Computer algebra for combinatorics

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Two exercises from last time

1. Compute a telescoper for the diagonal of the rational power series

$$\frac{1}{1-x-y} = \sum_{i,j\geq 0} \binom{i+j}{i} x^i y^j$$

in two different ways:

- using the Almkvist-Zeilberger (2G) creative telescoping algorithm;
- using the Hermite reduction-based (4G) creative telescoping algorithm.

2. Let $f, g \in \mathbb{Q}[x]$ be two coprime polynomials. Let $h \in \mathbb{Q}[x]$ be another polynomial such that $\deg h < \deg f + \deg g$.

- Show that the equation

$$sf + tg = h$$

admits an unique solution $(s,t) \in \mathbb{Q}[x]^2$ s.t. $\deg s < \deg g, \quad \deg t < \deg f$.
- Design an algorithm for computing the solution (s,t) starting from (f,g,h) in quasi-optimal complexity.

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   - Design an algorithm for computing the solution (s, t) starting from (f, g, h) in quasi-optimal complexity.

▷ **Uniqueness:** If $(\tilde{s}, \tilde{t})$ is another solution, then $(s - \tilde{s})f = (\tilde{t} - t)g$, so g divides $s - \tilde{s}$ and since $\deg g > \deg(s - \tilde{s})$, we get $s = \tilde{s}$, and also $t = \tilde{t}$.

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▷ Denote by t the remainder of the Euclidean division of T by f:

$$T = qf + t, \qquad \deg t < \deg f, \quad \deg q = \deg T - \deg f$$

and let $s := S + qg$. In other words:

$$(s, t) := (S, T) + q(g, -f).$$

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$$\deg(sf) = \deg(h - tg) < \deg f + \deg g.$$

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▷ **Algorithm** of complexity $O(\mathsf{M}(n) \log n)$, where $n = \deg f + \deg g$.

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▷ Conclusion: $(1-4t)\cdot H_t - 2\cdot H = \partial_x\left(\frac{2x-1}{x-x^2-t}\right)$, where $H_t := \partial_t H$

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▷ I.e., we search for $c_0(t) \in \mathbb{C}(t)$ and $G(t,x) \in \mathbb{C}(t,x)$, such that

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$$H_t + c_0(t) \cdot H = G_x \cdot H + G \cdot H_x$$

▷ Dividing by H and switching LHS and RHS yields the equivalent equation

$$G_x + G \cdot \frac{H_x}{H} = \frac{H_t}{H} + c_0(t),$$

i.e.

$$G_x + G \cdot \frac{2\,x - 1}{x - x^2 - t} = \frac{1}{x - x^2 - t} + c_0(t)$$

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▷ Possible poles of G at $X_\pm = \frac{1}{2} \cdot \left(1 \pm \sqrt{1-4t}\right)$, with local behavior:

$$G = (x - X_\pm)^v + (\text{h.o.t.}), \quad G_x = v \cdot (x - X_\pm)^{v-1} + (\text{h.o.t.})$$

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▷ Around $x = X_\pm$, LHS of (1) writes

$$\left(v\cdot(x-X_\pm)^{v-1}+\cdots\right) + \left(\frac{1-2x}{x-X_\mp}\cdot(x-X_\pm)^{v-1}+\cdots\right)$$

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▷ Look for polynomial $G = g_N(t) \cdot x^N + (\text{l.o.t.})$ with $g_N \neq 0$. Then

$$G_x = N \cdot g_N(t) \cdot x^{N-1} + (\text{l.o.t.})$$

so numer(LHS): $\left(-x^2 \cdot N \cdot g_N(t) \cdot x^{N-1} + \cdots\right) + \left((2x-1) \cdot g_N(t) \cdot x^N + \cdots\right)$
has degree $N+1$ if $N \neq 2$, while numer(RHS) has degree ≤ 2

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has degree $N + 1$ if $N \neq 2$, while numer(RHS) has degree ≤ 2

▷ Thus $N \leq 2$, and $G = g_2 x^2 + g_1 x + g_0$, so (1) yields a linear system / $\mathbb{C}(t)$

$$c_0 + g_1 + g_2 = 0, \quad 2\,t\,g_2 + c_0 - 2\,g_0 = 0, \quad t \cdot (c_0 - g_1) - 1 - g_0 = 0$$

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▷ Conclusion: $(4t-1)H_t + 2H = \partial_x\left(\frac{1-2x}{x-x^2-t}\right)$.

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$$H_t = \frac{1}{(x-x^2-t)^2} = \frac{2/(1-4t)}{x-x^2-t} + \partial_x\left(\frac{(2x-1)/(1-4t)}{x-x^2-t}\right).$$

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▷ Therefore, $(1-4t)H_t - 2H = \partial_x\left(\frac{2x-1}{x-x^2-t}\right)$.

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▷ Hermite reduction computation: extended gcd of $g = x - x^2 - t$ and g_x :

$$g + \left(\frac{1}{4} - \frac{x}{2}\right) g_x = \delta := \frac{1}{4} - t.$$

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▷ As a consequence

$$\frac{1}{g^2} = \frac{1}{\delta} \cdot \frac{g + \frac{1-2x}{4} g_x}{g^2} = \frac{1}{\delta} \cdot \frac{1}{g} + \frac{1-2x}{4\delta} \cdot \frac{g_x}{g^2} = \frac{1}{\delta} \cdot \frac{1}{g} + \frac{2x-1}{4\delta} \cdot \left(\frac{1}{g}\right)_x.$$

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▷ Integrating by parts yields:

$$\frac{1}{g^2} = \frac{1}{2\delta} \cdot \frac{1}{g} + \partial_x \left(\frac{2x-1}{4\delta \cdot g}\right).$$

i.e.,

$$\frac{1}{g^2} = \frac{2}{1-4t} \cdot \frac{1}{g} + \partial_x \left(\frac{2x-1}{(1-4t)g}\right).$$

Enumerative Combinatorics: science of counting

Area of mathematics primarily concerned with counting discrete objects.

▷ Main outcome: theorems

Computer Algebra: effective mathematics

Area of computer science primarily concerned with the algorithmic manipulation of algebraic objects.

▷ Main outcome: algorithms

Computer Algebra for Enumerative Combinatorics

Today: Algorithms for proving Theorems on Lattice Paths Combinatorics.

An (innocent looking) combinatorial question

Let $\mathscr{S} = \{\uparrow, \leftarrow, \searrow\}$. An $\mathscr{S}$-walk is a path in $\mathbb{Z}^2$ using only steps from $\mathscr{S}$. Show that, for any integer n, the following quantities are equal:

(i) number a_n of n-steps $\mathscr{S}$-walks confined to the upper half plane $\mathbb{Z} \times \mathbb{N}$ that start and finish at the origin $(0,0)$ (*excursions*);

(ii) number b_n of n-steps $\mathscr{S}$-walks confined to the quarter plane $\mathbb{N}^2$ that start at the origin $(0,0)$ and finish on the diagonal of $\mathbb{N}^2$ (*diagonal walks*).

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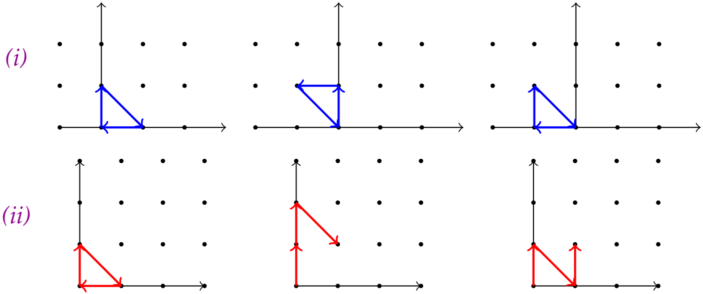
For instance, for $n = 3$, this common value is $a_3 = b_3 = 3$:



Teaser 1: This “exercise” is non-trivial

Teaser 2: It can be solved using Experimental Math and Computer Algebra

Teaser 3: . . . by two robust and efficient algorithmic techniques,
Guess-and-Prove and Creative Telescoping

Why care about counting walks?

Many objects can be encoded by walks:

- probability theory (voting, games of chance, branching processes, . . .)
- discrete mathematics (permutations, trees, words, urns, . . .)
- statistical physics (Ising model, . . .)
- operations research (queueing theory, . . .)

Counting walks is an old topic: the ballot problem [Bertrand, 1887]

Suppose that candidates A and B are running in an election. If a votes are cast for A and b votes are cast for B, where $a > b$, then the probability that A stays ahead of B throughout the counting of the ballots is $(a-b)/(a+b)$.

Lattice path reformulation: find the number of paths in $\mathbb{Z}^2$ with a upsteps $\nearrow$ and b downsteps $\searrow$ that start at the origin and never touch the x-axis

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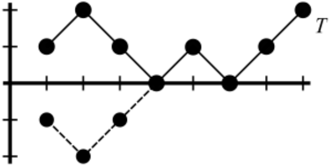
Lattice path reformulation: find the number of paths in $\mathbb{Z}^2$ with $a-1$ upsteps $\nearrow$ and b downsteps $\searrow$ that start at $(1,1)$ and never touch the x-axis

Reflection principle [Aebly, 1923]: *paths in* $\mathbb{Z}^2$ *from* $(1,1)$ *to* $T(a+b, a-b)$ *that do touch the x-axis* are in bijection with *paths in* $\mathbb{Z}^2$ *from* $(1,-1)$ *to* T



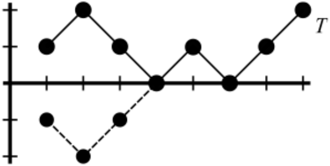
Answer: $\underbrace{(\textit{paths in } \mathbb{Z}^2 \textit{ from } (1,1) \textit{ to } T)}_{\binom{a+b-1}{a-1}} - \underbrace{(\textit{paths in } \mathbb{Z}^2 \textit{ from } (1,-1) \textit{ to } T)}_{\binom{a+b-1}{b-1}}$

Counting walks is an old topic: the ballot problem [Bertrand, 1887]

Suppose that candidates A and B are running in an election. If a votes are cast for A and b votes are cast for B, where $a > b$, then the probability that A stays ahead of B throughout the counting of the ballots is $(a-b)/(a+b)$.

