{S}$-walk is a path in $\mathbb{Z}^2$ using only steps from $\mathfrak{S}$. Show that, for any integer n, the following quantities are equal:

(i) the number of $\mathfrak{S}$-walks of length n confined to the upper half plane $\mathbb{Z} \times \mathbb{N}$ that start and end at the origin $(0,0)$;

(ii) the number of $\mathfrak{S}$-walks of length n confined to the quarter plane $\mathbb{N}^2$ that start at the origin $(0,0)$ and finish on the diagonal $x = y$.

An exercise involving the model 

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For instance, for $n = 3$, this common value is 3:

(i) $(0,0) \mapsto (-1,0) \mapsto (-1,1) \mapsto (0,0), (0,0) \mapsto (0,1) \mapsto (-1,1) \mapsto (0,0)$ and $(0,0) \mapsto (0,1) \mapsto (1,0) \mapsto (0,0)$, i.e., W–N–SE, N–W–SE, N–SE–W

(ii) $(0,0) \mapsto (0,1) \mapsto (1,0) \mapsto (0,0), (0,0) \mapsto (0,1) \mapsto (0,2) \mapsto (1,1)$ and $(0,0) \mapsto (0,1) \mapsto (1,0) \mapsto (1,1)$, i.e., N–SE–W, N–N–SE, N–SE–N

Thanks for your attention!