Computer Algebra for Lattice Path Combinatorics

Alin Bostan

JNCF 2017

January 16th 2017

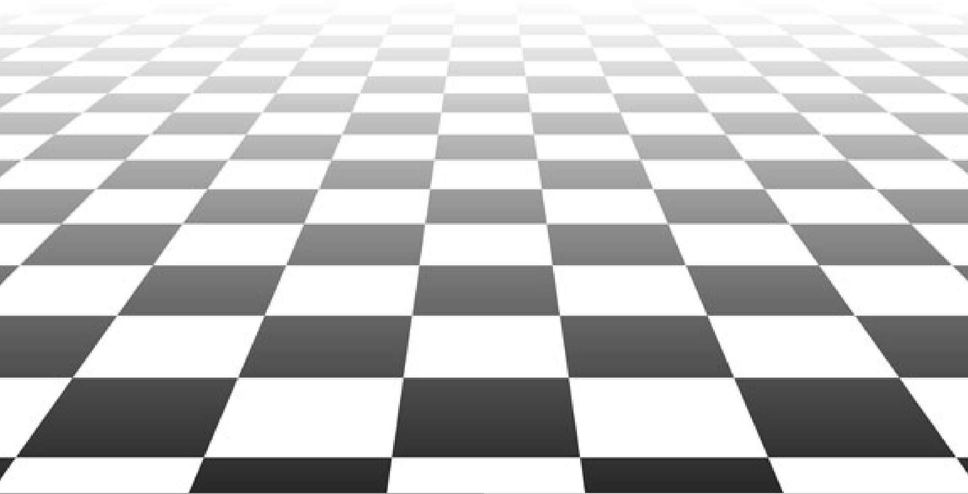
Overview

Part 1: General presentation

Part 2: Guess'n'Prove

Part 3: Creative telescoping

Part 1: General presentation



An (innocent looking) exercise

Let $\mathfrak{S} = \{\uparrow, \leftarrow, \searrow\}$. A $\mathfrak{S}$-walk is a path in $\mathbb{Z}^2$ using only steps from $\mathfrak{S}$. Show that, for any integer n, the following quantities are equal:

(i) the number a_n of $\mathfrak{S}$-walks of length n confined to the upper half plane $\mathbb{Z} \times \mathbb{N}$ that start and end at the origin $(0,0)$;

(ii) the number b_n of $\mathfrak{S}$-walks of length n confined to the quarter plane $\mathbb{N}^2$ that start at the origin $(0,0)$ and finish on the diagonal $x = y$.

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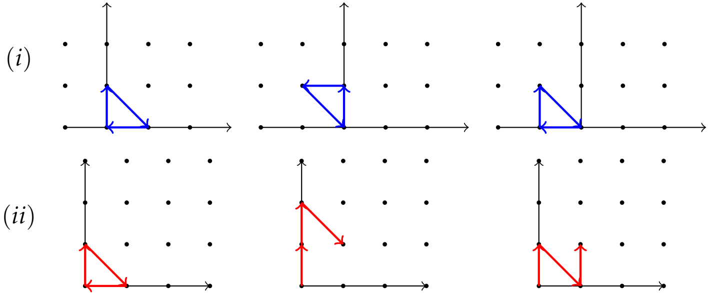
For instance, for $n = 3$, this common value is $a_3 = b_3 = 3$:



Teasers

Teaser 1: This exercise can be solved using computer algebra!

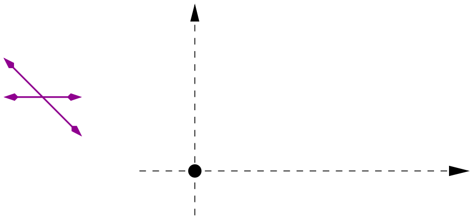
Teaser 2: The answer has a nice closed form!

$$a_{3n} = b_{3n} = \frac{(3n)!}{n!^2 \cdot (n+1)!}, \qquad \text{and} \quad a_m = b_m = 0 \quad \text{if 3 does not divide } m.$$

Teaser 3: A certain group attached to the step set $\{\uparrow, \leftarrow, \searrow\}$ is finite!

Let $\mathfrak{S}$ be a subset of $\mathbb{Z}^d$ (step set, or model) and $p_0 \in \mathbb{Z}^d$ (starting point).

Example: $\mathfrak{S} = \{(1,0), (-1,0), (1,-1), (-1,1)\}$, $p_0 = (0,0)$



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A path (walk) of length n starting at p_0 is a sequence $(p_0, p_1, \ldots, p_n)$ of elements in $\mathbb{Z}^d$ such that $p_{i+1} - p_i \in \mathfrak{S}$ for all i.

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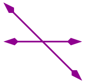

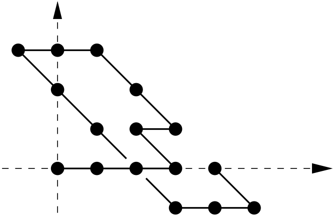

General context: lattice paths confined to cones

Let $\mathfrak{S}$ be a subset of $\mathbb{Z}^d$ (step set, or model) and $p_0 \in \mathbb{Z}^d$ (starting point).

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Let $\mathfrak{C}$ be a cone of $\mathbb{R}^d$ (if $x \in \mathfrak{C}$ and $r \geq 0$ then $r \cdot x \in \mathfrak{C}$).

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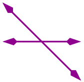

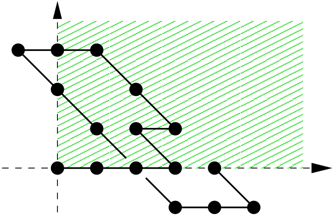

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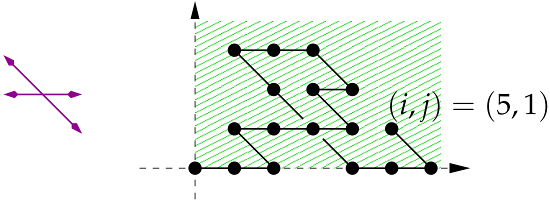

Questions

- What is the number a_n of n-step walks contained in $\mathfrak{C}$?
- For $\boldsymbol{i} \in \mathfrak{C}$, what is the number $a_{n;\boldsymbol{i}}$ of such walks that end at $\boldsymbol{i}$?
- What about their GF's $A(t) = \sum_n a_n t^n$ and $A(t;\boldsymbol{x}) = \sum_{n,\boldsymbol{i}} a_{n;\boldsymbol{i}} \boldsymbol{x}^{\boldsymbol{i}} t^n$?

Why count walks in cones?

Many discrete objects can be encoded in that way:

- discrete mathematics (permutations, trees, words, urns, ...)
- statistical physics (Ising model, ...)
- probability theory (branching processes, games of chance, ...)
- operations research (queueing theory, ...)

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Journal of Statistical Planning and Inference 140 (2010) 2237–2254

Contents lists available at ScienceDirect

Journal of Statistical Planning and Inference

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journal homepage: www.elsevier.com/locate/jspi

A history and a survey of lattice path enumeration

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ARTICLE INFO

Available online 21 January 2010

Keywords:
Lattice path
Reflection principle
Method of images

ABSTRACT

In celebration of the Sixth International Conference on Lattice Path Counting and Applications, it is befitting to review the history of lattice path enumeration and to survey how the topic has progressed thus far.

We start the history with early games of chance specifically the ruin problem which later appears as the ballot problem. We discuss André's Reflection Principle and its misnomer, its relation with the method of images and possible origins from physics and Brownian motion, and the earliest evidence of lattice path techniques and solutions.

In the survey, we give representative articles on lattice path enumeration found in the literature in the last 35 years by the lattice, step set, boundary, characteristics counted, and solution method. Some of this work appears in the author's 2005 dissertation.

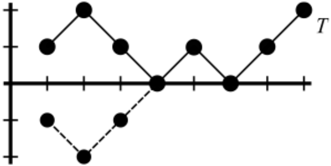
An old topic: The ballot problem and the reflection principle

Ballot problem [Bertrand, 1887]

> Suppose that candidates A and B are running in an election. If a votes are cast for A and b votes are cast for B, where $a > b$, then the probability that A stays ahead of B throughout the counting of the ballots is $(a-b)/(a+b)$.

Lattice path reformulation: given positive integers a, b with $a > b$, find the number of Dyck paths starting at the origin and consisting of a upsteps $\nearrow$ and b downsteps $\searrow$ such that no step ends on the x-axis.

