Computer Algebra for Lattice Path Combinatorics

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informatics mathematics Inria

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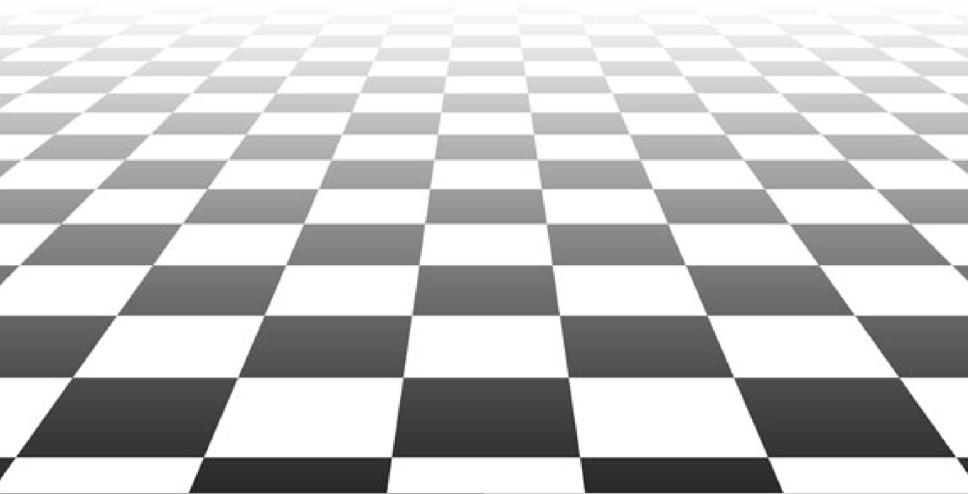
Overview

Part 1: General presentation

Part 2: Guess'n'Prove

Part 3: Creative telescoping

Part 2: Guess'n'Prove



quadrant models: 79

$|G| < \infty$: 23

$|G| = \infty$: 56

nonzero orbit sum: 19

zero orbit sum: 4

asymptotics + GB

Kernel method + CT

Guess'n'Prove

not D-finite

D-finite

algebraic

The Main Theorem Let $\mathfrak{S}$ be a 2D non-singular model with small steps. The following assertions are equivalent:

(1) The full generating function $F_{\mathfrak{S}}(t;x,y)$ is D-finite

(2) the excursions generating function $F_{\mathfrak{S}}(t;0,0)$ is D-finite

(3) the excursions sequence $[t^n]\,F_{\mathfrak{S}}(t;0,0)$ is $\sim K \cdot \rho^n \cdot n^\alpha$, with $\alpha \in \mathbb{Q}$

(4) the group $\mathcal{G}_{\mathfrak{S}}$ is finite (and $|\mathcal{G}_{\mathfrak{S}}| = 2 \cdot \min\{\ell \in \mathbb{N}^\star \mid \frac{\ell}{\alpha+1} \in \mathbb{Z}\}$)

(5) the step set $\mathfrak{S}$ has either an axial symmetry, or zero drift and cardinal different from 5.

Proof

(1) $\Rightarrow$ (2) Easy

(2) $\Rightarrow$ (3) [Denisov & Wachtel 2013] + [Chudnovsky'85, André'89, Katz'70]

(3) $\Rightarrow$ (4) [B., Raschel & Salvy 2013]

(4) $\Rightarrow$ (1) [Bousquet-Mélou & Mishna 2010] + [B. & Kauers 2010]

(5) $\Leftrightarrow$ (4) A posteriori observation

Summary of Part 1: Classification of 2D non-singular walks

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(5) the step set $\mathfrak{S}$ has either an axial symmetry, or zero drift and cardinal different from 5.

Moreover, under (1)–(5), $F_{\mathfrak{S}}(t;x,y)$ is algebraic if and only if the model $\mathfrak{S}$ has positive covariance $\displaystyle\sum_{(i,j)\in\mathfrak{S}} ij - \sum_{(i,j)\in\mathfrak{S}} i \cdot \sum_{(i,j)\in\mathfrak{S}} j > 0$, and iff it has $\mathrm{OS}=0$.

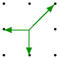
In this case, $F_{\mathfrak{S}}(t;x,y)$ is expressible using nested radicals.
If not, $F_{\mathfrak{S}}(t;x,y)$ is expressible using iterated integrals of ${}_2F_1$ expressions.