Lattice path reformulation: find the number of paths in $\mathbb{Z}^2$ with $a-1$ upsteps $\nearrow$ and b downsteps $\searrow$ that start at $(1,1)$ and never touch the x-axis

Reflection principle [Aebly, 1923]: *paths in* $\mathbb{Z}^2$ *from* $(1,1)$ *to* $T(a+b, a-b)$ *that do touch the x-axis* are in bijection with *paths in* $\mathbb{Z}^2$ *from* $(1,-1)$ *to* T



Answer: $\underbrace{(\textit{paths in } \mathbb{Z}^2 \textit{ from } (1,1) \textit{ to } T) - (\textit{paths in } \mathbb{Z}^2 \textit{ from } (1,-1) \textit{ to } T)}$

$$\binom{a+b-1}{a-1} - \binom{a+b-1}{b-1} = \frac{a-b}{a+b}\binom{a+b}{a}$$

. . . but it is still a very hot topic

Lot of recent activity; many recent contributors:

Arquès, Bacher, Banderier, Beaton, Bernardi, Biane, Bostan, Bousquet-Mélou, Buchacher, Budd, Chyzak, Cori, Courtiel, Denisov, Dreyfus, Du, Duchon, Dulucq, Duraj, Fayolle, Fisher, Flajolet, Fusy, Garbit, Gessel, Gouyou-Beauchamps, Guttmann, Guy, Hardouin, van Hoeij, Hou, Iasnogorodski, Johnson, Kauers, Kenyon, Koutschan, Krattenthaler, Kreweras, Kurkova, Lecouvey, Malyshev, Melczer, Miller, Mishna, Niederhausen, Owczarek, Pech, Petkovšek, Prellberg, Raschel, Rechnitzer, Roques, Sagan, Salvy, Sheffield, Singer, Tarrago, Trotignon, Verron, Viennot, Wachtel, Wallner, Wang, Wilf, D. Wilson, M. Wilson, Xu, Yatchak, Yeats, Zeilberger, . . .

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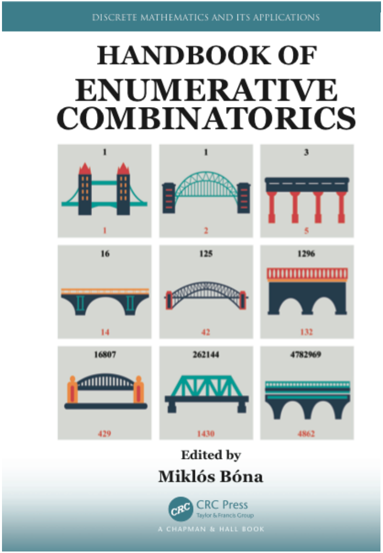
etc.

~~**Specific question**~~

~~**Ad hoc solution**~~ ⟶ **Systematic approach**

. . . but it is still a very hot topic



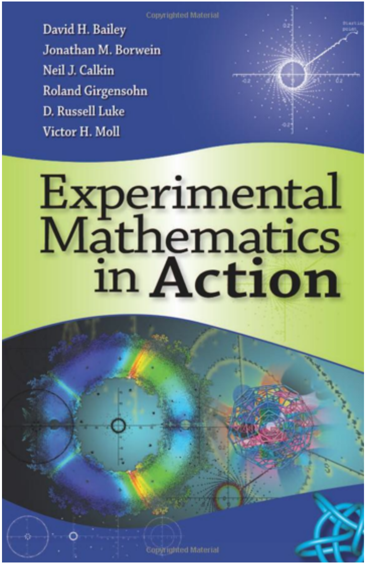
Chapter 10

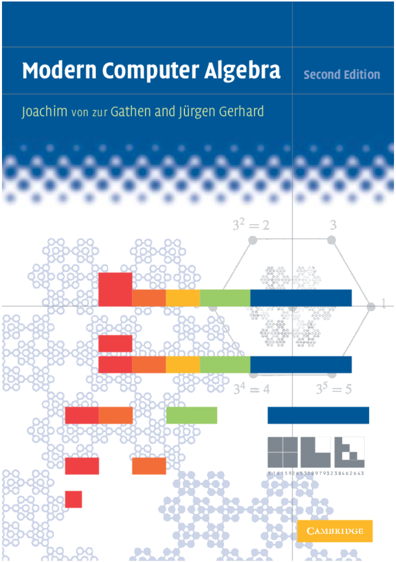
Lattice Path Enumeration

Christian Krattenthaler
Universität Wien

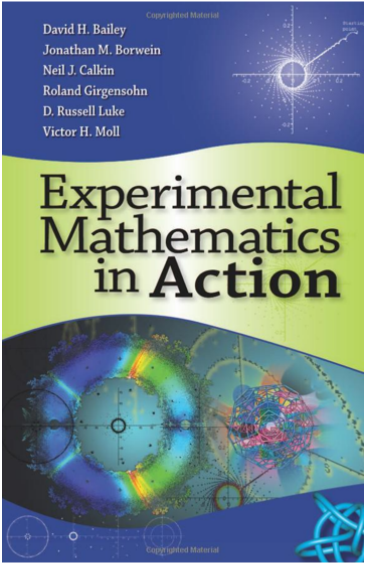
CONTENTS

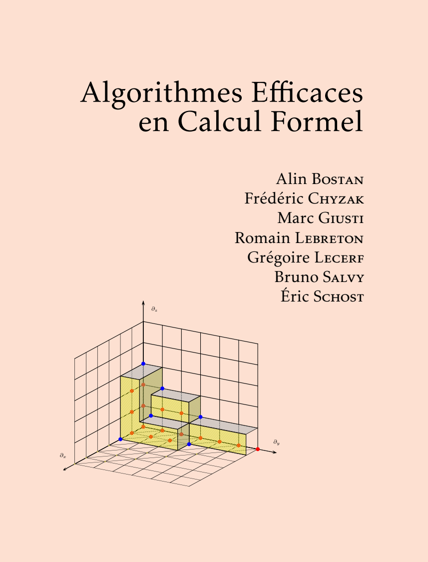
Our approach: Experimental Mathematics using Computer Algebra





Our approach: Experimental Mathematics using Computer Algebra





▷ Nearest-neighbor walks in the quarter plane:
$\mathscr{S}$-walks in $\mathbb{N}^2$: starting at $(0,0)$ and using steps in a *fixed* subset $\mathscr{S}$ of

$$\{\swarrow, \leftarrow, \nwarrow, \uparrow, \nearrow, \rightarrow, \searrow, \downarrow\}$$

▷ Counting sequence $q_{\mathscr{S}}(n)$: number of $\mathscr{S}$-walks of length n

▷ Generating function:

$$Q_{\mathscr{S}}(t) = \sum_{n=0}^{\infty} q_{\mathscr{S}}(n) t^n \ \in \mathbb{Z}[[t]]$$

Lattice walks with small steps in the quarter plane

▷ Nearest-neighbor walks in the quarter plane:
$\mathscr{S}$-walks in $\mathbb{N}^2$: starting at $(0,0)$ and using steps in a *fixed* subset $\mathscr{S}$ of

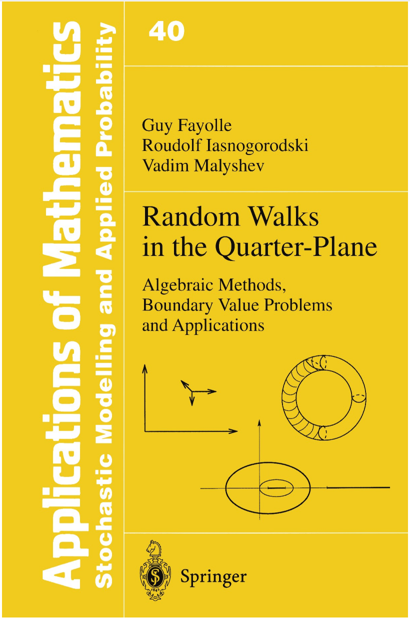
$$\{\swarrow, \leftarrow, \nwarrow, \uparrow, \nearrow, \rightarrow, \searrow, \downarrow\}$$

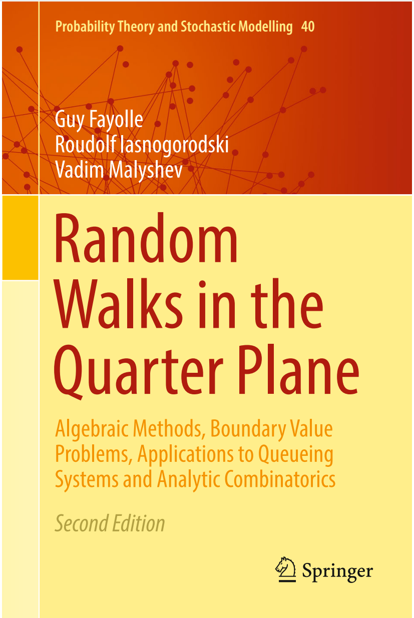
▷ Counting sequence $q_{\mathscr{S}}(i,j;n)$: number of walks of length n ending at (i,j)

▷ Complete generating function (with "catalytic " variables x, y):

$$Q_{\mathscr{S}}(x,y;t) = \sum_{i,j,n=0}^{\infty} q_{\mathscr{S}}(i,j;n) x^i y^j t^n \;\in \mathbb{Z}[[x,y,t]]$$

Entire books dedicated to small step walks in the quarter plane!





Among the 2^8 step sets $\mathscr{S} \subseteq \{-1,0,1\}^2 \setminus \{(0,0)\}$, some are:

Small-step models of interest

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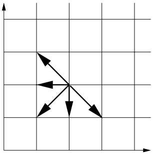
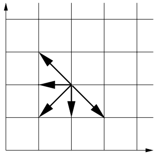
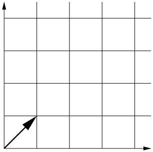


trivial,

Among the 2^8 step sets $\mathscr{S} \subseteq \{-1,0,1\}^2 \setminus \{(0,0)\}$, some are:

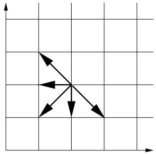


trivial,



simple,

Among the 2^8 step sets $\mathscr{S} \subseteq \{-1,0,1\}^2 \setminus \{(0,0)\}$, some are:



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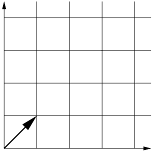
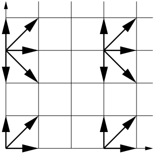
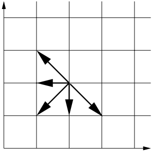


simple,



intrinsic to the half plane,

Among the 2^8 step sets $\mathscr{S} \subseteq \{-1,0,1\}^2 \setminus \{(0,0)\}$, some are:



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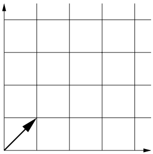
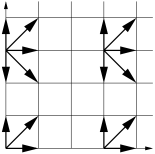


simple,



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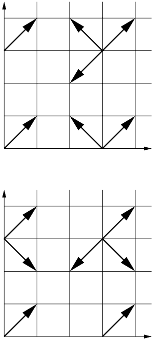
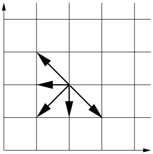


symmetrical.

