Computer Algebra for Combinatorics

Alin Bostan

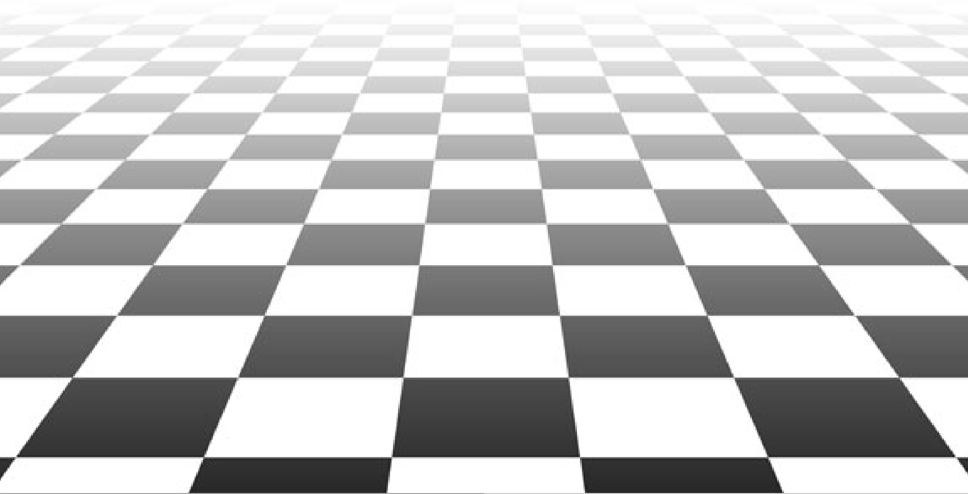
Inria

MPRI, C-2-22, January 24, 2019

Overview

Part 1:	General presentation
Part 2:	Guess'n'Prove
Part 3:	Creative telescoping

Part 1: General presentation



Computer Algebra for Enumerative Combinatorics

Enumerative Combinatorics: the science of counting

Area of mathematics primarily concerned with counting discrete objects.

▷ Main outcome: theorems

Computer Algebra: effective mathematics

Area of computer science primarily concerned with the algorithmic manipulation of mathematical objects.

▷ Main outcome: algorithms

This lecture: Computer Algebra for Enumerative Combinatorics

$\longrightarrow$ Algorithms for proving Theorems on Lattice Paths Combinatorics.

An (innocent looking) combinatorial question

Let $\mathscr{S} = \{\uparrow, \leftarrow, \searrow\}$. An $\mathscr{S}$-walk is a path in $\mathbb{Z}^2$ using only steps from $\mathscr{S}$. Show that, for any integer n, the following quantities are equal:

(i) number a_n of $\mathscr{S}$-walks of length n confined to the upper half plane $\mathbb{Z} \times \mathbb{N}$ that start and end at the origin $(0,0)$ (*excursions*);

(ii) number b_n of $\mathscr{S}$-walks of length n confined to the quarter plane $\mathbb{N}^2$ that start at the origin $(0,0)$ and finish on the diagonal of $\mathbb{N}^2$ (*diagonal walks*).

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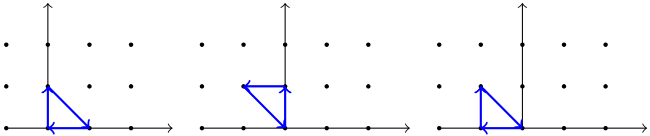
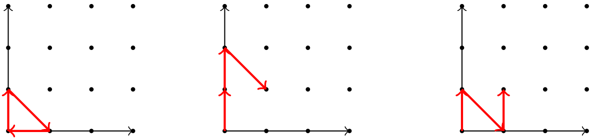
(ii) number b_n of $\mathscr{S}$-walks of length n confined to the quarter plane $\mathbb{N}^2$ that start at the origin $(0,0)$ and finish on the diagonal of $\mathbb{N}^2$ (*diagonal walks*).

For instance, for $n = 3$, this common value is $a_3 = b_3 = 3$:

(i)



(ii)



Teaser 1: This problem can be solved using computer algebra!

Teaser 2: The answer has a beautiful formula!

$$a_{3n} = b_{3n} = \frac{(3n)!}{n!^2 \cdot (n+1)!} \qquad \text{and} \quad a_m = b_m = 0 \quad \text{if } m \text{ is not a multiple of 3.}$$

Teaser 3: A certain group attached to the step set $\{\uparrow, \leftarrow, \searrow\}$ is finite!