Reflection principle: *Dyck paths in $\mathbb{N}^2$ from $(1,1)$ to $T(a+b, a-b)$ that touch the x-axis* are in bijection with *Dyck paths in $\mathbb{Z}^2$ from $(1,-1)$ to T*



Answer: good paths = paths from $(1,1)$ to T that never touch the x-axis

$$\binom{a+b-1}{a-1} - \binom{a+b-1}{b-1} = \frac{a-b}{a+b}\binom{a+b}{a}$$

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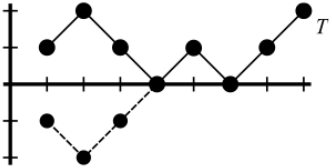


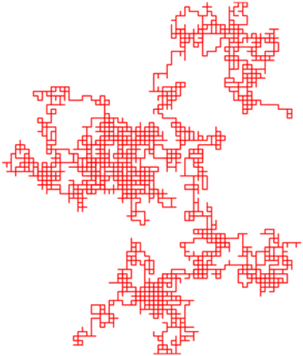
Answer: when $a = n+1$ and $b = n$, this is the Catalan number

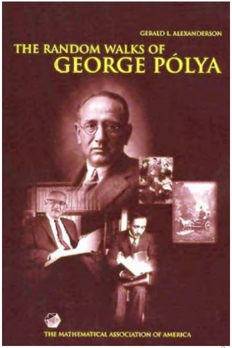
$$C_n = \frac{1}{2n+1}\binom{2n+1}{n+1} = \frac{1}{n+1}\binom{2n}{n}$$

Motto: Drunkard: "Will I ever, ever get home again?"
Polya (1921): "You can't miss; just keep going and stay out of 3D!" (Adam and Delbruck, 1968)

[Pólya, 1921] The simple random walk on $\mathbb{Z}^d$ is recurrent in dimensions $d = 1, 2$ ("Alle Wege fuehren nach Rom"), and transient in dimension $d \geq 3$

Über eine Aufgabe der Wahrscheinlichkeitsrechnung betreffend die Irrfahrt im Straßennetz.





Still a topical issue

Many recent contributors:

Adan, Bacher, Banderier, Bernardi, Bostan, Bousquet-Mélou, Chyzak, Cori, Denisov, Du, Duchon, Dulucq, Fayolle, Fisher, Flajolet, Garbit, Gessel, Gouyou-Beauchamps, Guttmann, Guy, van Hoeij, Iasnogorodski, Johnson, Kauers, Koutschan, Krattenthaler, Kreweras, Kurkova, van Leeuwarden, Malyshev, Melczer, Mishna, Niederhausen, Pech, Petkovšek, Prellberg, Raschel, Rechnitzer, Sagan, Salvy, Viennot, Wachtel, Wang, Wilf, Wilson, Yatchak, Yeats, Zeilberger...

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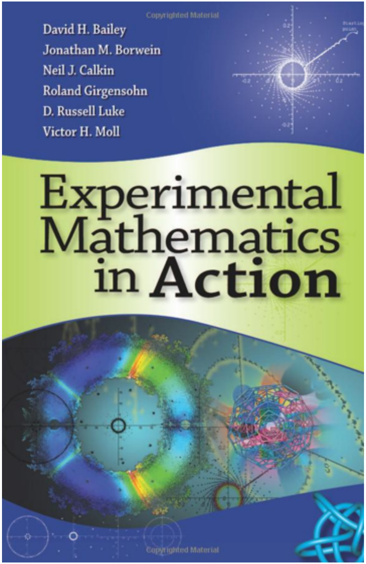
~~**Specific question**~~

~~**Ad hoc solution**~~ ⟶ **Systematic approach**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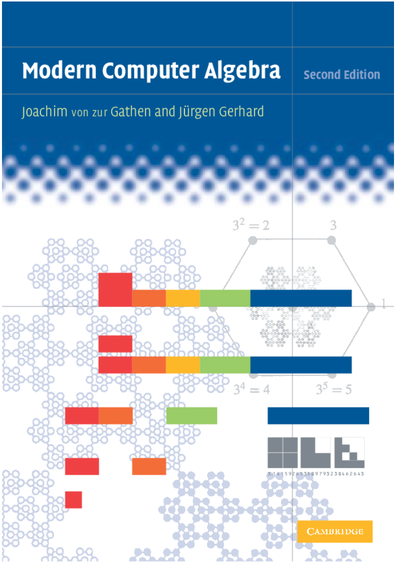
Modern Computer Algebra
Second Edition
Joachim von zur Gathen and Jürgen Gerhard
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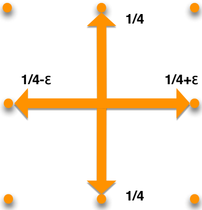
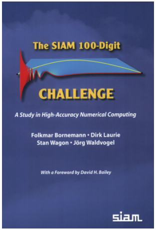
Example: From the SIAM 100-Digit Challenge [Trefethen 2002]



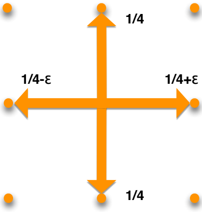


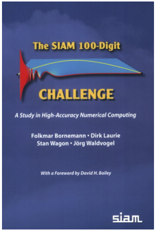
Problem 6

A flea starts at $(0,0)$ on the infinite two-dimensional integer lattice and executes a biased random walk: At each step it hops north or south with probability $1/4$, east with probability $1/4+\epsilon$, and west with probability $1/4-\epsilon$. The probability that the flea returns to $(0,0)$ sometime during its wanderings is $1/2$. What is ϵ?

- Computer algebra conjectures and proves

$$p(\epsilon) = 1 - \sqrt{\frac{A}{2}} \cdot {}_2F_1\left(\begin{matrix} \frac{1}{2}, \frac{1}{2} \\ 1 \end{matrix} \,\middle|\, \frac{2\sqrt{1-16\epsilon^2}}{A} \right)^{-1}, \quad \text{with } A = 1 + 8\epsilon^2 + \sqrt{1-16\epsilon^2}.$$





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$$\epsilon \approx 0.06191395447399094284817521647321217699963877499836207606146725885993101029759615845907105645752087861\ldots$$

A (very) basic cone: the full space

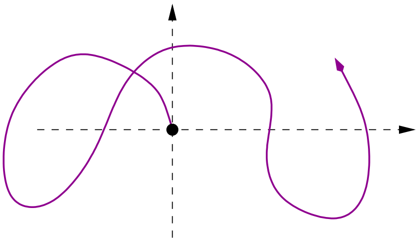
Rational series

If $\mathfrak{S} \subset \mathbb{Z}^d$ is finite and $\mathfrak{C} = \mathbb{R}^d$, then

$$a_n = |\mathfrak{S}|^n, \text{ i.e. } \quad A(t) = \sum_{n \geq 0} a_n t^n = \frac{1}{1 - |\mathfrak{S}|\, t}$$

More generally:

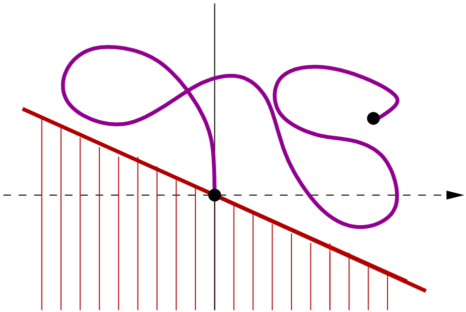
$$A(t; \boldsymbol{x}) = \sum_{n, \boldsymbol{i}} a_{n;\boldsymbol{i}} \boldsymbol{x}^{\boldsymbol{i}} t^n = \frac{1}{1 - t \sum_{\boldsymbol{s} \in \mathfrak{S}} \boldsymbol{x}^{\boldsymbol{s}}}.$$



Also well-known: a (rational) half-space

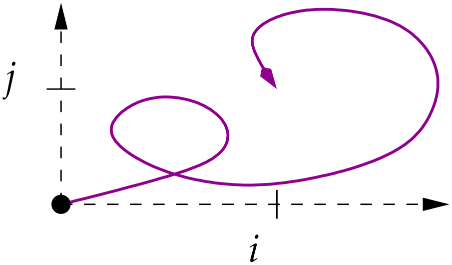
Algebraic series [Bousquet-Mélou & Petkovšek, 2000]

If $\mathfrak{S} \subset \mathbb{Z}^d$ is finite and $\mathfrak{C}$ is a rational half-space, then $A(t;\boldsymbol{x})$ is algebraic, given by an explicit system of polynomial equations.