Among the 2^8 step sets $\mathscr{S} \subseteq \{-1,0,1\}^2 \setminus \{(0,0)\}$, some are:



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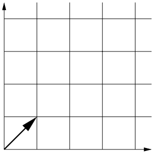

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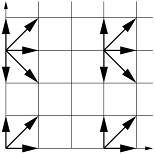

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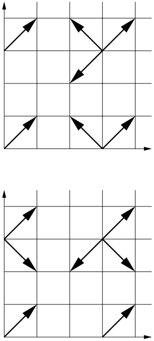
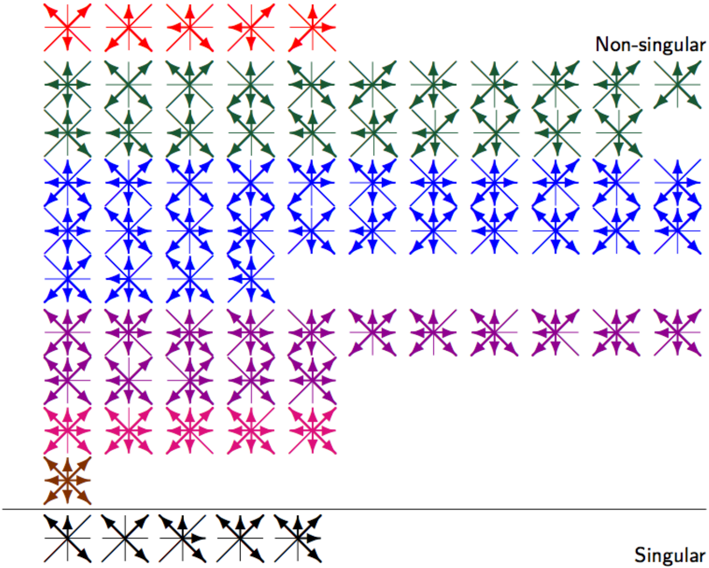


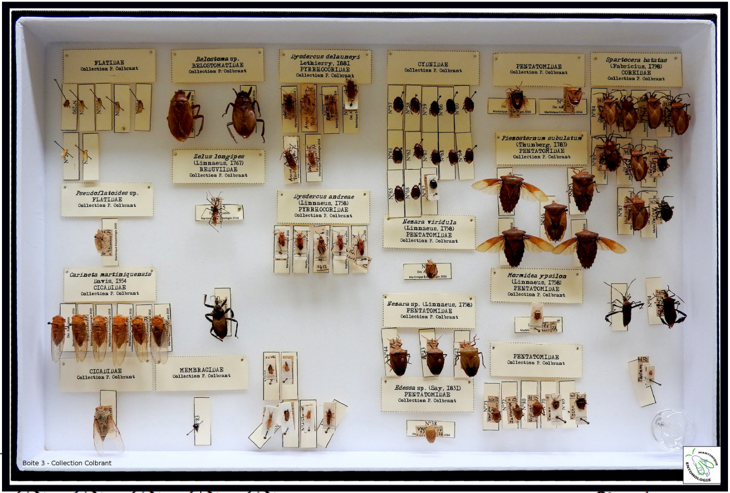
symmetrical.

One is left with 79 interesting distinct models.


Non-singular
Singular

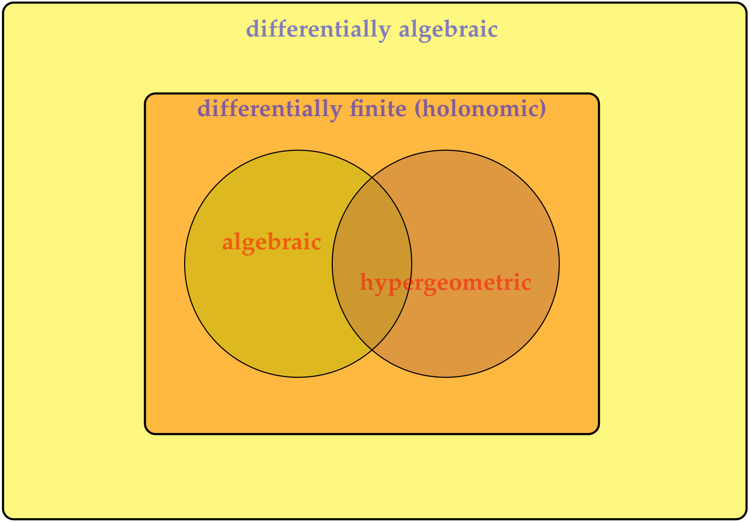
Task: classify their generating functions!

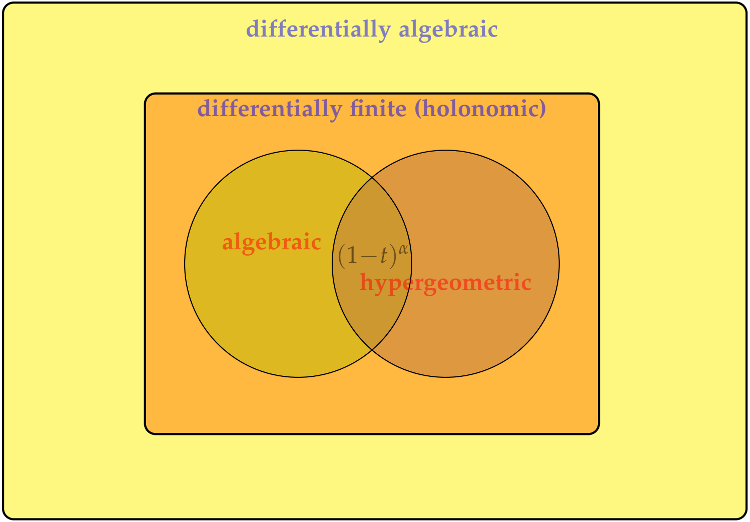
Non-singular

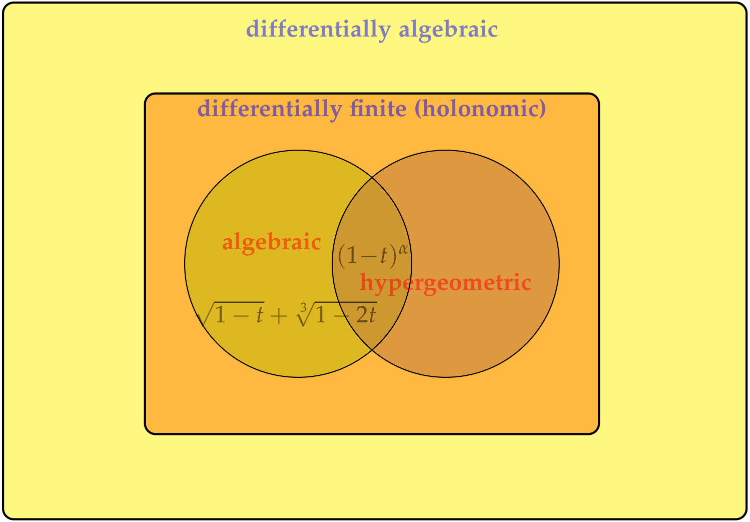


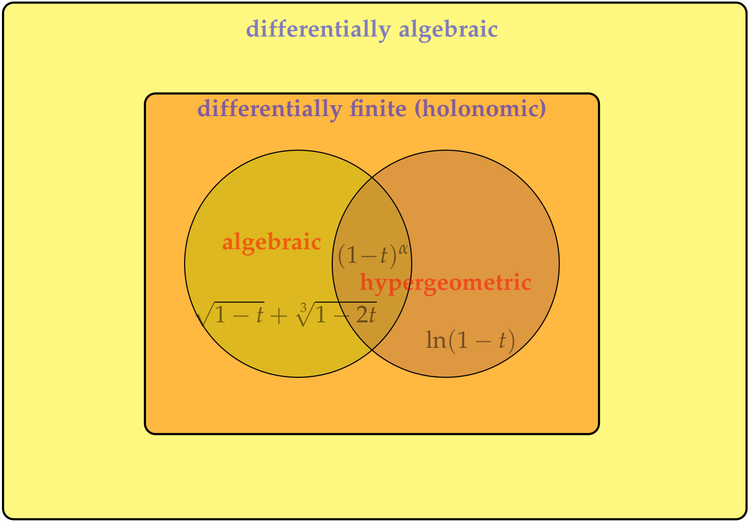
Singular

Classification criterion: properties of generating functions




differentially algebraic
differentially finite (holonomic)
algebraic
$(1-t)^{\alpha}$
hypergeometric


differentially algebraic
differentially finite (holonomic)
algebraic
$(1-t)^\alpha$
hypergeometric
$\sqrt{1-t}+\sqrt[3]{1-2t}$


differentially algebraic
differentially finite (holonomic)
algebraic
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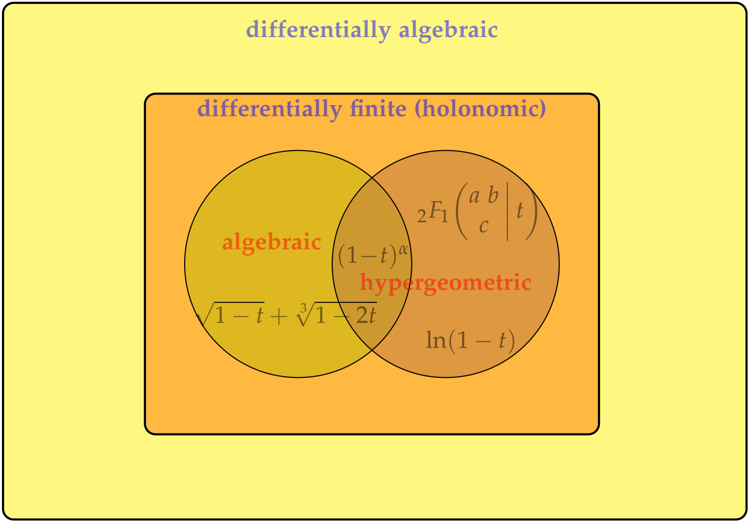
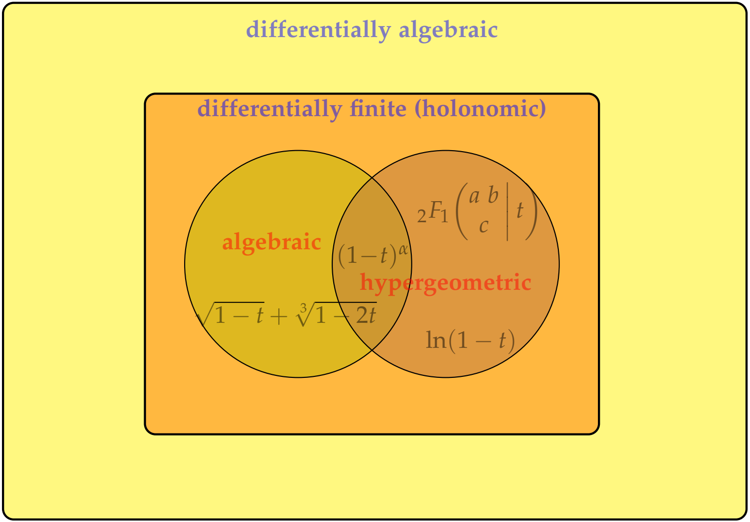
hypergeometric
$\sqrt{1-t}+\sqrt[3]{1-2t}$
$\ln(1-t)$