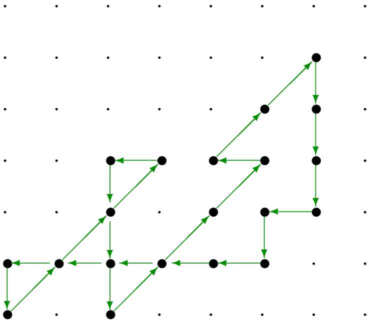
▷ Proof of the last statements: [B., Chyzak, van Hoeij, Kauers & Pech 2017]

Two important models: Kreweras and Gessel walks

$\mathfrak{S} = \{\downarrow, \leftarrow, \nearrow\}$ $\qquad F_{\mathfrak{S}}(t;x,y) \equiv K(t;x,y)$

$\mathfrak{S} = \{\nearrow, \swarrow, \leftarrow, \rightarrow\}$ $\qquad F_{\mathfrak{S}}(t;x,y) \equiv G(t;x,y)$





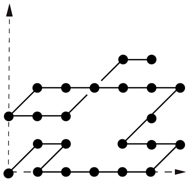
Example: A Kreweras excursion.

Gessel's conjecture

- Gessel walks: walks in $\mathbb{N}^2$ using only steps in $\mathfrak{S} = \{\nearrow, \swarrow, \leftarrow, \rightarrow\}$
- $g(n; i, j)$ = number of walks from $(0, 0)$ to (i, j) with n steps in $\mathfrak{S}$

Question: Find the nature of the generating function

$$G(t; x, y) = \sum_{i,j,n=0}^{\infty} g(n; i, j)\, x^i y^j t^n \ \in \mathbb{Q}[[x, y, t]]$$

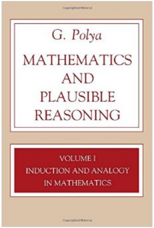


Theorem (B.-Kauers 2010) $G(t; x, y)$ is an algebraic function†.

$\rightarrow$ Effective, computer-driven discovery and proof

† Minimal polynomial $P(x, y, t, G(t; x, y)) = 0$ has $> 10^{11}$ terms; ≈ 30 Gb (!)

First guess, then prove [Pólya, 1954]



Guessing and Proving

George Pólya

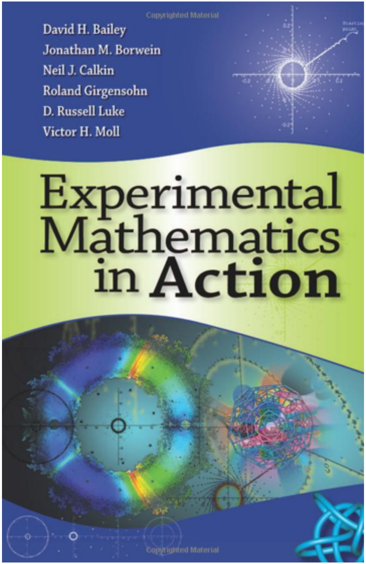
What is "scientific method"? Philosophers and non-philosophers have discussed this question and have not yet finished discussing it. Yet as a first introduction it can be described in three syllables:

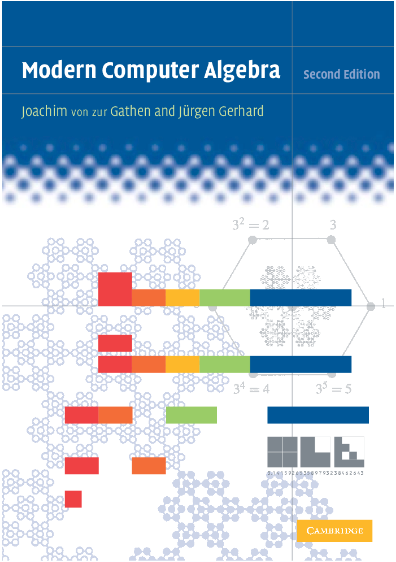
Guess and test.

Mathematicians too follow this advice in their research although they sometimes refuse to confess it. They have, however, something which the other scientists cannot really have. For mathematicians the advice is

First guess, then prove.

Personal bias: Experimental Mathematics using Computer Algebra





Methodology for proving algebraicity

Experimental mathematics –Guess'n'Prove– approach:

(S1) Generate data

(S2) Conjecture

(S3) Prove

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compute a high order expansion of the series $F_{\mathfrak{S}}(t;x,y)$;

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guess a candidate for the minimal polynomial of $F_{\mathfrak{S}}(t;x,y)$, using Hermite-Padé approximation;

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rigorously certify the minimal polynomials, using (exact) polynomial computations.

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+ Efficient Computer Algebra

Step (S1): high order series expansions

$f_{\mathfrak{S}}(n;i,j)$ satisfies the recurrence with constant coefficients

$$f_{\mathfrak{S}}(n+1;i,j) = \sum_{(u,v)\in\mathfrak{S}} f_{\mathfrak{S}}(n;i-u,j-v) \quad \text{for} \quad n,i,j \geq 0$$

+ initial conditions $f_{\mathfrak{S}}(0;i,j) = \delta_{0,i,j}$ and $f_{\mathfrak{S}}(n;-1,j) = f_{\mathfrak{S}}(n;i,-1) = 0$.