Example: For Dyck paths (ballot problem), $A(t;1) = \sum_{n\geq 0} C_n t^n = \frac{1-\sqrt{1-4t}}{2t}$

The "next" case: intersection of two half-spaces



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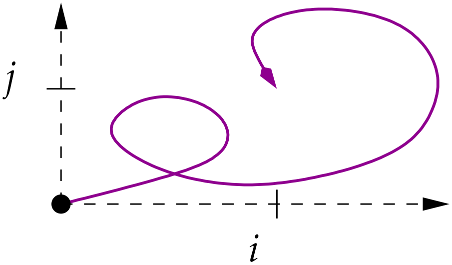
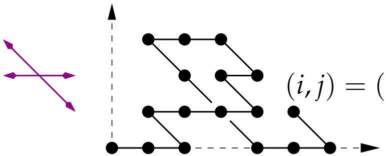
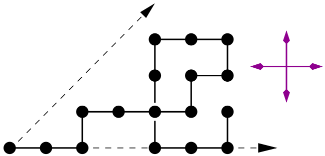




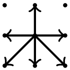
$(i, j) = (5, 1)$ $\simeq$

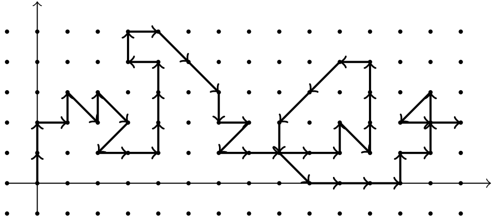


▷ From now on: we focus on nearest-neighbor walks in the quarter plane, i.e. walks in $\mathbb{N}^2$ starting at $(0,0)$ and using steps in a *fixed* subset $\mathfrak{S}$ of

$$\{\swarrow, \leftarrow, \nwarrow, \uparrow, \nearrow, \rightarrow, \searrow, \downarrow\}.$$

▷ Example with $n = 45$, $i = 14$, $j = 2$ for:

$\mathfrak{S} =$ 



Lattice walks with small steps in the quarter plane

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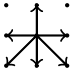
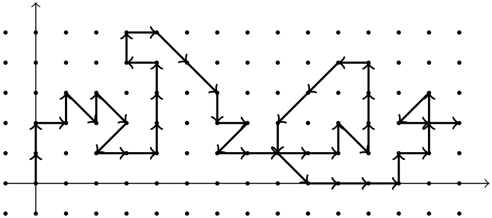
$\mathfrak{S} =$ 



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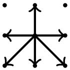
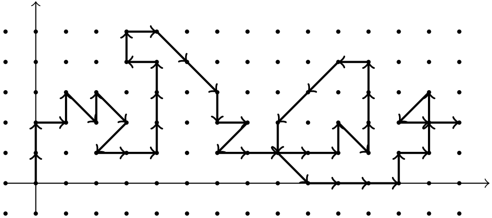
$\mathfrak{S} =$ 



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▷ Specializations:

- $f_{n;0,0}$ = number of walks of length n returning to origin ("excursions");
- $f_n = \sum_{i,j\geq 0} f_{n;i,j}$ = number of walks with prescribed length n.

Generating functions and combinatorial problems

▷ Complete generating function:

$$F(t;x,y) = \sum_{n=0}^{\infty} \left(\sum_{i,j=0}^{\infty} f_{n;i,j} x^i y^j \right) t^n \ \in \mathbb{Q}[x,y][[t]].$$

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- Walks returning to the origin (“excursions”): $F(t;0,0)$;
- Walks with prescribed length: $F(t;1,1) = \sum_{n \geq 0} f_n t^n$;
- Walks ending on the horizontal axis: $F(t;1,0)$;
- Walks ending on the diagonal: “$F(t;0,\infty)$“ $:= \left[x^0\right] F(t;x,1/x)$.

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Combinatorial questions:

Given $\mathfrak{S}$, what can be said about $F(t;x,y)$, resp. $f_{n;i,j}$, and their variants?

- Structure of F: algebraic? transcendental?
- Explicit form: of F? of f?
- Asymptotics of f?

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▷ Specializations:

- Walks returning to the origin ("excursions"): $F(t;0,0)$;
- Walks with prescribed length: $F(t;1,1) = \sum_{n \geq 0} f_n t^n$;
- Walks ending on the horizontal axis: $F(t;1,0)$;
- Walks ending on the diagonal: "$F(t;0,\infty)$" $:= \left[x^0\right] F(t;x,1/x)$.

Combinatorial questions:

Given $\mathfrak{S}$, what can be said about $F(t;x,y)$, resp. $f_{n;i,j}$, and their variants?

- Structure of F: algebraic? transcendental?
- Explicit form: of F? of f?
- Asymptotics of f?

Our goal: Use computer algebra to give computational answers.

Small-step models of interest

From the 2^8 step sets $\mathfrak{S} \subseteq \{-1,0,1\}^2 \setminus \{(0,0)\}$, some are:

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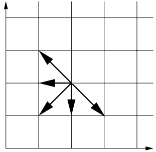

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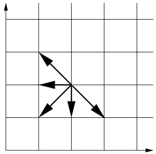

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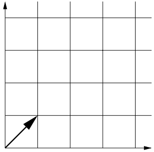

simple,

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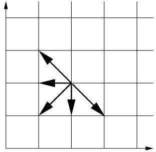

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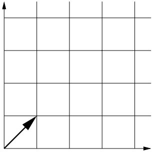
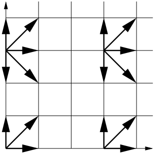


simple,



intrinsic to the half plane,

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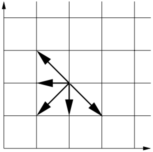
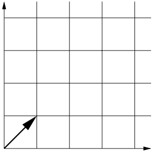


trivial,



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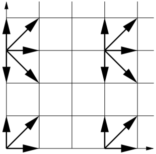

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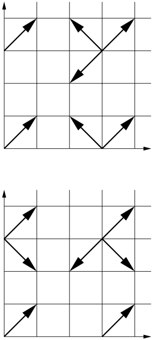
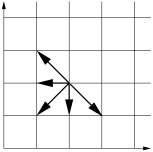


symmetrical.

Small-step models of interest

From the 2^8 step sets $\mathfrak{S} \subseteq \{-1,0,1\}^2 \setminus \{(0,0)\}$, some are:



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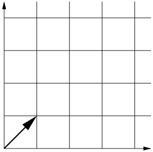

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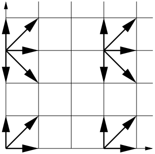
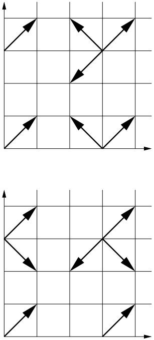


intrinsic to the half plane,



symmetrical.

One is left with 79 interesting distinct models.

Small-step models of interest

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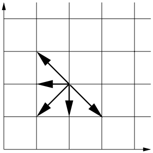

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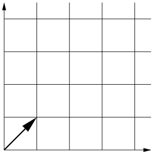

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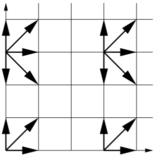

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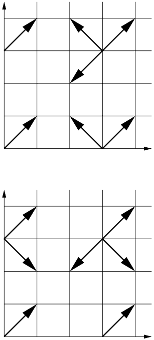
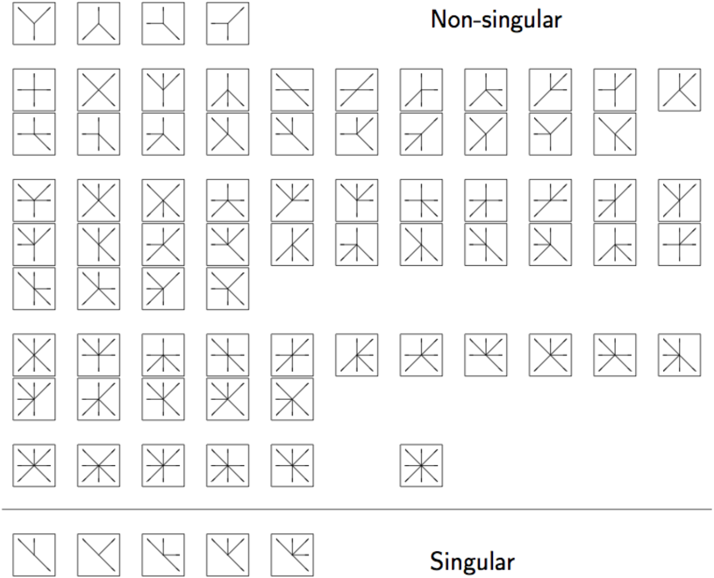


symmetrical.

One is left with 79 interesting distinct models.

Is any further classification possible?

The 79 models



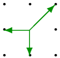
The 79 models

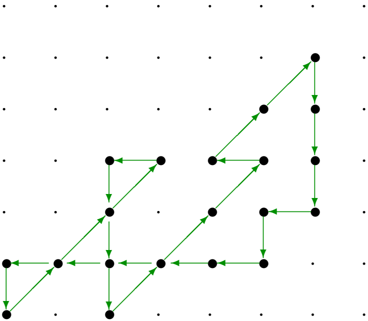
Non-singular

Singular

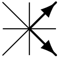
$\mathfrak{S} = \{\downarrow, \leftarrow, \nearrow\}$ $\qquad F_{\mathfrak{S}}(t;x,y) \equiv K(t;x,y)$

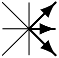
$\mathfrak{S} = \{\nearrow, \swarrow, \leftarrow, \rightarrow\}$ $\qquad F_{\mathfrak{S}}(t;x,y) \equiv G(t;x,y)$





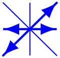
Example: A Kreweras excursion.