Classification criterion: properties of generating functions



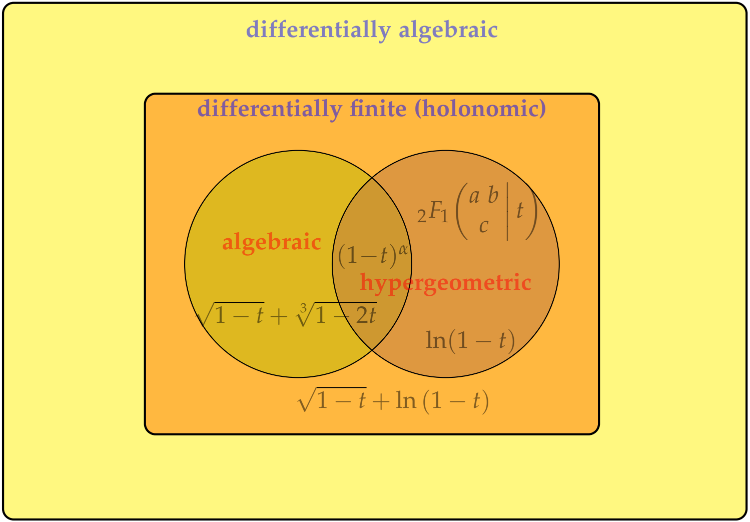
$${}_2F_1\left(\begin{matrix} a\ b \\ c \end{matrix} \middle| t\right) := \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{t^n}{n!}, \quad \text{where} \quad (a)_n = a(a+1)\cdots(a+n-1).$$

Classification criterion: properties of generating functions



E.g., $(1-t)^\alpha = {}_2F_1\left(\begin{matrix} -\alpha\ 1 \\ 1 \end{matrix} \middle| t\right), \quad \ln(1-t) = -t \cdot {}_2F_1\left(\begin{matrix} 1\ 1 \\ 2 \end{matrix} \middle| t\right) = -\sum_{n=1}^{\infty} \frac{t^n}{n}$

Classification criterion: properties of generating functions



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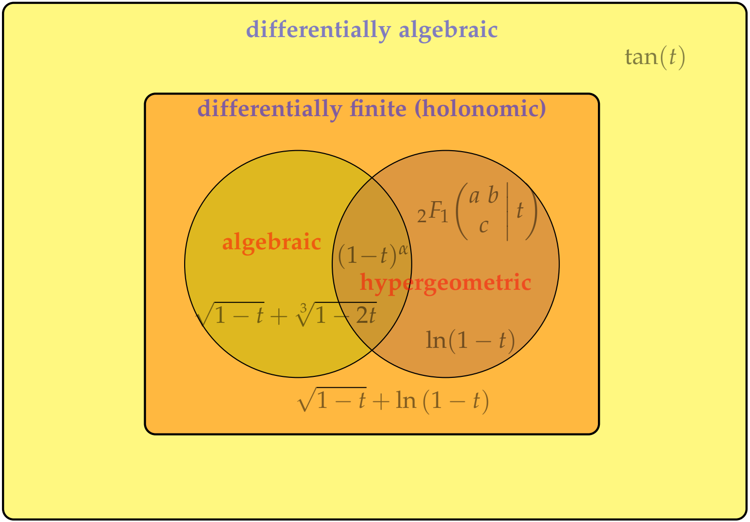
Classification criterion: properties of generating functions



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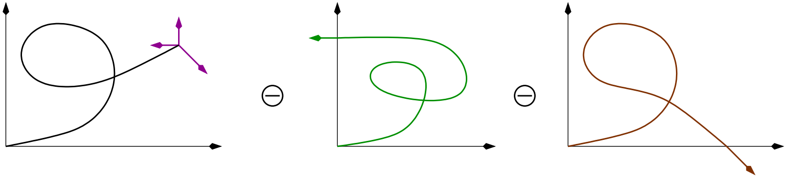
Generating function: $Q(x,y) \equiv Q(x,y;t) = \sum_{i,j,n=0}^{\infty} q(i,j;n) x^i y^j t^n \in \mathbb{Z}[[x,y,t]]$

Recursive construction yields the *kernel equation*

$$Q(x,y) = 1 + t\left(y + \frac{1}{x} + x\frac{1}{y}\right) Q(x,y) - t\frac{1}{x} Q(0,y) - tx\frac{1}{y} Q(x,0)$$

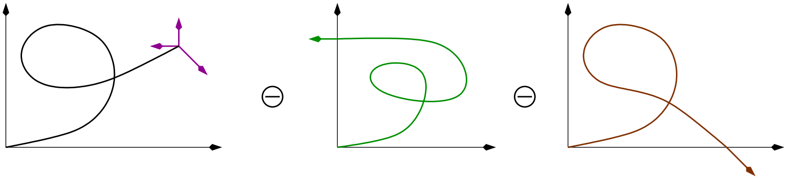


Algebraic reformulation of main task: solving a functional equation

Generating function: $Q(x,y) \equiv Q(x,y;t) = \sum_{i,j,n=0}^{\infty} q(i,j;n)x^i y^j t^n \in \mathbb{Z}[[x,y,t]]$

Recursive construction yields the *kernel equation*

$$\left(1 - t\left(y + \frac{1}{x} + x\frac{1}{y}\right)\right) xyQ(x,y) = xy - tyQ(0,y) - tx^2Q(x,0)$$



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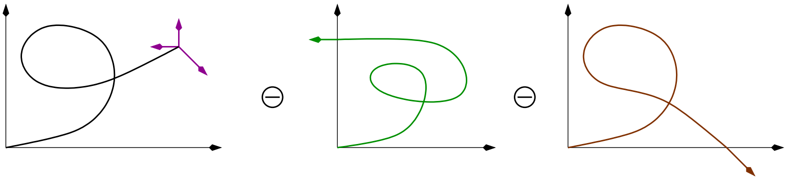

New task: Solve this functional equation!

Algebraic reformulation of main task: solving a functional equation

Generating function: $Q(x,y) \equiv Q(x,y;t) = \sum_{i,j,n=0}^{\infty} q(i,j;n)x^i y^j t^n \in \mathbb{Z}[[x,y,t]]$

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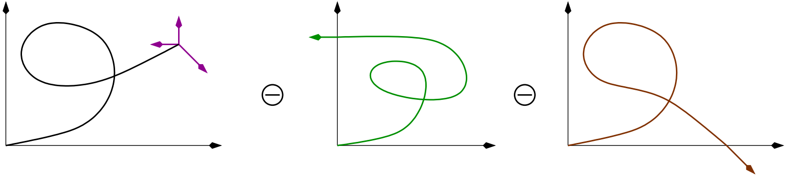
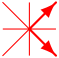
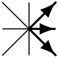


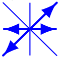
New task: For the other models – solve 78 similar equations!

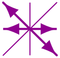
Dyck: 

Motzkin: 

Pólya: 

Kreweras: 

Gessel: 

Gouyou-Beauchamps: 

King walks: 

Tandem walks: 

1. Let $M_{n,k}$ be the number of $\{(1,1),(1,-1)\}$-walks in $\mathbb{N}^2$ of length n that start at $(0,0)$ and end at vertical altitude k. Let $M(x,y) = \sum_{n,k} M_{n,k} x^n y^k$.

   (a) Show that $(y - x(1+y^2)) \cdot M(x,y) = y - x \cdot M(x,0)$

   (b) Deduce that $M(x,y) = \dfrac{\sqrt{1-4x^2} + 2xy - 1}{2x(y - x(1+y^2))}$

$$\mathscr{S} = \{(1,1),(1,-1)\}$$

$$M_{n+1,k} = M_{n,k-1} + M_{n,k+1}, \quad M_{0,0} = 1,\ M_{-1,k} = M_{n,-1} = 0 \text{ for } k,n \geq 0$$

Multiply by $x^{n+1}y^{k+1}$, and sum over $n,k \in \mathbb{N}$

$$\Longrightarrow \qquad y \cdot \Big(M(x,y) - \underbrace{\sum_{k\geq 0} M_{0,k}y^k}_{M(0,y)=1}\Big) = y^2x \cdot M(x,y) + x \cdot \Big(M - \underbrace{\sum_{n\geq 0} M_{n,0}x^n}_{M(x,0)}\Big)$$

$$\Longrightarrow \qquad (y - x(1+y^2)) \cdot M(x,y) = y - x \cdot M(x,0) \qquad \text{(kernel equation)}$$

Solution, ex. 1(b)

$$\mathscr{S} = \{(1,1),(1,-1)\}$$

$$(y - x(1+y^2)) \cdot M(x,y) = y - x \cdot M(x,0) \qquad \text{(kernel equation)}$$

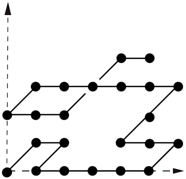
Kernel method: let $y_0 \in \mathbb{Q}[[x]]$ the power series root of $K = y - x(1+y^2)$

$$y_0 = \frac{1-\sqrt{1-4x^2}}{2x} = x + x^3 + 2x^5 + \cdots \in \mathbb{Q}[[x]]$$

Plugging $y = y_0$ in the (kernel equation) $\implies$ $E(x) = M(x,0) = \dfrac{y_0}{x}$

$$\implies \qquad M(x,y) = \frac{y-y_0}{K(x,y)} = \frac{\sqrt{1-4x^2}+2xy-1}{2x(y-x(1+y^2))}$$

- $g(n)$ = number of n-steps $\{\nearrow, \swarrow, \leftarrow, \rightarrow\}$-walks in $\mathbb{N}^2$
  1, 2, 7, 21, 78, 260, 988, 3458, 13300, 47880, ...



Question: What is the nature of the generating function

$$G(t) = \sum_{n=0}^{\infty} g(n)\, t^n \ ?$$

Gessel walks (2000)

- $g(i,j;n)$ = number of n-steps $\{\nearrow, \swarrow, \leftarrow, \rightarrow\}$-walks in $\mathbb{N}^2$ from $(0,0)$ to (i,j)

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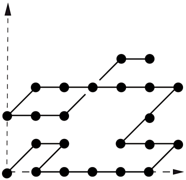

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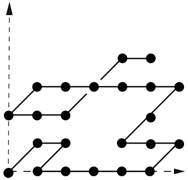

Theorem [B., Kauers, 2010]

$G(x,y;t)$ is an algebraic function†.