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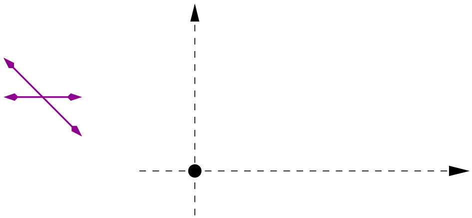
Teaser 3: A certain group attached to the step set $\{\uparrow, \leftarrow, \searrow\}$ is finite!

Let $\mathscr{S}$ be a subset of $\mathbb{Z}^d$ (step set, or model) and $p_0 \in \mathbb{Z}^d$ (starting point).

Example: $\mathscr{S} = \{(1,0), (-1,0), (1,-1), (-1,1)\}$, $p_0 = (0,0)$



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A path (walk) of length n starting at p_0 is a sequence $(p_0, p_1, \ldots, p_n)$ of elements in $\mathbb{Z}^d$ such that $p_{i+1} - p_i \in \mathscr{S}$ for all i.

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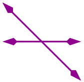

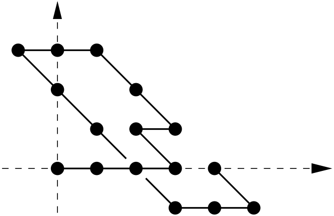

Combinatorial context: lattice paths confined to cones

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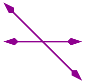
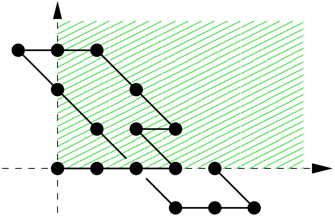
Let $\mathfrak{C}$ be a cone of $\mathbb{R}^d$ (if $x \in \mathfrak{C}$ and $r \geq 0$ then $r \cdot x \in \mathfrak{C}$).

Example: $\mathscr{S} = \{(1,0), (-1,0), (1,-1), (-1,1)\}$, $p_0 = (0,0)$ and $\mathfrak{C} = \mathbb{R}^2_+$





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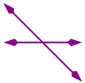

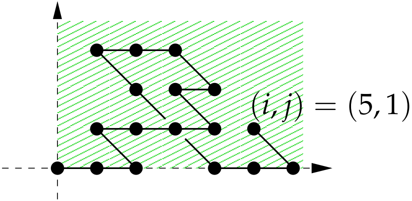

Questions

- What is the number a_n of n-step walks contained in $\mathfrak{C}$?
- For $\boldsymbol{i} \in \mathfrak{C}$, what is the number $a_{n;\boldsymbol{i}}$ of such walks that end at $\boldsymbol{i}$?
- What about their GF's $A(t) = \sum_n a_n t^n$ and $A(t;\boldsymbol{x}) = \sum_{n,\boldsymbol{i}} a_{n;\boldsymbol{i}} \boldsymbol{x}^{\boldsymbol{i}} t^n$?

Why should we care about counting walks?

Many objects from the *real world* can be encoded by walks:

- probability theory (*voting*, games of chance, branching processes, . . .)
- discrete mathematics (permutations, trees, words, urns, . . .)
- statistical physics (Ising model, . . .)
- operations research (queueing theory, . . .)

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A history and a survey of lattice path enumeration

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ABSTRACT

In celebration of the Sixth International Conference on Lattice Path Counting and Applications, it is befitting to review the history of lattice path enumeration and to survey how the topic has progressed thus far.

We start the history with early games of chance specifically the ruin problem which later appears as the ballot problem. We discuss André's Reflection Principle and its misnomer, its relation with the method of images and possible origins from physics and Brownian motion, and the earliest evidence of lattice path techniques and solutions.

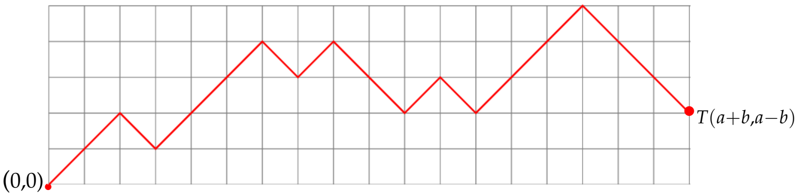
In the survey, we give representative articles on lattice path enumeration found in the literature in the last 35 years by the lattice, step set, boundary, characteristics counted, and solution method. Some of this work appears in the author's 2005 dissertation.