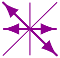
Dyck: 

Motzkin: 

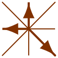
Pólya: 

Kreweras: 

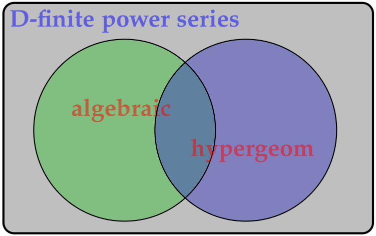
Gessel: 

Gouyou-Beauchamps: 

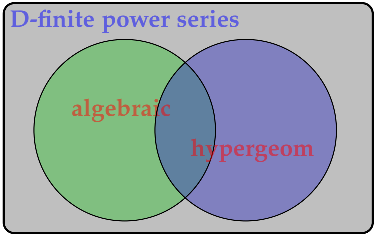
King: 

Exercise: 

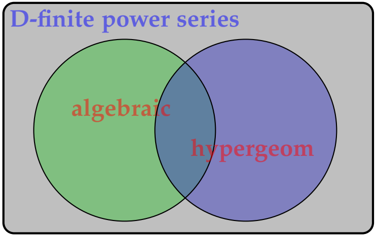
Classification of univariate power series



Classification of univariate power series



▷ *Algebraic*: $S(t) \in \mathbb{Q}[[t]]$ root of a polynomial $P \in \mathbb{Q}[t, T]$, i.e., $P\big(t, S(t)\big) = 0$.



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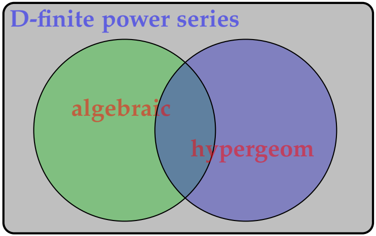
▷ *D-finite*: $S(t) \in \mathbb{Q}[[t]]$ satisfying a linear differential equation with polynomial coefficients $c_r(t)S^{(r)}(t) + \cdots + c_0(t)S(t) = 0$.

Classification of univariate power series



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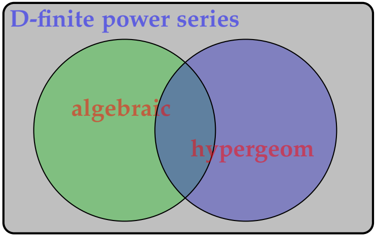
▷ *Hypergeometric*: $S(t) = \sum_{n=0}^{\infty} s_n t^n$ such that $\frac{s_{n+1}}{s_n} \in \mathbb{Q}(n)$. E.g.,

$$ {}_2F_1\left(\begin{matrix} a\ b \\ c \end{matrix} \,\middle|\, t\right) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{t^n}{n!}, \quad (a)_n = a(a+1)\cdots(a+n-1). $$

Classification of univariate power series



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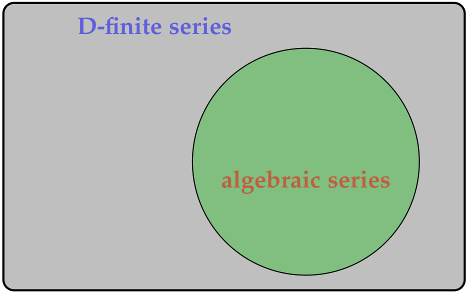
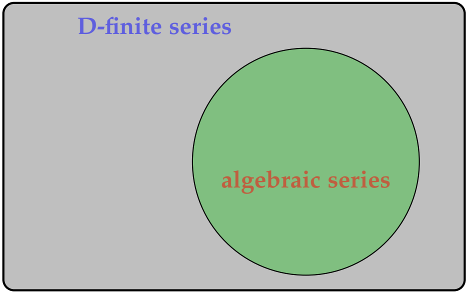
$$ {}_3F_2\left(\left.\begin{matrix} a\ b\ c \\ d\ e \end{matrix}\,\right|\, t\right) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n(c)_n}{(d)_n(e)_n}\,\frac{t^n}{n!}, \quad (a)_n = a(a+1)\cdots(a+n-1). $$



▷ $S \in \mathbb{Q}[[x, y, t]]$ is *algebraic* if it is the root of a polynomial $P \in \mathbb{Q}[x, y, t, T]$.



▷ $S \in \mathbb{Q}[[x,y,t]]$ is *algebraic* if it is the root of a polynomial $P \in \mathbb{Q}[x,y,t,T]$.

▷ $S \in \mathbb{Q}[[x,y,t]]$ is *D-finite* if it satisfies a system of linear partial differential equations with polynomial coefficients

$$\sum_i a_i(t,x,y)\frac{\partial^i S}{\partial x^i} = 0, \quad \sum_i b_i(t,x,y)\frac{\partial^i S}{\partial y^i} = 0, \quad \sum_i c_i(t,x,y)\frac{\partial^i S}{\partial t^i} = 0.$$

Gessel's walks

$$\mathfrak{S} = \{\nearrow, \swarrow, \leftarrow, \rightarrow\}$$

THE ON–LINE ENCYCLOPEDIA
OF INTEGER SEQUENCES®

founded in 1964 by N. J. A. Sloane

1,2,11,85 Search Hints

(Greetings from The On-Line Encyclopedia of Integer Sequences!)

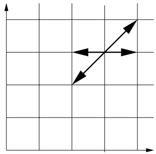
Search: **seq:1,2,11,85**

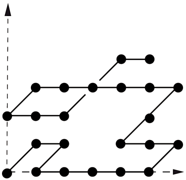
Displaying 1-1 of 1 result found. page 1

Sort: relevance | references | number | modified | created Format: long | short | data

A135404 Gessel sequence: the number of paths of length 2m in the plane, starting and ending at (0,1), with unit steps in the four directions (north, east, south, west) and staying in the region y>0, x>-y. +20 6

1, 2, 11, 85, 782, 8004, 88044, 1020162, 12294260, 152787976, 1946310467, 25302036071, 334560525538, 4488007049900, 60955295750460, 836838395382645, 11597595644244186, 162074575606984788, 2281839419729917410, 32340239369121304038, 461109219391987625316, 6610306991283738684600 (list; graph; refs; listen; history; text; internal format)





Conjecture 1 The generating function of Gessel excursions is equal to

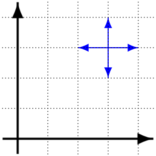
$$G(t;0,0) = {}_3F_2\left(\left.\begin{matrix}5/6 & 1/2 & 1\\ & 5/3 & 2\end{matrix}\,\right|\, 16t^2\right)$$

$$= \sum_{n=0}^{\infty} \frac{(5/6)_n(1/2)_n}{(5/3)_n(2)_n}(4t)^{2n}$$

$$= 1 + 2t^2 + 11t^4 + 85t^6 + 782t^7 + \cdots$$

Conjecture 2
The full generating function $G(t;x,y)$ is not D-finite.

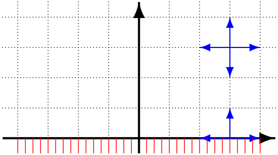
The simple walk in the plane

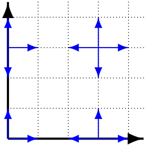


[Pólya, 1921]:

- ▷ Formula $\binom{2n}{n}^2$ for $2n$-excursions
- ▷ Rational generating function

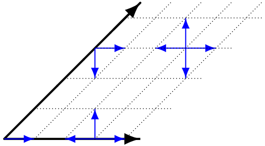
The simple walk in the half-plane and in the quarter-plane

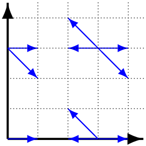




▷ Formulas $\binom{2n+1}{n}C_n$, resp. C_nC_{n+1}, for $2n$-excursions [Arquès, 1986]
▷ Full generating functions: algebraic [Bousquet-Mélou & Petkovšek, 2000], resp. D-finite [Bousquet-Mélou, 2002]

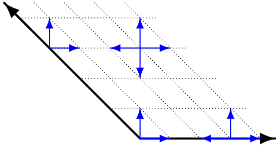
The simple walk in the cone with angle 45°

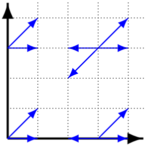




▷ Formula $C_n C_{n+2} - C_{n+1}^2$ for $2n$-excursions [Gouyou-Beauchamps, 1986]
▷ D-finite generating function [Gessel & Zeilberger, 1992]

What about the simple walk in the cone with angle 135°?