▷ computer-driven discovery/proof via *algorithmic Guess-and-Prove*

† Minimal polynomial $P(G(x,y;t);x,y,t) = 0$ has $> 10^{11}$ terms; $\approx 30\,\text{Gb}$ (6 DVDs!)

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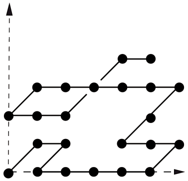
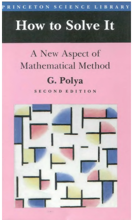


Corollary [B., Kauers, 2010] (former conjecture of Gessel's)

$$(3n+1)\, g(2n) = (12n+2)\, g(2n-1) \text{ and } (n+1)\, g(2n+1) = (4n+2)\, g(2n)$$

▷ computer-driven discovery/proof via *algorithmic Guess-and-Prove*



Guessing and Proving

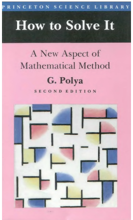
George Pólya

What is "scientific method"? Philosophers and non-philosophers have discussed this question and have not yet finished discussing it. Yet as a first introduction it can be described in three syllables:

Guess and test.

Mathematicians too follow this advice in their research although they sometimes refuse to confess it. They have, however, something which the other scientists cannot really have. For mathematicians the advice is

First guess, then prove.



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First guess, then prove.

generate data → make conjectures → prove them

Guess-and-Prove: a toy example

Question: Find $B_{i,j}$:= the number of $\{\rightarrow, \uparrow\}$-walks in $\mathbb{N}^2$ from $(0,0)$ to (i,j)

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1. There are 2 ways to get to (i,j), either from $(i-1,j)$, or from $(i,j-1)$:

$$B_{i,j} = B_{i-1,j} + B_{i,j-1}$$

2. There is only one way to get to a point on an axis: $B_{i,0} = B_{0,j} = 1$

▷ These two rules completely determine all the numbers $B_{i,j}$

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(I) Generate data:

⋮							
1	7	28	84	210	462	924	
1	6	21	56	126	252	462	
1	5	15	35	70	126	210	
1	4	10	20	35	56	84	
1	3	6	10	15	21	28	
1	2	3	4	5	6	7	
1	1	1	1	1	1	1	…

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1	3	6	10	15	21	28	$\longrightarrow$	$\frac{(i+1)(i+2)}{2}$
1	2	3	4	5	6	7	$\longrightarrow$	$i+1$
1	1	1	1	1	1	1	$\longrightarrow$	1

(II) Guess:

Guess-and-Prove: a toy example

Question: Find $B_{i,j}$:= the number of $\{\rightarrow,\uparrow\}$-walks in $\mathbb{N}^2$ from $(0,0)$ to (i,j)

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$$B_{i,j} \stackrel{?}{=} \frac{(i+j)!}{i!j!}$$

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1	2	3	4	5	6	7	
1	1	1	1	1	1	1	…

(III) Prove: If $C_{i,j} \stackrel{\text{def}}{=} \frac{(i+j)!}{i!j!}$, then

$$\frac{C_{i-1,j}}{C_{i,j}} + \frac{C_{i,j-1}}{C_{i,j}} = \frac{i}{i+j} + \frac{j}{i+j} = 1$$

and $C_{i,0} = C_{0,j} = 1$.

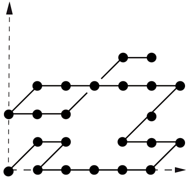
Thus $B_{i,j} = C_{i,j}$

Guess-and-Prove for Gessel walks

- $g(i,j;n)$ = number of n-steps $\{\nearrow, \swarrow, \leftarrow, \rightarrow\}$-walks in $\mathbb{N}^2$ from $(0,0)$ to (i,j)

Question: What is the nature of the generating function

$$G(x,y;t) = \sum_{i,j,n=0}^{\infty} g(i,j;n)\, x^i y^j t^n \ ?$$



Answer: [B., Kauers, 2010] $G(x,y;t)$ is an algebraic function†.

Approach:

1. Generate data: compute G to precision t^{1200} ($\approx$ 1.5 billion coeffs!)
2. Guess: conjecture polynomial equations for $G(x,0;t)$ and $G(0,y;t)$ (degree 24 each, coeffs. of degree $(46,56)$, with 80-bits digits coeffs.)
3. Prove: multivariate resultants of (very big) polynomials (30 pages each)

† Minimal polynomial $P(G(x,y;t);x,y,t) = 0$ has $> 10^{11}$ terms; $\approx 30\,$Gb (6 DVDs!)

A typical Guess-and-Prove algorithmic proof

Theorem ["Gessel excursions are algebraic"]

$$g(t) := G(0,0;\sqrt{t}) = \sum_{n=0}^{\infty} \frac{(5/6)_n (1/2)_n}{(5/3)_n (2)_n} (16t)^n \quad \text{is algebraic.}$$

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Proof: First guess a polynomial $P(t,T)$ in $\mathbb{Q}[t,T]$, then prove that P admits the power series $g(t) = \sum_{n=0}^{\infty} g_n t^n$ as a root.

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3. $r(t) = \sum_{n=0}^{\infty} r_n t^n$ being algebraic, it is D-finite, and so (r_n) is P-recursive:

$$(n+2)(3n+5) r_{n+1} - 4(6n+5)(2n+1) r_n = 0, \qquad r_0 = 1$$

$\Rightarrow$ solution $r_n = \frac{(5/6)_n (1/2)_n}{(5/3)_n (2)_n} 16^n = g_n$, thus $g(t) = r(t)$ is algebraic.