Counting walks is an old topic: the ballot problem [Bertrand, 1887]

CALCUL DES PROBABILITÉS. — *Solution d'un problème;*
par M. J. BERTRAND.

« On suppose que deux candidats A et B soient soumis à un scrutin de ballottage. Le nombre des votants est μ. A obtient m suffrages et est élu, B en obtient $\mu - m$. On demande la probabilité pour que, pendant le dépouillement du scrutin, le nombre des voix de A ne cesse pas une seule fois de surpasser celles de son concurrent.

Lattice path reformulation: find the number of paths with a upsteps $\nearrow$ and b downsteps $\searrow$ that start at the origin and never touch the x-axis again



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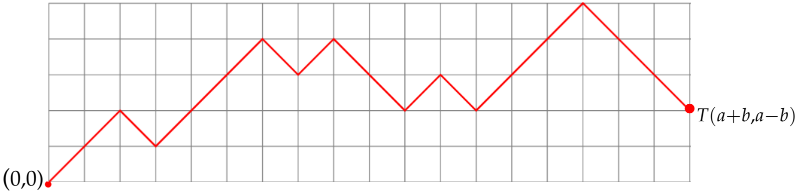


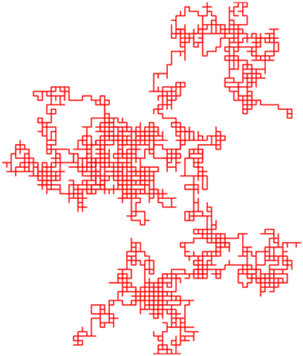
Exercise 2: Prove that the coordinates of the endpoint are indeed $(a+b, a-b)$

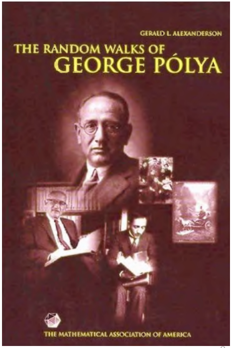
An old topic: Pólya's "promenade au hasard" / "Irrfahrt"

Motto: Drunkard: "Will I ever, ever get home again?"
Polya (1921): "You can't miss; just keep going and stay out of 3D!" (Adam and Delbruck, 1968)

[Pólya, 1921] Simple random walk $\{\pm 1\}^d$ on $\mathbb{Z}^d$ is recurrent in dimensions $d = 1, 2$ ("Alle Wege führen nach Rom"), and transient in dimension $d \geq 3$

Über eine Aufgabe der Wahrscheinlichkeitsrechnung betreffend die Irrfahrt im Straßennetz.





…but it is still a very hot topic

Lot of recent activity; many recent contributors:

Arquès, Bacher, Banderier, Bernardi, Bostan, Bousquet-Mélou, Budd, Chyzak, Cori, Courtiel, Denisov, Dreyfus, Du, Duchon, Dulucq, Duraj, Fayolle, Fisher, Flajolet, Fusy, Garbit, Gessel, Gouyou-Beauchamps, Guttmann, Guy, Hardouin, van Hoeij, Hou, Iasnogorodski, Johnson, Kauers, Kenyon, Koutschan, Krattenthaler, Kreweras, Kurkova, Malyshev, Melczer, Miller, Mishna, Niederhausen, Pech, Petkovšek, Prellberg, Raschel, Rechnitzer, Roques, Sagan, Salvy, Sheffield, Singer, Viennot, Wachtel, Wang, Wilf, D. Wilson, M. Wilson, Yatchak, Yeats, Zeilberger, …

etc.

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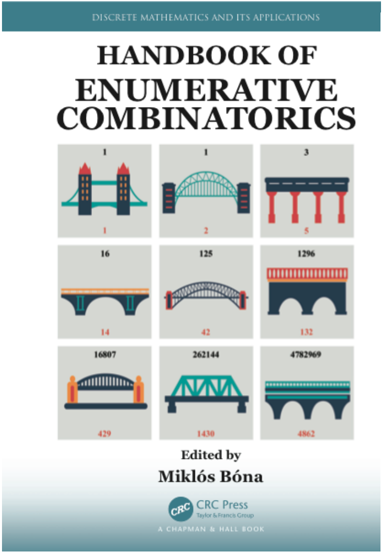
~~**Specific question**~~