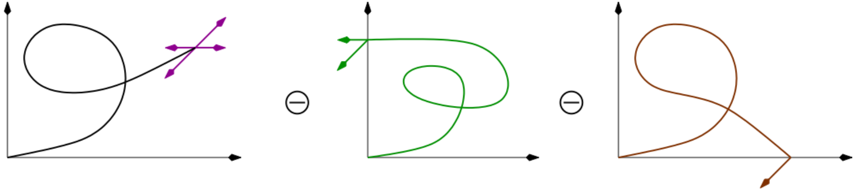


Algebraic reformulation: solving a functional equation

Generating function: $G(t;x,y) = \sum_{n=0}^{\infty} \sum_{i=0}^{n} \sum_{j=0}^{n} g(n;i,j) t^n x^i y^j \in \mathbb{Q}[x,y][[t]]$

"Kernel equation":

$$G(t;x,y) = 1 + t\left(xy + x + \frac{1}{xy} + \frac{1}{x}\right) G(t;x,y) - t\left(\frac{1}{x} + \frac{1}{x}\frac{1}{y}\right) G(t;0,y) - t\frac{1}{xy}\left(G(t;x,0) - G(t;0,0)\right)$$



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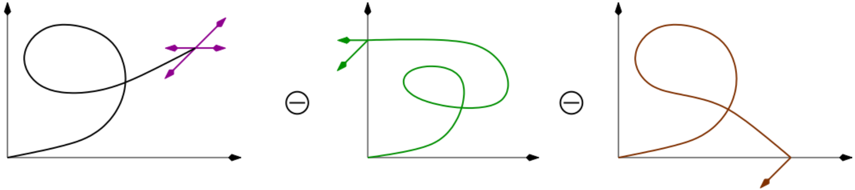
"Kernel equation":

$$
\begin{aligned}
G(t;x,y) = {} & 1 + t\left(xy + x + \frac{1}{xy} + \frac{1}{x}\right) G(t;x,y) \\
& - t\left(\frac{1}{x} + \frac{1}{x}\frac{1}{y}\right) G(t;0,y) - t\frac{1}{xy}\left(G(t;x,0) - G(t;0,0)\right)
\end{aligned}
$$



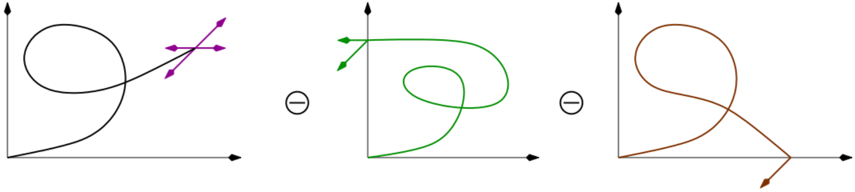
Task: Solve this functional equation!

Algebraic reformulation: solving a functional equation

Generating function: $G(t;x,y) = \sum_{n=0}^{\infty} \sum_{i=0}^{n} \sum_{j=0}^{n} g(n;i,j) t^n x^i y^j \in \mathbb{Q}[x,y][[t]]$

"Kernel equation":

$$\begin{aligned} G(t;x,y) =& 1 + t \left(xy + x + \frac{1}{xy} + \frac{1}{x} \right) G(t;x,y) \\ & - t \left(\frac{1}{x} + \frac{1}{x}\frac{1}{y} \right) G(t;0,y) - t \frac{1}{xy} \left(G(t;x,0) - G(t;0,0) \right) \end{aligned}$$



Task: For the other models: solve 78 similar equations!

Theorem [Kreweras 1965; 100 pages long combinatorial proof!]

$$K(t;0,0) = {}_3F_2\left(\left.\begin{matrix} 1/3 & 2/3 & 1 \\ & 3/2 & 2 \end{matrix}\,\right|\, 27\,t^3\right) = \sum_{n=0}^{\infty} \frac{4^n \binom{3n}{n}}{(n+1)(2n+1)} t^{3n}.$$

Theorem [Kauers, Koutschan & Zeilberger 2009: former Gessel's conj. 1]

$$G(t;0,0) = {}_3F_2\left(\left.\begin{matrix} 5/6 & 1/2 & 1 \\ & 5/3 & 2 \end{matrix}\,\right|\, 16t^2\right) = \sum_{n=0}^{\infty} \frac{(5/6)_n (1/2)_n}{(5/3)_n (2)_n} (4t)^{2n}.$$

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Main results (I): algebraicity of Gessel walks

Theorem [Kreweras 1965; 100 pages long combinatorial proof!]

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▷ Computer-driven discovery and proof.
▷ Guess'n'Prove method, using Hermite-Padé approximants† $\longrightarrow$ Part 2

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† Minimal polynomial $P(x,y,t,G(t;x,y)) = 0$ has $> 10^{11}$ terms; $\approx 30\,$Gb (!)

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▷ New (human) proofs [B., Kurkova & Raschel 2013], [Bousquet-Mélou 2015]

† Minimal polynomial $P(x,y,t,G(t;x,y)) = 0$ has $> 10^{11}$ terms; $\approx 30\,\text{Gb}$ (!)

Main results (II): Explicit form for $G(t;x,y)$

Theorem [B., Kauers & van Hoeij 2010]

Let $V = 1 + 4t^2 + 36t^4 + 396t^6 + \cdots$ be a root of

$$(V-1)(1+3/V)^3 = (16t)^2,$$

let $U = 1 + 2t^2 + 16t^4 + 2xt^5 + 2(x^2+83)t^6 + \cdots$ be a root of

$$x(V-1)(V+1)U^3 - 2V(3x+5xV-8Vt)U^2$$
$$-xV(V^2-24V-9)U + 2V^2(xV-9x-8Vt) = 0,$$

let $W = t^2 + (y+8)t^4 + 2(y^2+8y+41)t^6 + \cdots$ be a root of

$$y(1-V)W^3 + y(V+3)W^2 - (V+3)W + V - 1 = 0.$$

Then $G(t;x,y)$ is equal to

$$\frac{\frac{64(U(V+1)-2V)V^{3/2}}{x(U^2-V(U^2-8U+9-V))^2} - \frac{y(W-1)^4(1-Wy)V^{-3/2}}{t(y+1)(1-W)(W^2y+1)^2}}{(1+y+x^2y+x^2y^2)t - xy} - \frac{1}{tx(y+1)}.$$

▷ Computer-driven discovery and proof; no human proof yet.

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▷ Proof uses guessed minimal polynomials for $G(t;x,0)$ and $G(t;0,y)$.

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- Computer-driven discovery and proof; ~~no human proof yet~~
- Recent (human) proofs [B., Kurkova, Raschel '13], [Bousquet-Mélou '15]

Main results (III): Conjectured D-Finite $F(t;1,1)$ [B. & Kauers 2009]

	OEIS	$\mathfrak{S}$	Pol size	ODE size		OEIS	$\mathfrak{S}$	Pol size	ODE size
1	A005566		—	3, 4	13	A151275		—	5, 24
2	A018224		—	3, 5	14	A151314		—	5, 24
3	A151312		—	3, 8	15	A151255		—	4, 16
4	A151331		—	3, 6	16	A151287		—	5, 19
5	A151266		—	5, 16	17	A001006		2, 2	2, 3
6	A151307		—	5, 20	18	A129400		2, 2	2, 3
7	A151291		—	5, 15	19	A005558		—	3, 5
8	A151326		—	5, 18					
9	A151302		—	5, 24	20	A151265		6, 8	4, 9
10	A151329		—	5, 24	21	A151278		6, 8	4, 12
11	A151261		—	4, 15	22	A151323		4, 4	2, 3
12	A151297		—	5, 18	23	A060900		8, 9	3, 5

Equation sizes = {order, degree}@(algeq, diffeq)

- ▷ Computerized discovery by enumeration + Hermite–Padé
- ▷ 1–22: Confirmed by human proofs in [Bousquet-Mélou & Mishna 2010]
- ▷ 23: Confirmed by a human proof in [B., Kurkova & Raschel 2015]

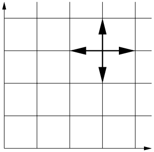
Main results (III): Conjectured D-Finite $F(t;1,1)$ [B. & Kauers 2009]