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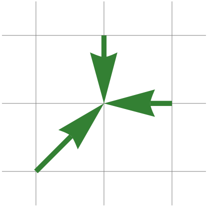
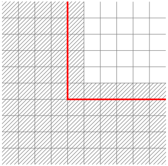
Example: for the Kreweras walks,

$$\begin{aligned} k(n+1;i,j) = k(n;i+1,j) \\ + k(n;i,j+1) \\ + k(n;i-1,j-1) \end{aligned}$$





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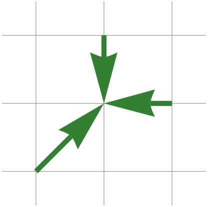

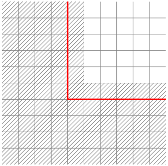

▷ Recurrence is used to compute $F_{\mathfrak{S}}(t;x,y) \bmod t^N$ for large N.

$$\begin{aligned} K(t;x,y) &= 1 + xyt + (x^2y^2 + y + x)t^2 + (x^3y^3 + 2xy^2 + 2x^2y + 2)t^3 \\ &+ (x^4y^4 + 3x^2y^3 + 3x^3y^2 + 2y^2 + 6xy + 2x^2)t^4 \\ &+ (x^5y^5 + 4x^3y^4 + 4x^4y^3 + 5xy^3 + 12x^2y^2 + 5x^3y + 8y + 8x)t^5 + \cdots \end{aligned}$$

Step (S2): guessing equations for $F_{\mathfrak{S}}(t;x,y)$, a first idea

In terms of generating functions, the recurrence on $k(n;i,j)$ reads

$$\boxed{\begin{aligned}&(xy-(x+y+x^2y^2)t)K(t;x,y)\\ &\qquad = xy - xt\,K(t;x,0) - yt\,K(t;0,y)\end{aligned}} \qquad \text{(KerEq)}$$

▷ A similar kernel equation holds for $F_{\mathfrak{S}}(t;x,y)$, for any $\mathfrak{S}$-walk.

Corollary. $F_{\mathfrak{S}}(t;x,y)$ is algebraic (resp. D-finite) if and only if $F_{\mathfrak{S}}(t;x,0)$ and $F_{\mathfrak{S}}(t;0,y)$ are both algebraic (resp. D-finite).

▷ Crucial simplification: equations for $G(t;x,y)$ are huge ($\approx 30\,$Gb)

Step (S2): guessing equations for $F_{\mathfrak{S}}(t;x,0)$ & $F_{\mathfrak{S}}(t;0,y)$

Task 1: Given the first N terms of $S = F_{\mathfrak{S}}(t;x,0) \in \mathbb{Q}[x][[t]]$, search for a differential equation satisfied by S at precision N:

$$c_r(x,t) \cdot \frac{\partial^r S}{\partial t^r} + \cdots + c_1(x,t) \cdot \frac{\partial S}{\partial t} + c_0(x,t) \cdot S = 0 \mod t^N.$$

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- Both tasks amount to linear algebra in size N over $\mathbb{Q}(x)$.
- In practice, we use modular Hermite-Padé approximation (Beckermann-Labahn algorithm) combined with (rational) evaluation-interpolation and rational number reconstruction.
- Fast (FFT-based) arithmetic in $\mathbb{F}_p[t]$ and $\mathbb{F}_p[t]\langle \frac{t}{\partial t} \rangle$.

Step (S2): guessing equations for $K(t; x, 0)$

Using $N = 80$ terms of $K(t; x, 0)$, one can guess

▷ a linear differential equation of order 4, degrees $(14, 11)$ in (t, x), such that

$$t^3 \cdot (3t-1) \cdot (9t^2+3t+1) \cdot (3t^2+24t^2x^3-3xt-2x^2) \cdot$$
$$\cdot (16t^2x^5+4x^4-72t^4x^3-18x^3t+5t^2x^2+18xt^3-9t^4) \cdot$$
$$\cdot (4t^2x^3-t^2+2xt-x^2) \cdot \frac{\partial^4 K(t; x, 0)}{\partial t^4} + \cdots$$
$$= 0 \bmod t^{80}$$

▷ a polynomial of tridegree $(6, 10, 6)$ in (T, t, x)

$$\mathcal{P}_{x,0} = x^6t^{10}T^6 - 3x^4t^8(x-2t)T^5 +$$
$$+ x^2t^6\left(12t^2+3t^2x^3-12xt+\frac{7}{2}x^2\right)T^4 + \cdots$$

such that $\mathcal{P}_{x,0}(K(t; x, 0), t, x) = 0 \bmod t^{80}$.