~~**Ad hoc solution**~~ ⟶ **Systematic approach**

…but it is still a very hot topic



Chapter 10

Lattice Path Enumeration

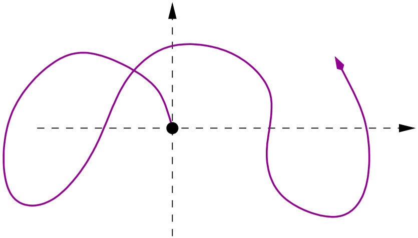
Christian Krattenthaler
Universität Wien

CONTENTS

Geometric sequences

The number a_n of n-step $\mathscr{S}$-walks in the whole plane $\mathbb{Z}^2$ is equal to

$$a_n = |\mathscr{S}|^n.$$



Proof: $a_n = |\mathscr{S}| \cdot a_{n-1} = |\mathscr{S}|^2 \cdot a_{n-2} = \cdots = |\mathscr{S}|^{n-1} \cdot a_1 = |\mathscr{S}|^n$. □

▷ Remark: For $\mathscr{S} = \{\rightarrow, \uparrow\}$, the sequence a_n is

$$1, 2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, \ldots$$

It satisfies the recurrence relation with constant coefficients $a_{n+1} - 2a_n = 0$.

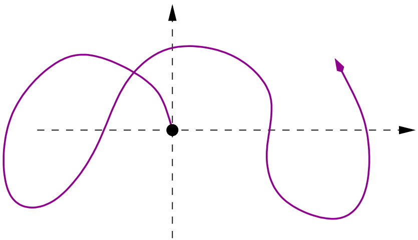
Rational series [folklore]

If $\mathscr{S} \subset \mathbb{Z}^d$ is finite and $\mathfrak{C} = \mathbb{R}^d$, then

$$a_n = |\mathscr{S}|^n, \text{ i.e. } \quad A(t) = \sum_{n \geq 0} a_n t^n = \frac{1}{1 - |\mathscr{S}|\, t}.$$

More generally:

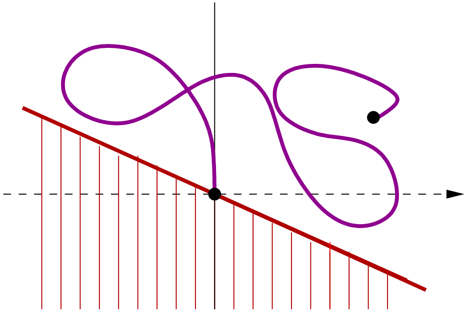
$$A(t; \boldsymbol{x}) = \sum_{n, \boldsymbol{i}} a_{n;\boldsymbol{i}} \boldsymbol{x}^{\boldsymbol{i}} t^n = \frac{1}{1 - t \sum_{\boldsymbol{s} \in \mathscr{S}} \boldsymbol{x}^{\boldsymbol{s}}}.$$



The next case: walks confined to a half-space

Recurrent sequences [Bousquet-Mélou, Petkovšek, 2000]

The sequence (a_n) of n-step $\mathscr{S}$-walks confined to a *half-space* still satisfies a *recurrence relation* (but not necessarily with constant coefficients).



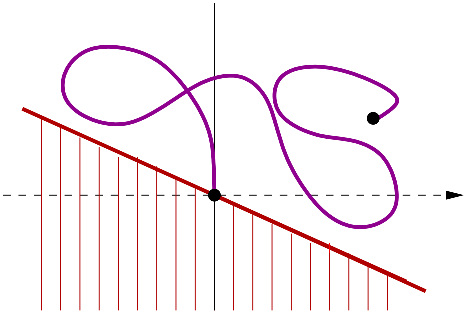
▷ Example: For $\{\rightarrow, \uparrow\}$-walks in the half-plane $\mathbb{Z} \times \mathbb{N}$, the sequence (a_n) is

$$1, 1, 2, 3, 6, 10, 20, 35, 70, 126, 252, 462, 924, 1716, \ldots$$

It is *not geometric*, but satisfies $(n+3)a_{n+2} - 2a_{n+1} - 4(n+1)a_n = 0$ for all n

Algebraic series [Bousquet-Mélou, Petkovšek, 2000]

If $\mathscr{S} \subset \mathbb{Z}^d$ is finite and $\mathfrak{C}$ is a rational half-space, then $A(t;\boldsymbol{x})$ is algebraic, given by an explicit system of polynomial equations.