	OEIS	$\mathfrak{S}$	alg	asympt		OEIS	$\mathfrak{S}$	alg	asympt
1	A005566		N	$\frac{4}{\pi}\frac{4^n}{n}$	13	A151275		N	$\frac{12\sqrt{30}}{\pi}\frac{(2\sqrt{6})^n}{n^2}$
2	A018224		N	$\frac{2}{\pi}\frac{4^n}{n}$	14	A151314		N	$\frac{\sqrt{6}\lambda\mu C^{5/2}}{5\pi}\frac{(2C)^n}{n^2}$
3	A151312		N	$\frac{\sqrt{6}}{\pi}\frac{6^n}{n}$	15	A151255		N	$\frac{24\sqrt{2}}{\pi}\frac{(2\sqrt{2})^n}{n^2}$
4	A151331		N	$\frac{8}{3\pi}\frac{8^n}{n}$	16	A151287		N	$\frac{2\sqrt{2}A^{7/2}}{\pi}\frac{(2A)^n}{n^2}$
5	A151266		N	$\frac{1}{2}\sqrt{\frac{3}{\pi}}\frac{3^n}{n^{1/2}}$	17	A001006		Y	$\frac{3}{2}\sqrt{\frac{3}{\pi}}\frac{3^n}{n^{3/2}}$
6	A151307		N	$\frac{1}{2}\sqrt{\frac{5}{2\pi}}\frac{5^n}{n^{1/2}}$	18	A129400		Y	$\frac{3}{2}\sqrt{\frac{3}{\pi}}\frac{6^n}{n^{3/2}}$
7	A151291		N	$\frac{4}{3\sqrt{\pi}}\frac{4^n}{n^{1/2}}$	19	A005558		N	$\frac{8}{\pi}\frac{4^n}{n^2}$
8	A151326		N	$\frac{2}{\sqrt{3\pi}}\frac{6^n}{n^{1/2}}$					
9	A151302		N	$\frac{1}{3}\sqrt{\frac{5}{2\pi}}\frac{5^n}{n^{1/2}}$	20	A151265		Y	$\frac{2\sqrt{2}}{\Gamma(1/4)}\frac{3^n}{n^{3/4}}$
10	A151329		N	$\frac{1}{3}\sqrt{\frac{7}{3\pi}}\frac{7^n}{n^{1/2}}$	21	A151278		Y	$\frac{3\sqrt{3}}{\sqrt{2}\Gamma(1/4)}\frac{3^n}{n^{3/4}}$
11	A151261		N	$\frac{12\sqrt{3}}{\pi}\frac{(2\sqrt{3})^n}{n^2}$	22	A151323		Y	$\frac{\sqrt{2}\,3^{3/4}}{\Gamma(1/4)}\frac{6^n}{n^{3/4}}$
12	A151297		N	$\frac{\sqrt{3}B^{7/2}}{2\pi}\frac{(2B)^n}{n^2}$	23	A060900		Y	$\frac{4\sqrt{3}}{3\Gamma(1/3)}\frac{4^n}{n^{2/3}}$

$$A = 1+\sqrt{2},\ B = 1+\sqrt{3},\ C = 1+\sqrt{6},\ \lambda = 7+3\sqrt{6},\ \mu = \sqrt{\tfrac{4\sqrt{6}-1}{19}}$$

- ▷ Computerized discovery by enumeration + Hermite–Padé + LLL/PSLQ.
- ▷ Confirmed by human proofs in [Melczer & Wilson, 2015]

The group of a model: the simple walk case



The characteristic polynomial $\chi_{\mathfrak{S}} := x + \frac{1}{x} + y + \frac{1}{y}$

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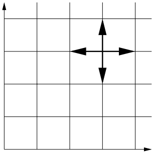
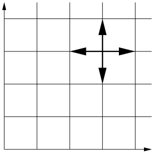


The characteristic polynomial $\chi_{\mathfrak{S}} := x + \frac{1}{x} + y + \frac{1}{y}$ is left invariant under

$$\psi(x,y) = \left(x, \frac{1}{y}\right), \quad \phi(x,y) = \left(\frac{1}{x}, y\right),$$



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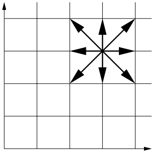
and thus under any element of the group

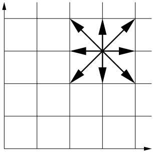
$$\langle \psi, \phi \rangle = \left\{ (x,y), \left(x, \frac{1}{y}\right), \left(\frac{1}{x}, \frac{1}{y}\right), \left(\frac{1}{x}, y\right) \right\}.$$

The group of a model: the general case



The polynomial $\chi_{\mathfrak{S}} := \sum_{(i,j)\in\mathfrak{S}} x^i y^j = \sum_{i=-1}^{1} B_i(y) x^i = \sum_{j=-1}^{1} A_j(x) y^j$

The group of a model: the general case



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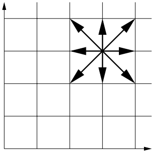

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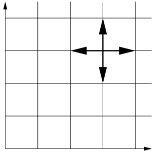
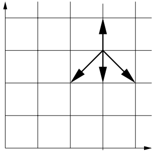
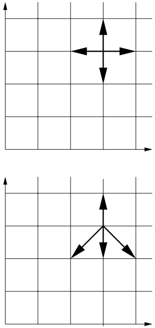
$$\mathcal{G}_{\mathfrak{S}} := \langle \psi, \phi \rangle.$$





Order 4,

Examples of groups



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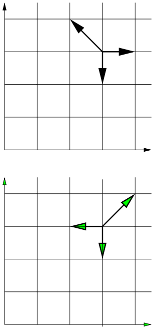
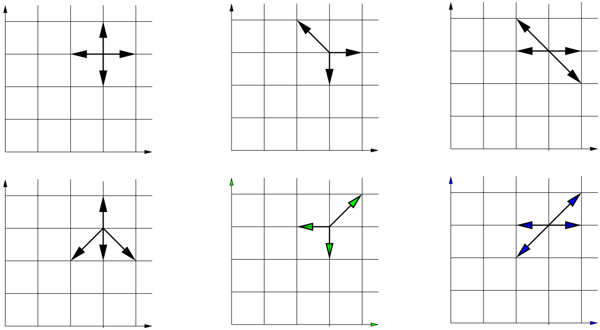


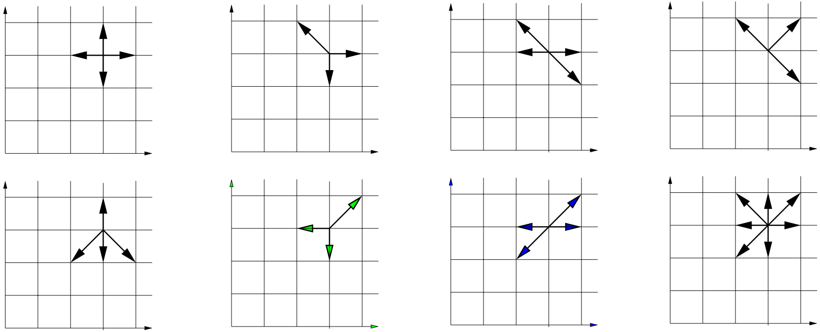
order 6,

Examples of groups



Order 4, order 6, order 8,

Examples of groups



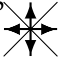
Order 4, order 6, order 8, order ∞.

An important concept: the orbit sum (OS)

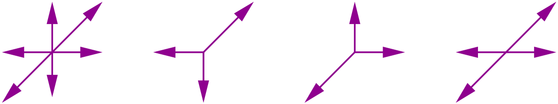
The orbit sum of a model $\mathfrak{S}$ is the following polynomial in $\mathbb{Q}[x, x^{-1}, y, y^{-1}]$:

$$\mathrm{OrbitSum}(\mathfrak{S}) := \sum_{\theta \in \mathcal{G}_{\mathfrak{S}}} (-1)^{\theta} \theta(xy)$$

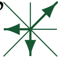
▷ E.g., for the simple walk:

$$\mathrm{OS}$$  $$= x \cdot y - \frac{1}{x} \cdot y + \frac{1}{x} \cdot \frac{1}{y} - x \cdot \frac{1}{y}$$

▷ For 4 models, the orbit sum is zero:



E.g. for the Kreweras model:

$$\mathrm{OS}$$  $$= x \cdot y - \frac{1}{xy} \cdot y + \frac{1}{xy} \cdot x - y \cdot x + y \cdot \frac{1}{xy} - x \cdot \frac{1}{xy} \quad = 0$$

The 79 models: finite and infinite groups

79 models

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23 admit a finite group

[Mishna'07]

79 models

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all $F(t;x,y)$ D-finite

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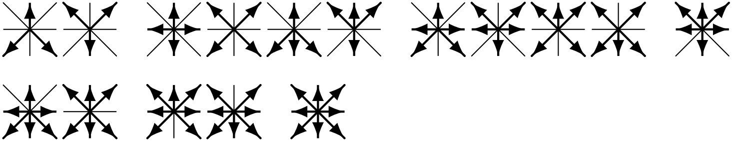
[Bousquet-Mélou & Mishna'10]

$\longrightarrow$ all non-D-finite

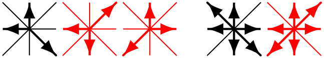
- [Mishna & Rechnitzer'07] and [Melczer & Mishna'13] for 5 singular models
- [Kurkova & Raschel'13] and [B., Raschel & Salvy'13] for all others