Step (S2): guessing equations for $G(t;x,0)$ and $G(t;0,y)$

Using $N = 1200$ terms of $G(t;x,y)$, our guesser found candidates

- $\mathcal{P}_{x,0}$ in $\mathbb{Z}[T,t,x]$ of degree $(24,43,32)$, coefficients of 21 digits
- $\mathcal{P}_{0,y}$ in $\mathbb{Z}[T,t,y]$ of degree $(24,44,40)$, coefficients of 23 digits

such that

$$\mathcal{P}_{x,0}(G(t;x,0),t,x) = 0 \mod t^{1200}, \quad \mathcal{P}_{0,y}(G(t;0,y),t,y) = 0 \mod t^{1200}.$$

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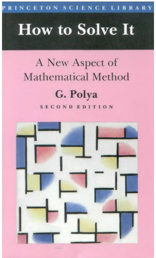
▷ Guessing $\mathcal{P}_{x,0}$ by undetermined coefficients would have required to solve a dense linear system of size $\approx 100\,000$, and ≈ 1000 digits entries!

▷ We actually first guessed differential equations†, then computed their p-curvatures to empirically certify them. This led us suspect the algebraicity of $G(t;x,0)$ and $G(t;0,y)$, using a conjecture of Grothendieck (on differential equations modulo p) as an oracle.

† of order 11, and bidegree $(96,78)$ for $G(t;x,0)$, and $(68,28)$ for $G(t;0,y)$

Guessing is good, proving is better [Pólya, 1957]



Guessing and Proving

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Proof: First guess a polynomial $P(t, T)$ in $\mathbb{Q}[t, T]$, then prove that P admits the power series $g(t) = \sum_{n=0}^{\infty} g_n t^n$ as a root.

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2. Implicit function theorem: $\exists!$ root $r(t) \in \mathbb{Q}[[t]]$ of P.
3. $r(t) = \sum_{n=0}^{\infty} r_n t^n$ being algebraic, it is D-finite, and so is (r_n):

$$(n+2)(3n+5)r_{n+1} - 4(6n+5)(2n+1)r_n = 0, \qquad r_0 = 1$$

$\Rightarrow$ solution $r_n = \frac{(5/6)_n (1/2)_n}{(5/3)_n (2)_n} 16^n = g_n$, thus $g(t) = r(t)$ is algebraic.

1. Setting $y_0 = \frac{x-t-\sqrt{x^2-2tx+t^2(1-4x^3)}}{2tx^2} = t + \frac{1}{x}t^2 + \frac{x^3+1}{x^2}t^3 + \cdots$ in the kernel equation

$$\underbrace{(xy-(x+y+x^2y^2)t)}_{\overset{!}{=}\,0}K(t;x,y) = xy - xtK(t;x,0) - ytK(t;0,y)$$

Step (S3): rigorous proof for Kreweras walks 

1. Setting $y_0 = \frac{x - t - \sqrt{x^2 - 2tx + t^2(1-4x^3)}}{2tx^2} = t + \frac{1}{x}t^2 + \frac{x^3+1}{x^2}t^3 + \cdots$ in the kernel equation (diagonal symmetry implies $K(t;y,x) = K(t;x,y)$)

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shows that $U = K(t;x,0)$ satisfies the reduced kernel equation

$$\boxed{0 = x \cdot y_0 - x \cdot t \cdot U(t,x) - y_0 \cdot t \cdot U(t,y_0)} \qquad \text{(RKerEq)}$$

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3. The guessed candidate $\mathcal{P}_{x,0}(T,t,x)$ has a root $H(t,x)$ in $\mathbb{Q}[[x,t]]$.

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shows that $U = K(t;x,0)$ satisfies the reduced kernel equation

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2. (RKerEq) admits a unique solution in $\mathbb{Q}[[x,t]]$, namely $U = K(t;x,0)$.

3. The guessed candidate $\mathcal{P}_{x,0}(T,t,x)$ has a root $H(t,x)$ in $\mathbb{Q}[[x,t]]$.

4. $U = H(t,x)$ also satisfies (RKerEq) Resultant computations!