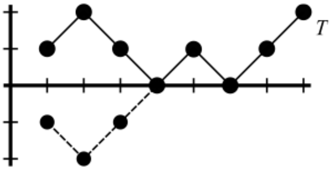
Example: For Dyck paths (ballot problem), $A(t;1) = \sum_{n\geq 0} C_n t^n = \frac{1-\sqrt{1-4t}}{2t}$

Back to the ballot problem [Bertrand, 1887]

Suppose that candidates A and B are running in an election. If a votes are cast for A and b votes are cast for B, where $a > b$, then the probability that A stays ahead of B throughout the counting of the ballots is $(a-b)/(a+b)$.

Lattice path reformulation: find the number of paths with a upsteps $\nearrow$ and b downsteps $\searrow$ that start at the origin and never touch the x-axis again

Reflection principle [Aebly, 1923]: *paths in $\mathbb{N}^2$ from $(1,1)$ to $T(a+b, a-b)$ that do touch the x-axis* are in bijection with *paths in $\mathbb{Z}^2$ from $(1,-1)$ to T*



Answer: (*paths in $\mathbb{Z}^2$ from $(1,1)$ to T*) $-$ (*paths in $\mathbb{Z}^2$ from $(1,-1)$ to T*)

$$\binom{a+b-1}{a-1} - \binom{a+b-1}{b-1} = \frac{a-b}{a+b}\binom{a+b}{a}$$

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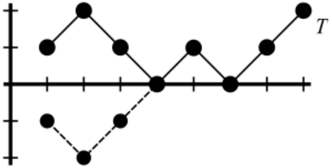
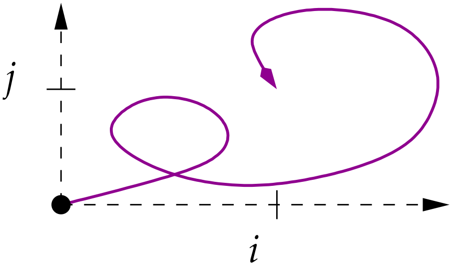


Answer: when $a = n+1$ and $b = n$, this is the Catalan number

$$C_n = \frac{1}{2n+1}\binom{2n+1}{n+1} = \frac{1}{n+1}\binom{2n}{n}$$

The next case: walks confined to a half-space



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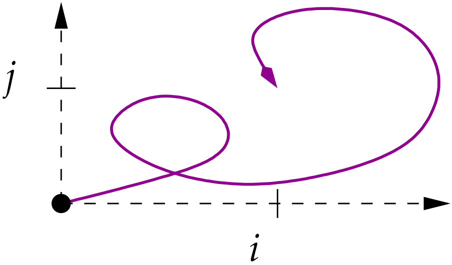