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```
> P:=gfun:-listtoalgeq([seq(pochhammer(5/6,n)*pochhammer(1/2,n)/
      pochhammer(5/3,n)/pochhammer(2,n)*16^n, n=0..100)], g(t)):
> gfun:-diffeqtorec(gfun:-algeqtodiffeq(P[1], g(t)), g(t), r(n));
```

Algorithmic classification of models with D-Finite $Q_{\mathscr{S}}(t) := Q_{\mathscr{S}}(1,1;t)$

	OEIS	$\mathscr{S}$	Pol size	LDE size	Rec size		OEIS	$\mathscr{S}$	Pol size	LDE size	Rec size
1	A005566		—	(3, 4)	(2, 2)	13	A151275		—	(5, 24)	(9, 18)
2	A018224		—	(3, 5)	(2, 3)	14	A151314		—	(5, 24)	(9, 18)
3	A151312		—	(3, 8)	(4, 5)	15	A151255		—	(4, 16)	(6, 8)
4	A151331		—	(3, 6)	(3, 4)	16	A151287		—	(5, 19)	(7, 11)
5	A151266		—	(5, 16)	(7, 10)	17	A001006		(2, 2)	(2, 3)	(2, 1)
6	A151307		—	(5, 20)	(8, 15)	18	A129400		(2, 2)	(2, 3)	(2, 1)
7	A151291		—	(5, 15)	(6, 10)	19	A005558		—	(3, 5)	(2, 3)
8	A151326		—	(5, 18)	(7, 14)						
9	A151302		—	(5, 24)	(9, 18)	20	A151265		(6, 8)	(4, 9)	(6, 4)
10	A151329		—	(5, 24)	(9, 18)	21	A151278		(6, 8)	(4, 12)	(7, 4)
11	A151261		—	(4, 15)	(5, 8)	22	A151323		(4, 4)	(2, 3)	(2, 1)
12	A151297		—	(5, 18)	(7, 11)	23	A060900		(8, 9)	(3, 5)	(2, 3)

Equation sizes = (order, degree)

▷ Computerized discovery: enumeration + guessing [B., Kauers, 2009]

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Equation sizes = (order, degree)

- ▷ Computerized discovery: enumeration + guessing [B., Kauers, 2009]
- ▷ 1–22: DF confirmed by human proofs in [Bousquet-Mélou, Mishna, 2010]
- ▷ 23: DF confirmed by a human proof in [B., Kurkova, Raschel, 2017]
- ▷ All: explicit eqs. proved via CA [B., Chyzak, van Hoeij, Kauers, Pech, 2017]

Algorithmic classification of models with D-Finite $Q_{\mathscr{S}}(t) := Q_{\mathscr{S}}(1,1;t)$

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2	A018224		N	$\frac{2}{\pi}\frac{4^n}{n}$	14	A151314		N	$\frac{\sqrt{6}\lambda\mu C^{5/2}}{5\pi}\frac{(2C)^n}{n^2}$
3	A151312		N	$\frac{\sqrt{6}}{\pi}\frac{6^n}{n}$	15	A151255		N	$\frac{24\sqrt{2}}{\pi}\frac{(2\sqrt{2})^n}{n^2}$
4	A151331		N	$\frac{8}{3\pi}\frac{8^n}{n}$	16	A151287		N	$\frac{2\sqrt{2}A^{7/2}}{\pi}\frac{(2A)^n}{n^2}$
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8	A151326		N	$\frac{2}{\sqrt{3\pi}}\frac{6^n}{n^{1/2}}$		$A=1+\sqrt{2},\ B=1+\sqrt{3},\ C=1+\sqrt{6},\ \lambda=7+3\sqrt{6},$			$\mu=\sqrt{\frac{4\sqrt{6}-1}{19}}$
9	A151302		N	$\frac{1}{3}\sqrt{\frac{5}{2\pi}}\frac{5^n}{n^{1/2}}$	20	A151265		Y	$\frac{2\sqrt{2}}{\Gamma(1/4)}\frac{3^n}{n^{3/4}}$
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11	A151261		N	$\frac{12\sqrt{3}}{\pi}\frac{(2\sqrt{3})^n}{n^2}$	22	A151323		Y	$\frac{\sqrt{2}3^{3/4}}{\Gamma(1/4)}\frac{6^n}{n^{3/4}}$
12	A151297		N	$\frac{\sqrt{3}B^{7/2}}{2\pi}\frac{(2B)^n}{n^2}$	23	A060900		Y	$\frac{4\sqrt{3}}{3\Gamma(1/3)}\frac{4^n}{n^{2/3}}$

▷ Computerized discovery: convergence acceleration + LLL [B., Kauers, '09]

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3	A151312		N	$\frac{\sqrt{6}}{\pi}\frac{6^n}{n}$	15	A151255		N	$\frac{24\sqrt{2}}{\pi}\frac{(2\sqrt{2})^n}{n^2}$
4	A151331		N	$\frac{8}{3\pi}\frac{8^n}{n}$	16	A151287		N	$\frac{2\sqrt{2}A^{7/2}}{\pi}\frac{(2A)^n}{n^2}$
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7	A151291		N	$\frac{4}{3\sqrt{\pi}}\frac{4^n}{n^{1/2}}$	19	A005558		N	$\frac{8}{\pi}\frac{4^n}{n^2}$
8	A151326		N	$\frac{2}{\sqrt{3\pi}}\frac{6^n}{n^{1/2}}$		$A = 1+\sqrt{2},\ B = 1+\sqrt{3},\ C = 1+\sqrt{6},\ \lambda = 7+3\sqrt{6},$			$\mu = \sqrt{\frac{4\sqrt{6}-1}{19}}$
9	A151302		N	$\frac{1}{3}\sqrt{\frac{5}{2\pi}}\frac{5^n}{n^{1/2}}$	20	A151265		Y	$\frac{2\sqrt{2}}{\Gamma(1/4)}\frac{3^n}{n^{3/4}}$
10	A151329		N	$\frac{1}{3}\sqrt{\frac{7}{3\pi}}\frac{7^n}{n^{1/2}}$	21	A151278		Y	$\frac{3\sqrt{3}}{\sqrt{2}\Gamma(1/4)}\frac{3^n}{n^{3/4}}$
11	A151261		N	$\frac{12\sqrt{3}}{\pi}\frac{(2\sqrt{3})^n}{n^2}$	22	A151323		Y	$\frac{\sqrt{2}3^{3/4}}{\Gamma(1/4)}\frac{6^n}{n^{3/4}}$
12	A151297		N	$\frac{\sqrt{3}B^{7/2}}{2\pi}\frac{(2B)^n}{n^2}$	23	A060900		Y	$\frac{4\sqrt{3}}{3\Gamma(1/3)}\frac{4^n}{n^{2/3}}$

- ▷ Computerized discovery: convergence acceleration + LLL [B., Kauers, '09]
- ▷ Asympt. confirmed by human proofs via ACSV in [Melczer, Wilson, 2016]
- ▷ Transcendence proofs via CA [B., Chyzak, van Hoeij, Kauers, Pech, 2017]

Theorem [B., Chyzak, van Hoeij, Kauers, Pech, 2017]

Let $\mathscr{S}$ be one of the models 1–19. Then

- $Q_{\mathscr{S}}(x, y; t)$ is expressible using iterated integrals of ${}_2F_1$ expressions.
- $Q_{\mathscr{S}}(x, y; t)$ is transcendental.

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Example (King walks in the quarter plane, A151331)

Q$(t) = \frac{1}{t}\int_0^t \frac{1}{(1+4x)^3} \cdot {}_2F_1\left(\begin{matrix}\frac{3}{2} & \frac{3}{2}\\ & 2\end{matrix}\,\middle|\, \frac{16x(1+x)}{(1+4x)^2}\right) dx$

$$= 1 + 3t + 18t^2 + 105t^3 + 684t^4 + 4550t^5 + 31340t^6 + 219555t^7 + \cdots$$

Models 1–19: proofs, explicit expressions and transcendence

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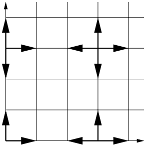
▷ Computer-driven discovery and proof; no human proof yet.
▷ Proof uses: (1) kernel method + (2) creative telescoping.



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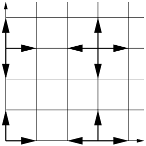
(1) Kernel method [Bousquet-Mélou, Mishna, 2010]



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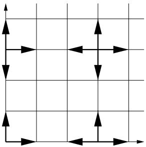

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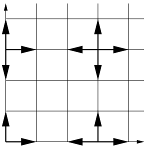

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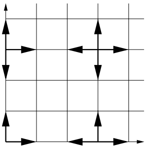

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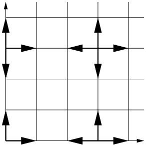

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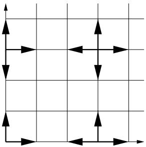
Summing up yields the orbit equation:

$$\sum_{\theta\in\mathcal{G}} (-1)^{\theta}\theta\big(xy\,Q(x,y;t)\big) = \frac{xy - \frac{1}{x}y + \frac{1}{x}\frac{1}{y} - x\frac{1}{y}}{K(x,y;t)}$$



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Taking positive parts yields:

$$[x^> y^>] \sum_{\theta\in\mathcal{G}} (-1)^\theta \theta\big(xy\,Q(x,y;t)\big) = [x^> y^>] \frac{xy - \frac{1}{x}y + \frac{1}{x}\frac{1}{y} - x\frac{1}{y}}{K(x,y;t)}$$

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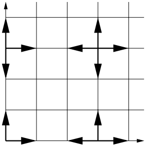

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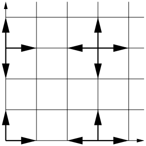

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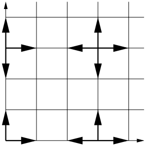
$$\mathsf{GF} = \mathsf{PosPart}\left(\frac{\mathsf{OS}}{\mathsf{kernel}}\right) = \oiint \mathsf{RatFrac}$$



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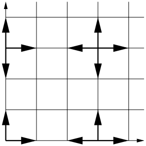

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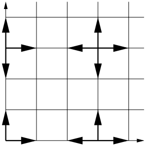
▷ Argument works if $\text{OS} \neq 0$: algebraic version of the reflection principle

(1) Kernel method [Bousquet-Mélou, Mishna, 2010]



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▷ Creative Telescoping finds a differential equation for GF $= \oint\!\!\oint$ RatFrac

(2) Creative Telescoping

"An algorithmic toolbox for multiple sums and integrals with parameters"

Example [Apéry 1978]: $A_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2$ satisfies the recurrence

$$(n+1)^3 A_{n+1} + n^3 A_{n-1} = (2n+1)(17n^2+17n+5) A_n.$$

▷ Key fact used to prove that $\zeta(3) := \sum_{n \geq 1} \frac{1}{n^3} \approx 1.202056903\ldots$ is irrational.

1. Journées Arithmétiques de Marseille-Luminy, June 1978

The board of programme changes informed us that R. Apéry (Caen) would speak Thursday, 14.00 "Sur l'irrationalité de $\zeta(3)$." Though there had been earlier rumours of his claiming a proof, scepticism was general. The lecture tended to strengthen this view to rank disbelief. Those who listened casually, or who were afflicted with being non-Francophone, appeared to hear only a sequence of unlikely assertions.

7. ICM '78, Helsinki, August 1978

Neither Cohen nor I had been able to prove (5) or (5′) in the intervening 2 months. After a few days of fruitless effort the specific problem was mentioned to Don Zagier (Bonn), and with irritating speed he showed that indeed the sequence $\{b'_n\}$ satisfies the recurrence (4). This more or less broke the dam and (5′) and (5) were quickly conquered. Henri Cohen addressed a very well-attended meeting at 17.00 on Friday, August 18 in the language of the majority, proving (5) and explaining how this implied the

[Van der Poorten, 1979: "A proof that Euler missed"]