Step (S3): rigorous proof for Kreweras walks 

1. Setting $y_0 = \frac{x-t-\sqrt{x^2-2tx+t^2(1-4x^3)}}{2tx^2} = t + \frac{1}{x}t^2 + \frac{x^3+1}{x^2}t^3 + \cdots$ in the kernel equation

$$\underbrace{(xy - (x+y+x^2y^2)t)}_{\stackrel{!}{=}\,0} K(t;x,y) = xy - xtK(t;x,0) - ytK(t;y,0)$$

shows that $U = K(t;x,0)$ satisfies the reduced kernel equation

$$\boxed{0 = x \cdot y_0 - x \cdot t \cdot U(t,x) - y_0 \cdot t \cdot U(t,y_0)} \qquad \text{(RKerEq)}$$

2. (RKerEq) admits a unique solution in $\mathbb{Q}[[x,t]]$, namely $U = K(t;x,0)$.

3. The guessed candidate $\mathcal{P}_{x,0}(T,t,x)$ has a root $H(t,x)$ in $\mathbb{Q}[[x,t]]$.

4. $U = H(t,x)$ also satisfies (RKerEq) Resultant computations!

5. Uniqueness $\implies H(t,x) = K(t;x,0) \implies K(t;x,0)$ is algebraic!

Algebraicity of Kreweras walks: a computer proof in a nutshell

```
[bostan@inria ~]$ maple
    |\^/|     Maple 19 (APPLE UNIVERSAL OSX)
._|\|   |/|_. Copyright (c) Maplesoft, a division of Waterloo Maple Inc. 2014
 \  MAPLE  /  All rights reserved. Maple is a trademark of
 <____ ____>  Waterloo Maple Inc.
      |       Type ? for help.

# HIGH ORDER EXPANSION (S1)
> st,bu:=time(),kernelopts(bytesused):
> f:=proc(n,i,j)
  option remember;
    if i<0 or j<0 or n<0 then 0
    elif n=0 then if i=0 and j=0 then 1 else 0 fi
    else f(n-1,i-1,j-1)+f(n-1,i,j+1)+f(n-1,i+1,j) fi
  end:
> S:=series(add(add(f(k,i,0)*x^i,i=0..k)*t^k,k=0..80),t,80):

# GUESSING (S2)
> libname:=".",libname:gfun:-version();
                                       3.62
> gfun:-seriestoalgeq(S,Fx(t)):
> P:=collect(numer(subs(Fx(t)=T,%[1])),T):

# RIGOROUS PROOF (S3)
> ker := (T,t,x) -> (x+T+x^2*T^2)*t-x*T:
> pol := unapply(P,T,t,x):
> p1 := resultant(pol(z-T,t,x),ker(t*z,t,x),z):
> p2 := subs(T=x*T,resultant(numer(pol(T/z,t,z)),ker(z,t,x),z)):
> normal(primpart(p1,T)/primpart(p2,T));
                                         1

# time (in sec) and memory consumption (in Mb)
> trunc(time()-st),trunc((kernelopts(bytesused)-bu)/1000^2);
                                     7, 617
```

Step (S3): rigorous proof for Gessel walks

Same strategy, but several complications:

- stepset diagonal symmetry is lost: $G(t;x,y) \neq G(t;y,x)$;
- $G(t;0,0)$ occurs in (KerEq) (because of the step $\swarrow$);
- equations are $\approx 5\,000$ times bigger.

$\longrightarrow$ replace equation (RKerEq) by a system of 2 reduced kernel equations.

$\longrightarrow$ fast algorithms needed (e.g., [B., Flajolet, Salvy & Schost 2006] for computations with algebraic series).

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Fast computation of special resultants

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Summary

☺ Guess'n'Prove is a powerful method, especially when combined with efficient computer algebra

☺ It is robust: it can be used to uniformly prove

- ☺ D-finiteness in all the cases with finite group
- ☺ algebraicity in all the cases with finite group and zero orbit sum

☺ Brute-force and/or use of naive algorithms = hopeless.
E.g. size of algebraic equations for $G(t;x,y) \approx 30\,\text{Gb}$.

INSIDE THE BOX

–Hermite-Padé approximants–

Definition: Given a column vector $\mathbf{F} = (f_1, \dots, f_n)^T \in \mathbb{K}[[x]]^n$ and an n-tuple $\mathbf{d} = (d_1, \dots, d_n) \in \mathbb{N}^n$, a Hermite-Padé approximant of type $\mathbf{d}$ for $\mathbf{F}$ is a row vector $\mathbf{P} = (P_1, \dots, P_n) \in \mathbb{K}[x]^n$, ($\mathbf{P} \neq 0$), such that:

(1) $\mathbf{P} \cdot \mathbf{F} = P_1 f_1 + \cdots + P_n f_n = O(x^\sigma)$ with $\sigma = \sum_i (d_i + 1) - 1$,

(2) $\deg(P_i) \leq d_i$ for all i.

σ is called the order of the approximant $\mathbf{P}$.