The 23 models with a finite group

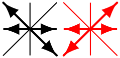
(i) 16 with a vertical symmetry, and group isomorphic to D_2



(ii) 5 with a diagonal or anti-diagonal symmetry, and group isomorphic to D_3



(iii) 2 with group isomorphic to D_4



(i): vertical symmetry; (ii)+(iii): zero drift $\sum_{s \in \mathfrak{S}} s$

In red, models with $\mathrm{OS} = 0$ and algebraic GF

Main results (IV): explicit expressions for the 19 D-finite transcendental models

Theorem [B., Chyzak, van Hoeij, Kauers & Pech, 2016]

Let $\mathfrak{S}$ be one of the 19 models with finite group $\mathcal{G}_{\mathfrak{S}}$, and non-zero orbit sum. Then

- $F_{\mathfrak{S}}$ is expressible using iterated integrals of ${}_2F_1$ expressions.
- Among the 19×4 specializations of $F_{\mathfrak{S}}(t; x, y)$ at $(x, y) \in \{0, 1\}^2$, only 4 are algebraic: for $\mathfrak{S} =$  at $(1, 1)$, and $\mathfrak{S} =$  at $(1, 0), (0, 1), (1, 1)$

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Example (King walks in the quarter plane, A025595)

F  $(t;1,1) = \frac{1}{t}\int_0^t \frac{1}{(1+4x)^3} \cdot {}_2F_1\left(\begin{matrix}\frac{3}{2} & \frac{3}{2}\\ & 2\end{matrix} \,\middle|\, \frac{16x(1+x)}{(1+4x)^2}\right) dx$

$$= 1 + 3t + 18t^2 + 105t^3 + 684t^4 + 4550t^5 + 31340t^6 + 219555t^7 + \cdots$$

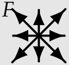
Main results (IV): explicit expressions for the 19 D-finite transcendental models

Theorem [B., Chyzak, van Hoeij, Kauers & Pech, 2016]

Let $\mathfrak{S}$ be one of the 19 models with finite group $\mathcal{G}_{\mathfrak{S}}$, and non-zero orbit sum. Then

- $F_{\mathfrak{S}}$ is expressible using iterated integrals of ${}_2F_1$ expressions.
- Among the 19×4 specializations of $F_{\mathfrak{S}}(t;x,y)$ at $(x,y) \in \{0,1\}^2$, only 4 are algebraic: for $\mathfrak{S} =$  at $(1,1)$, and $\mathfrak{S} =$  at $(1,0),(0,1),(1,1)$

Example (King walks in the quarter plane, A025595)

F 

$$F(t;1,1) = \frac{1}{t}\int_0^t \frac{1}{(1+4x)^3} \cdot {}_2F_1\left(\begin{matrix}\frac{3}{2} & \frac{3}{2}\\ & 2\end{matrix} \,\middle|\, \frac{16x(1+x)}{(1+4x)^2}\right) dx$$

$$= 1 + 3t + 18t^2 + 105t^3 + 684t^4 + 4550t^5 + 31340t^6 + 219555t^7 + \cdots$$

▷ Computer-driven discovery and proof; no human proof yet.

▷ Proof uses creative telescoping, ODE factorization, ODE solving.

⟶ Part 3

Hypergeometric Series Occurring in Explicit Expressions for $F(t;x,y)$

	$\mathfrak{S}$	occurring ${}_2F_1$	w		$\mathfrak{S}$	occurring ${}_2F_1$	w
1		${}_2F_1\left(\begin{matrix}\frac12,\frac12\\1\end{matrix}\middle\vert w\right)$	$16t^2$	11		${}_2F_1\left(\begin{matrix}\frac12,\frac12\\1\end{matrix}\middle\vert w\right)$	$\frac{16t^2}{4t^2+1}$
2		${}_2F_1\left(\begin{matrix}\frac12,\frac12\\1\end{matrix}\middle\vert w\right)$	$16t^2$	12		${}_2F_1\left(\begin{matrix}\frac14,\frac34\\1\end{matrix}\middle\vert w\right)$	$\frac{64t^3(2t+1)}{(8t^2-1)^2}$
3		${}_2F_1\left(\begin{matrix}\frac14,\frac34\\1\end{matrix}\middle\vert w\right)$	$\frac{64t^2}{(12t^2+1)^2}$	13		${}_2F_1\left(\begin{matrix}\frac14,\frac34\\1\end{matrix}\middle\vert w\right)$	$\frac{64t^2(t^2+1)}{(16t^2+1)^2}$
4		${}_2F_1\left(\begin{matrix}\frac12,\frac12\\1\end{matrix}\middle\vert w\right)$	$\frac{16t(t+1)}{(4t+1)^2}$	14		${}_2F_1\left(\begin{matrix}\frac14,\frac34\\1\end{matrix}\middle\vert w\right)$	$\frac{64t^2(t^2+t+1)}{(12t^2+1)^2}$
5		${}_2F_1\left(\begin{matrix}\frac14,\frac34\\1\end{matrix}\middle\vert w\right)$	$64t^4$	15		${}_2F_1\left(\begin{matrix}\frac14,\frac34\\1\end{matrix}\middle\vert w\right)$	$64t^4$
6		${}_2F_1\left(\begin{matrix}\frac14,\frac34\\1\end{matrix}\middle\vert w\right)$	$\frac{64t^3(t+1)}{(1-4t^2)^2}$	16		${}_2F_1\left(\begin{matrix}\frac14,\frac34\\1\end{matrix}\middle\vert w\right)$	$\frac{64t^3(t+1)}{(1-4t^2)^2}$
7		${}_2F_1\left(\begin{matrix}\frac12,\frac12\\1\end{matrix}\middle\vert w\right)$	$\frac{16t^2}{4t^2+1}$	17		${}_2F_1\left(\begin{matrix}\frac13,\frac23\\1\end{matrix}\middle\vert w\right)$	$27t^3$
8		${}_2F_1\left(\begin{matrix}\frac14,\frac34\\1\end{matrix}\middle\vert w\right)$	$\frac{64t^3(2t+1)}{(8t^2-1)^2}$	18		${}_2F_1\left(\begin{matrix}\frac13,\frac23\\1\end{matrix}\middle\vert w\right)$	$27t^2(2t+1)$
9		${}_2F_1\left(\begin{matrix}\frac14,\frac34\\1\end{matrix}\middle\vert w\right)$	$\frac{64t^2(t^2+1)}{(16t^2+1)^2}$	19		${}_2F_1\left(\begin{matrix}\frac12,\frac12\\1\end{matrix}\middle\vert w\right)$	$16t^2$
10		${}_2F_1\left(\begin{matrix}\frac14,\frac34\\1\end{matrix}\middle\vert w\right)$	$\frac{64t^2(t^2+t+1)}{(12t^2+1)^2}$				

▷ All related to the complete elliptic integrals $\int_0^{\pi/2}(1-k^2\sin^2\theta)^{\pm\frac12}\,d\theta$

Main results (V): non-D-finiteness in models with an infinite group

Theorem [B., Rachel & Salvy 2013]

Let $\mathfrak{S}$ be one of the 51 non-singular models with infinite group $\mathcal{G}_{\mathfrak{S}}$.
Then $F_{\mathfrak{S}}(t;0,0)$, and in particular $F_{\mathfrak{S}}(t;x,y)$, are non-D-finite.

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▷ Algorithmic proof. Uses Gröbner basis computations, polynomial factorization, cyclotomy testing.
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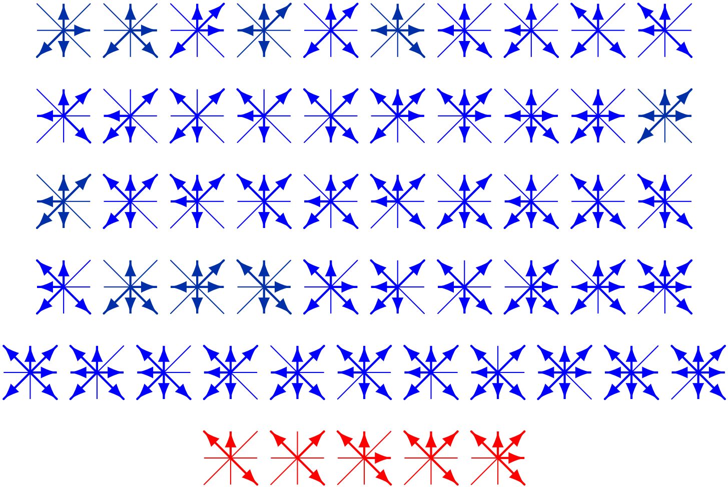
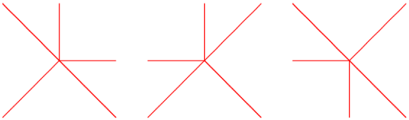
▷ [Bernardi, Bousquet-Mélou & Raschel 2016] For 9 of these 51 models, $F_{\mathfrak{S}}(t;x,y)$ is nevertheless D-algebraic!
▷ Upcoming talk by T. Dreyfus: this is false for the remaining 42 models.