"An algorithmic toolbox for multiple sums and integrals with parameters"

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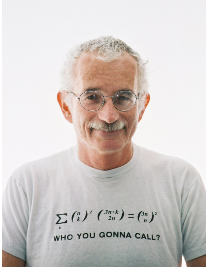
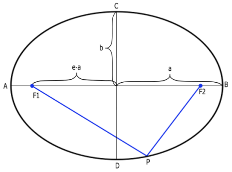


[Zeilberger, 1990: "The method of creative telescoping"]

"An algorithmic toolbox for multiple sums and integrals with parameters"

Example [Euler, 1733]: Perimeter of an ellipse of eccentricity e, semi-major axis 1

$$p(e) = 4\int_0^1 \sqrt{\frac{1-e^2u^2}{1-u^2}}\,\mathrm{d}u = 4\oint\!\!\oint \frac{\mathrm{d}u\mathrm{d}v}{1-\frac{1-e^2u^2}{(1-u^2)v^2}}$$



Principle: Find algorithmically

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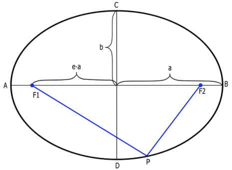

Principle: Find algorithmically

$$\left((e-e^3)\partial_e^2 + (1-e^2)\partial_e + e\right) \cdot \left(\frac{1}{1-\frac{1-e^2u^2}{(1-u^2)v^2}}\right) =$$

$$\partial_u\left(-\frac{e(-1-u+u^2+u^3)v^2(-3+2u+v^2+u^2(-2+3e^2-v^2))}{(-1+v^2+u^2(e^2-v^2))^2}\right)$$

$$+\partial_v\left(\frac{2e(-1+e^2)u(1+u^3)v^3}{(-1+v^2+u^2(e^2-v^2))^2}\right)$$

▷ Conclusion: $(e-e^3)\cdot p''(e) + (1-e^2)\cdot p'(e) + e\cdot p(e) = 0.$

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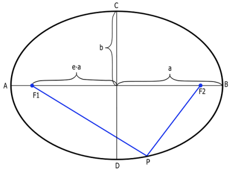

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$$+\,\partial_v\left(\frac{2e(-1+e^2)u(1+u^3)v^3}{(-1+v^2+u^2(e^2-v^2))^2}\right)$$

▷ Conclusion: $p(e) = \frac{\pi}{2}\cdot {}_2F_1\left(\left.\begin{matrix}-\frac{1}{2} & \frac{1}{2}\\ & 1\end{matrix}\right| e^2\right) = 2\pi - \frac{\pi}{2}e^2 - \frac{3\pi}{32}e^4 - \cdots.$

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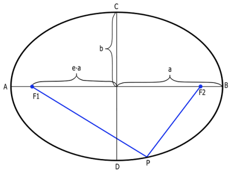

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▷ Drawback: Size(certificate) $\gg$ Size(telescoper).

Algorithm for the integration of rational functions [B., Lairez, Salvy, 2013]

- **Input:** $R(e, \mathbf{x})$ a rational function in e and $\mathbf{x} = x_1, \ldots, x_n$.
- **Output:** A linear ODE $T(e, \partial_e)y = 0$ satisfied by $y(e) = \oint\!\!\oint R(e, \mathbf{x})d\mathbf{x}$.
- **Complexity:** $\mathcal{O}(D^{8n+2})$, where $D = \deg R$.
- **Output size:** T has order $\leq D^n$ in ∂_e and degree $\leq D^{3n+2}$ in e.

▷ Avoids the (costly) computation of certificates, of size $\Omega(D^{n^2/2})$.

▷ Previous algorithms: complexity (at least) doubly exponential in n.

▷ Very efficient in practice.

A toy *transcendence proof*: blending Guess-and-Prove and CT

Theorem (Apéry's power series is transcendental)

$$f(t) = \sum_n A_n t^n, \quad \text{where } A_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2, \text{ is transcendental.}$$

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Proof:

1. Creative telescoping:

$$(n+1)^3 A_{n+1} + n^3 A_{n-1} = (2\,n+1)\,(17\,n^2 + 17\,n + 5) A_n, \quad A_0 = 1,\, A_1 = 5$$

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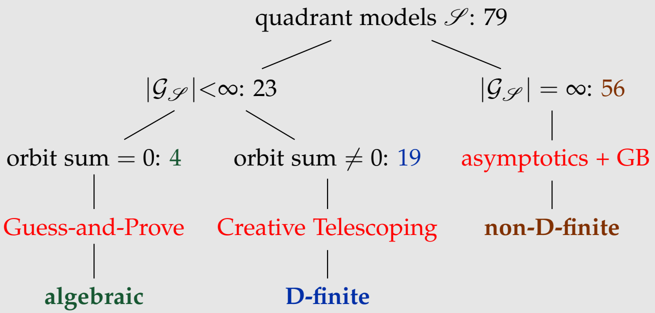
5. Conclusion: f is transcendental[†]

† f algebraic would imply a full basis of algebraic solutions for $L_f^{\min}$ [Tannery, 1875].

Summary: classification of walks with small steps in $\mathbb{N}^2$

$Q_{\mathscr{S}}$ is D-finite $\iff$ a certain group $\mathcal{G}_{\mathscr{S}}$ is finite (!)



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quadrant models $\mathscr{S}$: 79

- $|\mathcal{G}_{\mathscr{S}}| < \infty$: 23
  - OS $= 0$: 4 — Guess-and-Prove — **algebraic**
  - OS $\neq 0$: 19 — kernel + CT — **transcendental**
- $|\mathcal{G}_{\mathscr{S}}| = \infty$: 56
  - $\exists$ decoupling*: 9 — Tutte's invariants — **D-algebraic**
  - $\nexists$ decoupling: 47 — diff. Galois — **D-transcendental**

* $\exists U \in \mathbb{Q}(x,t), V \in \mathbb{Q}(y,t)$ s.t. $U(x) + V(y) = xy$ on the kernel $K(x,y;t) = 0$.

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▷ Many contributors (2010–2019): Bernardi, B., Bousquet-Mélou, Chyzak, Dreyfus, Hardouin, van Hoeij, Kauers, Kurkova, Mishna, Pech, Raschel, Roques, Salvy, Singer

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▷ Proofs use various tools: algebra, complex analysis, probability theory, differential Galois theory, computer algebra, etc.

Conclusion

☺ Enumerative Combinatorics and Computer Algebra enrich one another

☺ Classification of $Q(x, y; t)$ fully completed for 2D small step walks

☺ Robust algorithmic methods, based on efficient algorithms:

- Guess-and-Prove
- Creative Telescoping

☺ Brute-force and/or use of naive algorithms = hopeless.
E.g. size of algebraic equations for $G(x, y; t) \approx 30$Gb.

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- Brute-force and/or use of naive algorithms = hopeless.
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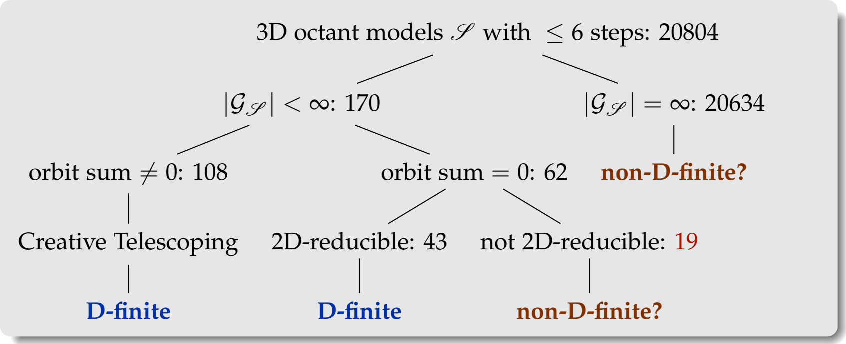
- Lack of "purely human" proofs for some results.
- Many beautiful open questions for 2D models with repeated or large steps, and in dimension > 2.

Bonus

Beyond dimension 2: walks with small steps in $\mathbb{N}^3$

- $2^{3^3-1} \approx 67$ million models, of which ≈ 11 million inherently 3D



[B., Bousquet-Mélou, Kauers, Melczer, 2016] + [Du, Hou, Wang, 2017];
completed by [Bacher, Kauers, Yatchak, 2016]

Beyond dimension 2: walks with small steps in $\mathbb{N}^3$

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3D octant models $\mathscr{S}$ with ≤ 6 steps: 20804

- $|\mathcal{G}_{\mathscr{S}}| < \infty$: 170
  - orbit sum $\neq 0$: 108
    - Creative Telescoping
      - **D-finite**
  - orbit sum $= 0$: 62
    - 2D-reducible: 43
      - **D-finite**
    - not 2D-reducible: 19
      - **non-D-finite?**
- $|\mathcal{G}_{\mathscr{S}}| = \infty$: 20634
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[B., Bousquet-Mélou, Kauers, Melczer, 2016] + [Du, Hou, Wang, 2017];
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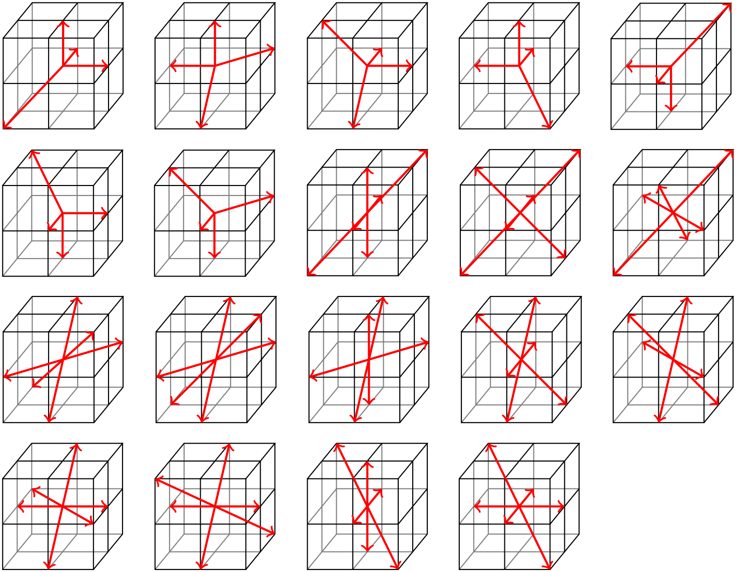
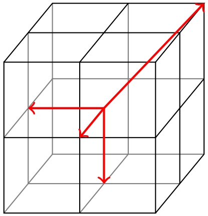
Question: differential finiteness $\Longleftrightarrow$ finiteness of the group?

Answer: probably no

19 mysterious 3D-models: finite $\mathcal{G}_{\mathscr{S}}$ and possibly non-D-finite $Q_{\mathscr{S}}$





Numerical computations [Dahne, Salvy, 2020] suggest:

$$k_{4n} = C \cdot 256^n / n^{\alpha}, \text{ for } \alpha = 3.3257570041744\ldots \notin \mathbb{Q},$$

so excursions are very probably non-D-finite quadrant models with steps in $\{-2,-1,0,1\}^2$: 13 110

- $|\mathcal{G}_{\mathscr{S}}| < \infty$: 240
  - $\text{OS} \neq 0$: 431 → **D-finite**
  - $\text{OS} = 0$: 9 → **D-finite?**
- $|\mathcal{G}_{\mathscr{S}}| = \infty$: 12 870
  - α rational: 16 → **non-D-finite?**
  - α irrational: 12 854 → **non-D-finite**

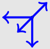
[B., Bousquet-Mélou, Melczer, 2018]

Question: differential finiteness $\Longleftrightarrow$ finiteness of the group?

Answer: ?

Two challenging models with large steps

Conjecture 1 [B., Bousquet-Mélou, Melczer, 2018]

For the model  the excursions generating function $Q(0,0;t^{1/2})$ equals

$$\frac{1}{3t} - \frac{1}{6t} \cdot \left(\frac{1-12t}{(1+36t)^{1/3}} \cdot {}_2F_1\left(\begin{matrix} \frac{1}{6} & \frac{2}{3} \\ & 1 \end{matrix} \,\middle|\, \frac{108t(1+4t)^2}{(1+36t)^2} \right) + \sqrt{1-12t} \cdot {}_2F_1\left(\begin{matrix} -\frac{1}{6} & \frac{2}{3} \\ 1 & \end{matrix} \,\middle|\, \frac{108t(1+4t)^2}{(1-12t)^2} \right) \right).$$

Conjecture 2 [B., Bousquet-Mélou, Melczer, 2018]

For the model  the excursions generating function $Q(0,0;t)$ equals

$$\frac{(1-24\,U+120\,U^2-144\,U^3)\,(1-4\,U)}{(1-3\,U)\,(1-2\,U)^{3/2}\,(1-6\,U)^{9/2}},$$

where $U = t^4 + 53\,t^8 + 4363\,t^{12} + \cdots$ is the unique series in $\mathbb{Q}[[t]]$ satisfying

$$U\,(1-2\,U)^3\,(1-3\,U)^3\,(1-6\,U)^9 = t^4\,(1-4\,U)^4.$$

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- Computer Algebra for Lattice Path Combinatorics, preprint, 2019.

Solution of the “exercise”

- The kernel equation reads (with $K(x,y) = 1 - t(y + \bar{x} + x\bar{y})$):

$$K(x,y)yH(x,y) = y - txH(x,0)$$

Excursions in $\mathbb{Z} \times \mathbb{N}$

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- Let

$$y_0 = \frac{x - t - \sqrt{(t-x)^2 - 4t^2x^3}}{2tx} \qquad = xt + t^2 + (x^2 + \bar{x})t^3 + (3x + \bar{x}^2)t^4 + \cdots$$

be the (unique) root in $\mathbb{Q}[x, \bar{x}][[t]]$ of $K(x, y_0) = 0$.

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$$0 = K(x,y_0)yH(x,y_0) = y_0 - txH(x,0),$$

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- Creative telescoping then proves:

$$(27t^4 - t)A''(t) + (108t^3 - 4)A'(t) + 54t^2A(t) = 0.$$