▷ Very useful concept in number theory (irrationality/transcendence):

- [Hermite 1873]: e is transcendent.
- [Lindemann 1882]: π is transcendent; so does e^α for any $\alpha \in \overline{\mathbb{Q}} \setminus \{0\}$.
- [Apéry 1978, Beukers 1981]: $\zeta(3) = \sum_{n \geq 1} \frac{1}{n^3}$ is irrational.
- [Rivoal 2000]: there exist infinite values of k such that $\zeta(2k+1) \notin \mathbb{Q}$.

Worked example

Let us compute a Hermite-Padé approximant of type $(1, 1, 1)$ for $(1, C, C^2)$, where $C(x) = 1 + x + 2x^2 + 5x^3 + 14x^4 + 42x^5 + O(x^6)$.
This boils down to finding $\alpha_0, \alpha_1, \beta_0, \beta_1, \gamma_0, \gamma_1$ (not all zero) such that

$$\alpha_0+\alpha_1 x+(\beta_0+\beta_1 x)(1+x+2x^2+5x^3+14x^4)+(\gamma_0+\gamma_1 x)(1+2x+5x^2+14x^3+42x^4)=O(x^5)$$

Identifying coefficients, this is equivalent to a homogeneous linear system:

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 2 & 1 \\ 0 & 0 & 2 & 1 & 5 & 2 \\ 0 & 0 & 5 & 2 & 14 & 5 \\ 0 & 0 & 14 & 5 & 42 & 14 \end{bmatrix} \times \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \beta_0 \\ \beta_1 \\ \gamma_0 \\ \gamma_1 \end{bmatrix} = 0 \iff \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 2 \\ 0 & 0 & 2 & 1 & 5 \\ 0 & 0 & 5 & 2 & 14 \\ 0 & 0 & 14 & 5 & 42 \end{bmatrix} \times \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \beta_0 \\ \beta_1 \\ \gamma_0 \end{bmatrix} = -\gamma_1 \begin{bmatrix} 0 \\ 1 \\ 2 \\ 5 \\ 14 \end{bmatrix}.$$

By homogeneity, one can choose $\gamma_1 = 1$.
Then, the violet minor shows that one can take $(\beta_0, \beta_1, \gamma_0) = (-1, 0, 0)$.
The other values are $\alpha_0 = 1$, $\alpha_1 = 0$.

Thus the approximant is $(1, -1, x)$, which corresponds to $P = 1 - y + xy^2$ such that $P(x, C(x)) = 0 \bmod x^5$.

Algebraic and differential approximation = guessing

- Hermite-Padé approximants of $n = 2$ power series are related to Padé approximants, i.e. to approximation of series by rational functions
- algebraic approximants = Hermite-Padé approximants for $f_\ell = A^{\ell-1}$, where $A \in \mathbb{K}[[x]]$ seriestoalgeq, listtoalgeq
- differential approximants = Hermite-Padé approximants for $f_\ell = A^{(\ell-1)}$, where $A \in \mathbb{K}[[x]]$ seriestodiffeq, listtodiffeq

```
> listtoalgeq([1,1,2,5,14,42,132,429],y(x));
                                                  2
                          [1 - y(x) + x y(x) , ogf]

> listtodiffeq([1,1,2,5,14,42,132,429],y(x));
                                         / 2       \
                    /d      \            |d        |
[{-2 y(x) + (2 - 4 x) |-- y(x)| + x |--- y(x)|, y(0) = 1, D(y)(0) = 1}, egf]
                    \dx     /            |  2      |
                                         \dx       /
```

Existence and naive computation

Theorem For any vector $\mathbf{F} = (f_1, \ldots, f_n)^T \in \mathbb{K}[[x]]^n$ and for any n-tuple $\mathbf{d} = (d_1, \ldots, d_n) \in \mathbb{N}^n$, there exists a Hermite-Padé approx. of type $\mathbf{d}$ for $\mathbf{F}$.

Proof: The undetermined coefficients of $P_i = \sum_{j=0}^{d_i} p_{i,j} x^j$ satisfy a linear homogeneous system with $\sigma = \sum_i (d_i + 1) - 1$ eqs and $\sigma + 1$ unknowns.

Corollary Computation in $O(\sigma^\omega)$, for $2 \leq \omega \leq 3$ (linear algebra exponent)

▷ There are better algorithms (the linear system is structured, Sylvester-like):

- Derksen's algorithm (Gaussian-like elimination) $O(\sigma^2)$
- Beckermann-Labahn's algorithm (DAC) $\tilde{O}(\sigma) = O(\sigma \log^2 \sigma)$
- structured linear algebra algorithms for Toeplitz-like matrices $\tilde{O}(\sigma)$

Theorem [Beckermann, Labahn, 1994] One can compute a Hermite-Padé approximant of type $(d, \ldots, d)$ for $\mathbf{F} = (f_1, \ldots, f_n)$ in $\tilde{O}(n^{\omega} d)$ ops. in $\mathbb{K}$.