The 56 models with infinite group



In blue, non-singular models, solved by [B., Raschel & Salvy 2013]
In red, singular models, solved by [Melczer & Mishna 2013]

[B., Raschel & Salvy 2013]: $F_{\mathfrak{S}}(t;0,0)$ is not D-finite for the models



For the 1st and the 3rd, the excursions sequence $[t^n]\, F_{\mathfrak{S}}(t;0,0)$

$$1, 0, 0, 2, 4, 8, 28, 108, 372, \ldots$$

is $\sim K \cdot 5^n \cdot n^{-\alpha}$, with $\alpha = 1 + \pi / \arccos(1/4) = 3.383396\ldots$

The irrationality of α prevents $F_{\mathfrak{S}}(t;0,0)$ from being D-finite.

The Main Theorem Let $\mathfrak{S}$ be one of the 74 non-singular models. The following assertions are equivalent:

(1) The full generating function $F_{\mathfrak{S}}(t;x,y)$ is D-finite

(2) the excursions generating function $F_{\mathfrak{S}}(t;0,0)$ is D-finite

(3) the excursions sequence $[t^n]\, F_{\mathfrak{S}}(t;0,0)$ is $\sim K \cdot \rho^n \cdot n^\alpha$, with $\alpha \in \mathbb{Q}$

(4) the group $\mathcal{G}_{\mathfrak{S}}$ is finite (and $|\mathcal{G}_{\mathfrak{S}}| = 2 \cdot \min\{\ell \in \mathbb{N}^\star \mid \frac{\ell}{\alpha+1} \in \mathbb{Z}\}$)

(5) the step set $\mathfrak{S}$ has either an axial symmetry, or zero drift and cardinal different from 5.

Summary: Classification of 2D non-singular walks

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In this case, $F_{\mathfrak{S}}(t;x,y)$ is expressible using nested radicals.
If not, $F_{\mathfrak{S}}(t;x,y)$ is expressible using iterated integrals of ${}_2F_1$ expressions.

Main methods

(1) for proving algebraicity / D-finiteness
- (1a) Guess'n'Prove
- (1b) Creative telescoping

(2) for proving non-D-finiteness
- (2a) Infinite number of singularities, or lacunary
- (2b) Asymptotics

Main methods

(1) for proving algebraicity / D-finiteness

- (1a) Guess'n'Prove — Hermite-Padé approximants
- (1b) Creative telescoping — Diagonals of rational functions

(2) for proving non-D-finiteness

- (2a) Infinite number of singularities, or lacunary
- (2b) Asymptotics

▷ All methods are algorithmic.

Summary: Walks with unit steps in $\mathbb{N}^2$

quadrant models $\mathfrak{S}$: 79

$|\mathcal{G}_{\mathfrak{S}}| < \infty$: 23

$|\mathcal{G}_{\mathfrak{S}}| = \infty$: 56

nonzero orbit sum: 19

zero orbit sum: 4

asymptotics + GB

Kernel method + CT

Guess'n'Prove

not D-finite

D-finite

algebraic

Extensions: Walks with unit steps in $\mathbb{N}^3$

$2^{3^3-1} \approx 67$ millions models, of which ≈ 11 million inherently 3D

3D octant models $\mathfrak{S}$ with ≤ 6 steps: 20804

$|\mathcal{G}_{\mathfrak{S}}| < \infty$: 170

$|\mathcal{G}_{\mathfrak{S}}| = \infty$?: 20634

orbit sum $\neq 0$: 108

orbit sum $= 0$: 62

not D-finite?

kernel method

2D-reducible: 43

not 2D-reducible: 19

D-finite

D-finite

not D-finite?

[B., Bousquet-Mélou, Kauers, Melczer 2015]

- ▷ Open question: are there non-D-finite models with a finite group?

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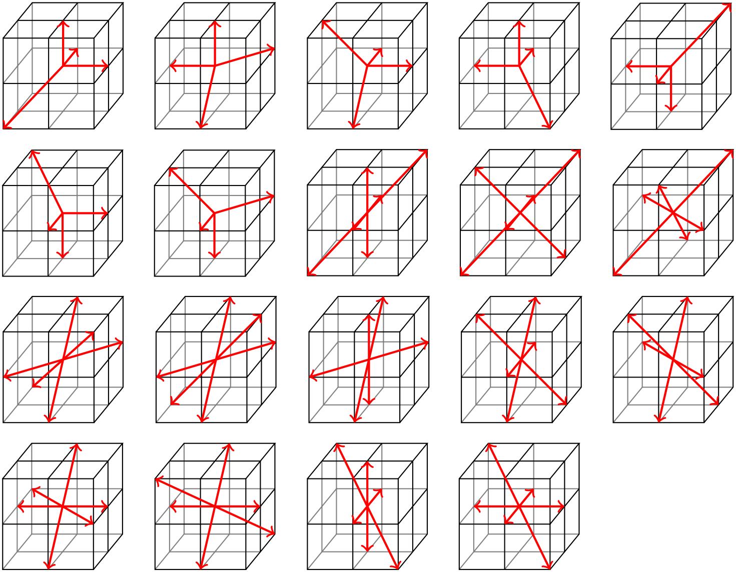
[B., Bousquet-Mélou, Kauers, Melczer 2015]

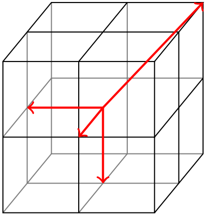
▷ Open question: are there non-D-finite models with a finite group?

▷ [Du, Hu, Wang, 2015]: proofs that groups are infinite in the 20634 cases

▷ [Bacher, Kauers, Yatchak, 2016]: extension to all 3D models; 170 models found with $|\mathcal{G}_\mathfrak{S}| < \infty$ and orbit sum 0 (instead of 19)

The 19 mysterious 3D-models





Two different computations suggest:

$$k_{4n} \approx C \cdot 256^n / n^{3.3257570041744\ldots},$$

so excursions are very probably transcendental (and even non-D-finite)

For  the sequence $f_n = [t^n]F(t;1,1)$ is $\sim \frac{4}{3\sqrt{\pi}} \frac{4^n}{\sqrt{n}}$. This implies

$$\int_0^{1/4} \left\{ \frac{(1-4v)^{1/2}(\frac{1}{2}+v)}{v^2} \left[1 + \frac{1}{2v(1+2v)(1+4v^2)^{1/2}} \times \left((1-v)\,{}_2F_1\left(\begin{matrix} \frac{3}{2}, \frac{1}{2} \\ 1 \end{matrix} \,\middle|\, \frac{16v^2}{1+4v^2} \right) - (1+v)(1-4v+8v^2)\,{}_2F_1\left(\begin{matrix} \frac{1}{2}, \frac{1}{2} \\ 1 \end{matrix} \,\middle|\, \frac{16v^2}{(1+4v^2)} \right) \right) \right] - \frac{1}{v^2} \right\} dv = -2$$

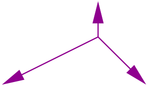
▷ Open question: can this be proved using Computer Algebra?

Extensions: Walks in $\mathbb{N}^2$ with longer steps

- Define (and use) a group $\mathcal{G}$ for models with larger steps?

- Example: When $\mathfrak{S} = \{(0,1),(1,-1),(-2,-1)\}$, there is an underlying group that is finite and

$$xyF(t;x,y) = [x^{>0}y^{>0}]\frac{(x-2x^{-2})(y-(x-x^{-2})y^{-1})}{1-t(xy^{-1}+y+x^{-2}y^{-1})}$$



[B., Bousquet-Mélou & Melczer, in preparation]

▷ Current status:

- 680 models with one large step, 643 proved non D-finite, 32 of 37 have differential equations guessed.

- 5910 models with two large steps, 5754 proved non D-finite, 69 of 156 have differential equations guessed.

Conclusion

- Computer algebra may solve difficult combinatorial problems
- Classification of $F(t; x, y)$ fully completed for 2D small step walks
- Robust algorithmic methods, based on efficient algorithms:
  - Guess'n'Prove
  - Creative Telescoping
- Brute-force and/or use of naive algorithms = hopeless.
  E.g. size of algebraic equations for $G(t; x, y) \approx 30$Gb.

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Classification of $F(t;x,y)$ fully completed for 2D small step walks

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E.g. size of algebraic equations for $G(t;x,y) \approx$ 30Gb.

Lack of “purely human” proofs for some results.

Still missing a unified proof of: finite group $\leftrightarrow$ D-finite.

Open: is $F(t;1,1)$ non-D-finite for all 56 models with infinite group?

Many open questions in dimension > 2.

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Thanks for your attention!