```
> Zeilberger(1/x * sqrt((t-x)^2 - 4*t^2*x^3)/(2*t^2*x^2), t, x, Dt);
```

The group of the model $\{\uparrow, \leftarrow, \searrow\}$

Step set $\mathscr{S} = \{(-1,0),(0,1),(1,-1)\}$, with characteristic polynomial

$$\chi(x,y) = \frac{1}{x} + y + x \cdot \frac{1}{y} = \bar{x} + y + x\bar{y}$$

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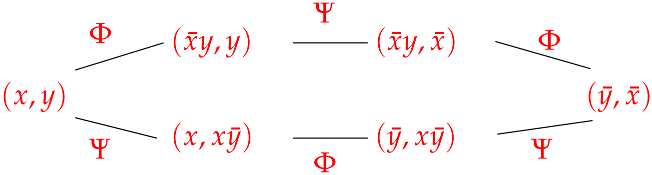
Φ and Ψ are involutions, and generate a finite dihedral group D_3 of order 6:



Diagonal walks in $\mathbb{N}^2$

- Orbit equation:

$$xyQ(x,y) - \bar{x}y^2Q(\bar{x}y,y) + \bar{x}^2yQ(\bar{x}y,\bar{x}) - \bar{x}\bar{y}Q(\bar{y},\bar{x}) + x\bar{y}^2Q(\bar{y},x\bar{y}) - x^2\bar{y}Q(x,x\bar{y}) = \frac{xy - \bar{x}y^2 + \bar{x}^2y - \bar{x}\bar{y} + x\bar{y}^2 - x^2\bar{y}}{1 - t(y + \bar{x} + x\bar{y})}$$

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- Corollary [Bousquet-Mélou & Mishna, 2010]:

$$xyQ(x,y) = [x^{>0}y^{>0}]\ \frac{xy - \bar{x}y^2 + \bar{x}^2y - \bar{x}\bar{y} + x\bar{y}^2 - x^2\bar{y}}{1 - t(y + \bar{x} + x\bar{y})}$$

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$$B(t) = [z^0]Q(z,\bar{z}) = [u^{-1}v^{-1}z^{-1}]\,\frac{\bar{u}\bar{v} - u\bar{v}^2 + u^2\bar{v} - uv + \bar{u}v^2 - \bar{u}^2v}{z(1-zu)(1-v\bar{z})(1-t(\bar{v}+u+\bar{u}v))}$$

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- Creative Telescoping gives a differential equation for $B(t)$:

$$(27t^4 - t)B''(t) + (108t^3 - 4)B'(t) + 54t^2B(t) = 0.$$

Conclusion

We have proved that $A(t)$ and $B(t)$ are both solutions of

$$(27t^4 - t)y''(t) + (108t^3 - 4)y'(t) + 54t^2 y(t) = 0.$$

Solving this equation proves:

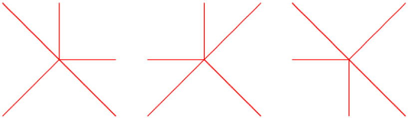
$$A(t) = B(t) = {}_2F_1\left(\begin{matrix}1/3 & 2/3\\ & 2\end{matrix}\,\middle|\, 27\,t^3\right) = \sum_{n=0}^{\infty} \frac{(3n)!}{n!^3}\frac{t^{3n}}{n+1}.$$

Thus the two sequences are equal to

$$a_{3n} = b_{3n} = \frac{(3n)!}{n!^2 \cdot (n+1)!}, \qquad \text{and} \quad a_m = b_m = 0 \quad \text{if 3 does not divide } m.$$

Example with infinite group: the scarecrows

[B., Raschel, Salvy, 2014]: $Q_{\mathscr{S}}(0,0;t)$ is not D-finite for the models



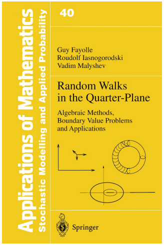
▷ For the 1st and the 3rd, the excursions sequence $[t^n]\, Q_{\mathscr{S}}(0,0;t)$

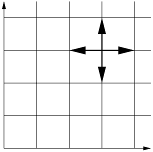
$$1, 0, 0, 2, 4, 8, 28, 108, 372, \ldots$$

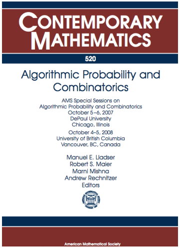
is $\sim K \cdot 5^n \cdot n^{-\alpha}$, with $\alpha = 1 + \pi / \arccos(1/4) = 3.383396\ldots$
[Denisov, Wachtel, 2015]

▷ The irrationality of α prevents $Q_{\mathscr{S}}(0,0;t)$ from being D-finite.
[Katz, 1970; Chudnovsky, 1985; André, 1989]

The group of a model: the simple walk case







The characteristic polynomial $\chi_{\mathscr{S}} := x + \frac{1}{x} + y + \frac{1}{y}$

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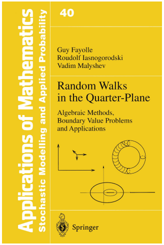

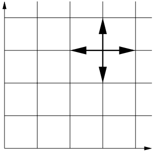

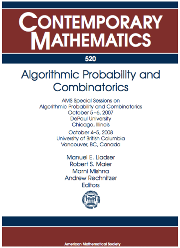

The characteristic polynomial $\chi_{\mathscr{S}} := x + \dfrac{1}{x} + y + \dfrac{1}{y}$ is left invariant under

$$\psi(x,y) = \left(x, \frac{1}{y}\right), \quad \phi(x,y) = \left(\frac{1}{x}, y\right),$$

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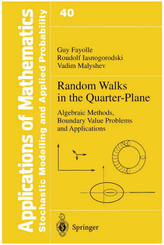

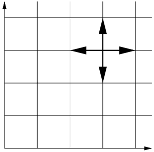

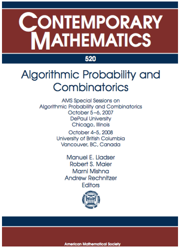

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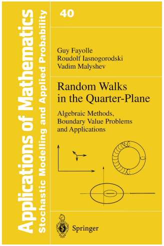
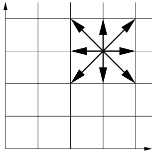
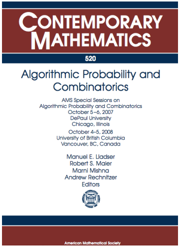
and thus under any element of the group

$$\langle \psi, \phi \rangle = \left\{ (x,y), \left(x, \frac{1}{y}\right), \left(\frac{1}{x}, \frac{1}{y}\right), \left(\frac{1}{x}, y\right) \right\}.$$

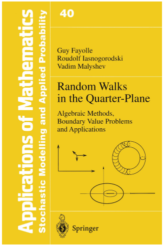
The group of a model

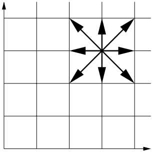


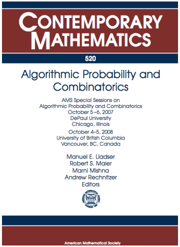




The generating polynomial $\chi_{\mathscr{S}} := \sum_{(i,j)\in\mathscr{S}} x^i y^j = \sum_{i=-1}^{1} B_i(y) x^i = \sum_{j=-1}^{1} A_j(x) y^j$







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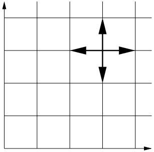
is left invariant under the birational involutions

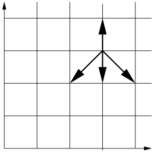
$$\psi(x,y) = \left(x, \frac{A_{-1}(x)}{A_{+1}(x)} \frac{1}{y}\right), \quad \phi(x,y) = \left(\frac{B_{-1}(y)}{B_{+1}(y)} \frac{1}{x}, y\right),$$

and thus under any element of the (dihedral) group

$$\mathcal{G}_{\mathscr{S}} := \langle \psi, \phi \rangle.$$

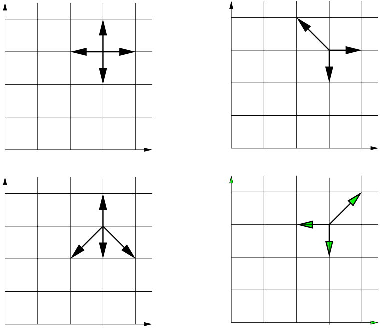
Examples of groups





Order 4,

Examples of groups



Order 4, order 6,

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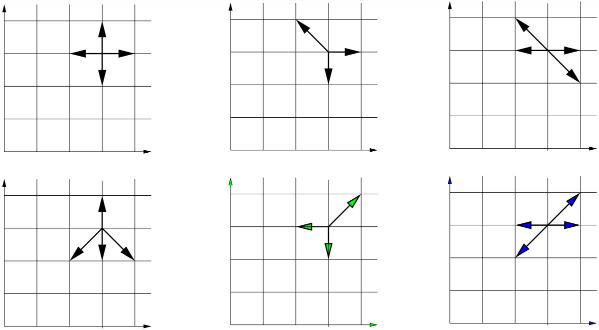
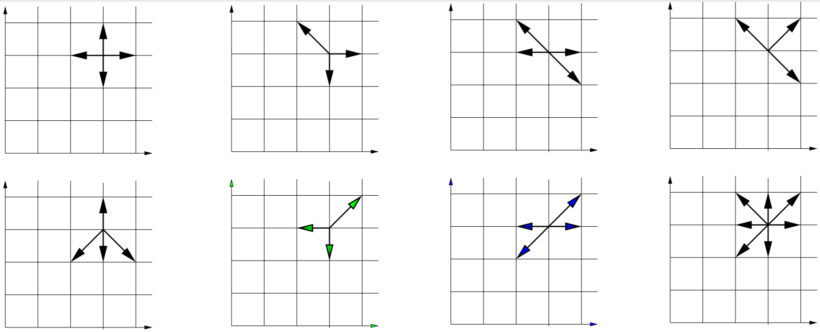
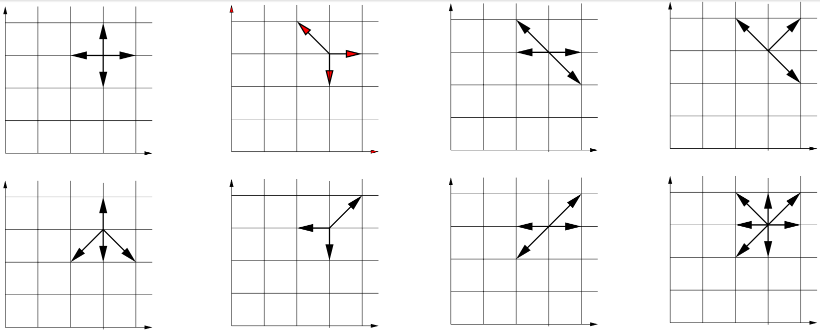


Order 4, order 6, order 8,

Examples of groups



Order 4, order 6, order 8, order ∞.



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