Ideas:

- Compute a whole matrix of approximants
- Exploit divide-and-conquer

Algorithm:

1. If $\sigma = n(d+1) - 1 \leq$ threshold, call the naive algorithm
2. Else:
   1. recursively compute $\mathbf{P}_1 \in \mathbb{K}[x]^{n \times n}$ s.t. $\mathbf{P}_1 \cdot \mathbf{F} = O(x^{\sigma/2})$, $\deg(\mathbf{P}_1) \approx \frac{d}{2}$
   2. compute "residue" $\mathbf{R}$ such that $\mathbf{P}_1 \cdot \mathbf{F} = x^{\sigma/2} \cdot \big(\mathbf{R} + O(x^{\sigma/2})\big)$
   3. recursively compute $\mathbf{P}_2 \in \mathbb{K}[x]^{n \times n}$ s.t. $\mathbf{P}_2 \cdot \mathbf{R} = O(x^{\sigma/2})$, $\deg(\mathbf{P}_2) \approx \frac{d}{2}$
   4. return $\mathbf{P} := \mathbf{P}_2 \cdot \mathbf{P}_1$

▷ The precise choices of degrees is a delicate issue
▷ Corollary: Gcd, extended gcd, Padé approximants in $\tilde{O}(d)$

BACK TO THE EXERCISE

–A hint–

The exercise

Let $\mathfrak{S} = \{\uparrow, \leftarrow, \searrow\}$. A $\mathfrak{S}$-walk is a path in $\mathbb{Z}^2$ using only steps from $\mathfrak{S}$. Show that, for any integer n, the following quantities are equal:

(i) the number a_n of $\mathfrak{S}$-walks of length n confined to the upper half plane $\mathbb{Z} \times \mathbb{N}$ that start and end at the origin $(0,0)$;

(ii) the number b_n of $\mathfrak{S}$-walks of length n confined to the quarter plane $\mathbb{N}^2$ that start at the origin $(0,0)$ and finish on the diagonal $x = y$.

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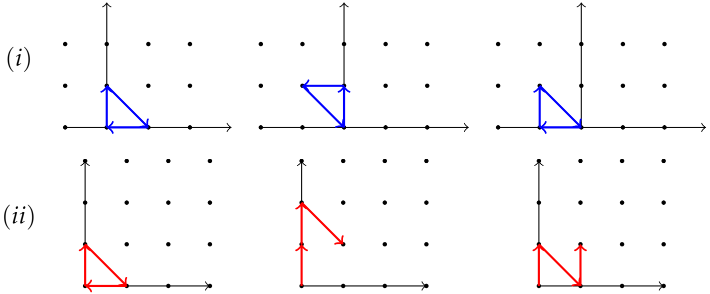
(i) the number a_n of $\mathfrak{S}$-walks of length n confined to the upper half plane $\mathbb{Z} \times \mathbb{N}$ that start and end at the origin $(0,0)$;

(ii) the number b_n of $\mathfrak{S}$-walks of length n confined to the quarter plane $\mathbb{N}^2$ that start at the origin $(0,0)$ and finish on the diagonal $x = y$.

For instance, for $n = 3$, this common value is $a_3 = b_3 = 3$:



$h(n; i, j)$ = nb. of $\{\uparrow, \leftarrow, \searrow\}$-walks in $\mathbb{Z} \times \mathbb{N}$ of length n from $(0,0)$ to (i, j)

The numbers $h(n; i, j)$ satisfy

$$h(n;i,j) = \begin{cases} 0 & \text{if } j < 0 \text{ or } n < 0, \\ \mathbb{1}_{i=j=0} & \text{if } n = 0, \\ \displaystyle\sum_{(i',j') \in \mathfrak{S}} h(n-1; i-i', j-j') & \text{otherwise.} \end{cases}$$

```
> h:=proc(n,i,j)
  option remember;
    if j<0 or n<0 then 0
    elif n=0 then if i=0 and j=0 then 1 else 0 fi
    else h(n-1,i,j-1)+h(n-1,i+1,j)+h(n-1,i-1,j+1) fi
  end:

> A:=series(add(h(n,0,0)*t^n,n=0..12),t,12);
```

$$A = 1 + 3t^3 + 30t^6 + 420t^9 + O(t^{12})$$

A recurrence relation for $\{\uparrow, \leftarrow, \searrow\}$-walks in $\mathbb{N}^2$

$q(n;i,j)$ = nb. of $\{\uparrow, \leftarrow, \searrow\}$-walks in $\mathbb{N}^2$ of length n from $(0,0)$ to (i,j)

The numbers $q(n;i,j)$ satisfy

$$q(n;i,j) = \begin{cases} 0 & \text{if } i < 0 \text{ or } j < 0 \text{ or } n < 0, \\ \mathbb{1}_{i=j=0} & \text{if } n = 0, \\ \displaystyle\sum_{(i',j')\in\mathfrak{S}} q(n-1; i-i', j-j') & \text{otherwise.} \end{cases}$$

```
> q:=proc(n,i,j)
  option remember;
    if i<0 or j<0 or n<0 then 0
    elif n=0 then if i=0 and j=0 then 1 else 0 fi
    else q(n-1,i,j-1)+q(n-1,i+1,j)+q(n-1,i-1,j+1) fi
  end:

> B:=series(add(add(q(n,k,k),k=0..n)*t^n,n=0..12),t,12);
```

$$B = 1 + 3t^3 + 30t^6 + 420t^9 + O(t^{12})$$

```
> seriestorec(series(add(h(n,0,0)*t^n,n=0..30),t,30), u(n))[1];
         2                       2
{(-27 n  - 81 n - 54) u(n) + (n  + 9 n + 18) u(n + 3),
                                        u(0) = 1, u(1) = 0, u(2) = 0}

> rsolve(%, u(n));

{   (n/3)                                        1/2
{ 27      GAMMA(n/3 + 2/3) GAMMA(n/3 + 1/3) 3    (n/3 + 1)
{ -----------------------------------------------------      irem(n, 3) = 0
{                                          2
{                2 Pi GAMMA(n/3 + 2)
{
{                          0                                 irem(n-1, 3) = 0
{
{                          0                                 irem(n-2, 3) = 0

> A:=sum(subs(n=3*n,op(2,%))*t^(3*n),n=0..infinity);
                                                  3
            A := hypergeom([1/3, 2/3], [2], 27 t )
```

▷ Thus, differential guessing predicts

$$A(t) = {}_2F_1\left(\begin{matrix}1/3 \quad 2/3\\ 2\end{matrix} \,\middle|\, 27\,t^3\right) = \sum_{n=0}^{\infty} \frac{(3n)!}{n!^3}\,\frac{t^{3n}}{n+1}.$$

```
> series(add(add(q(n,k,k),k=0..n)*t^n,n=0..30),t,30), u(n))[1];
           2                             2
{(-27 n  - 81 n - 54) u(n) + (n  + 9 n + 18) u(n + 3),
                                               u(0) = 1, u(1) = 0, u(2) = 0}

> rsolve(%, u(n));

{    (n/3)                                           1/2
{ 27      GAMMA(n/3 + 2/3) GAMMA(n/3 + 1/3) 3   (n/3 + 1)
{ -----------------------------------------------------     irem(n, 3) = 0
{                                         2
{                  2 Pi GAMMA(n/3 + 2)
{
{                              0                            irem(n-1, 3) = 0
{
{                              0                            irem(n-2, 3) = 0

> B:=sum(subs(n=3*n,op(2,%))*t^(3*n),n=0..infinity);
                                                      3
             B := hypergeom([1/3, 2/3], [2], 27 t )
```

▷ Thus, differential guessing predicts

$$A(t) = B(t) = {}_2F_1\left(\begin{matrix}1/3 \quad 2/3\\ 2\end{matrix}\,\middle|\, 27\,t^3\right) = \sum_{n=0}^{\infty} \frac{(3n)!}{n!^3}\,\frac{t^{3n}}{n+1}.$$

```
> series(add(add(q(n,k,k),k=0..n)*t^n,n=0..30),t,30), u(n))[1];
          2                       2
{(-27 n  - 81 n - 54) u(n) + (n  + 9 n + 18) u(n + 3),
                                          u(0) = 1, u(1) = 0, u(2) = 0}

> rsolve(%, u(n));

{   (n/3)                                        1/2
{ 27     GAMMA(n/3 + 2/3) GAMMA(n/3 + 1/3) 3   (n/3 + 1)
{ -----------------------------------------------------    irem(n, 3) = 0
{                                    2
{                 2 Pi GAMMA(n/3 + 2)
{
{                             0                            irem(n-1, 3) = 0
{
{                             0                            irem(n-2, 3) = 0

> B:=sum(subs(n=3*n,op(2,%))*t^(3*n),n=0..infinity);
                                                   3
              B := hypergeom([1/3, 2/3], [2], 27 t )
```

▷ In Part 3, we will prove this using creative telescoping

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Thanks for your attention!