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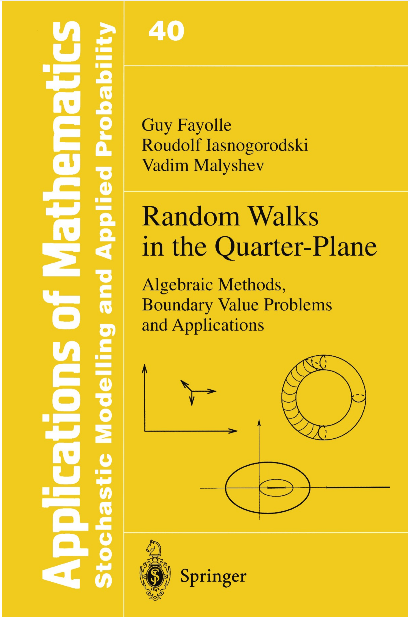
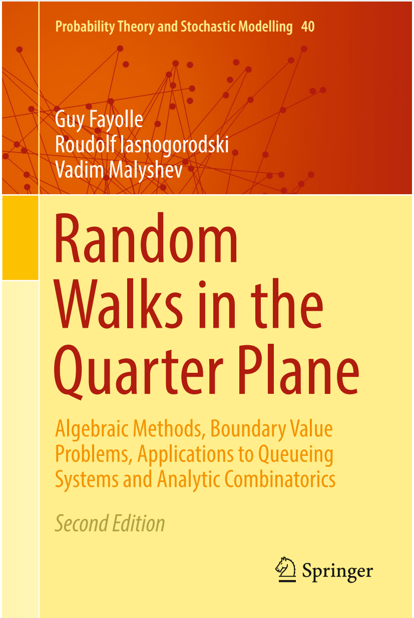
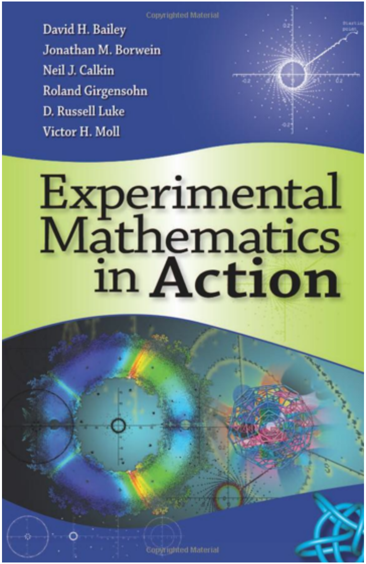
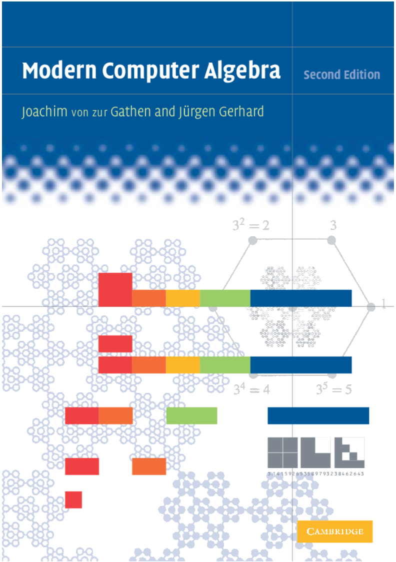
Entire books dedicated to walks in the quarter plane!





Our approach: Experimental Mathematics using Computer Algebra





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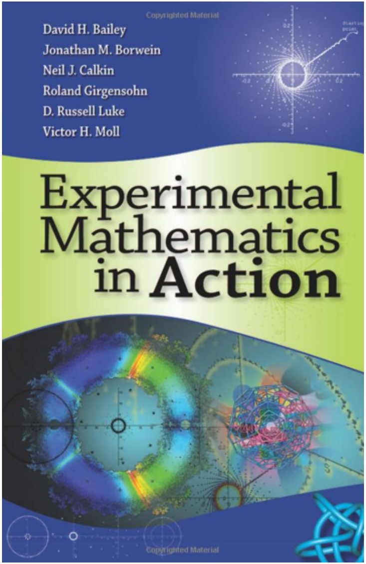
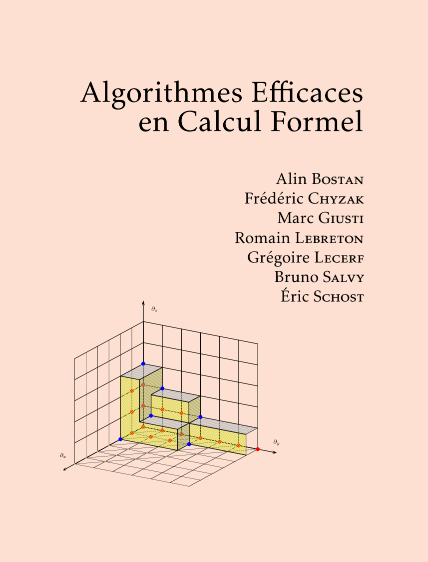
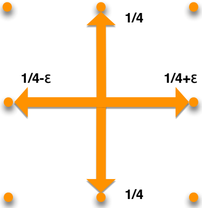
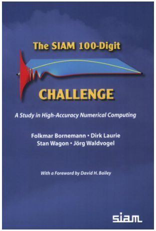


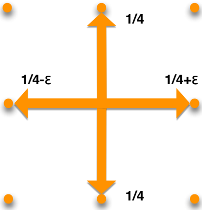


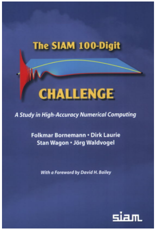




Problem 6

A flea starts at $(0,0)$ *on the infinite two-dimensional integer lattice and executes a biased random walk: At each step it hops north or south with probability* $1/4$*, east with probability* $1/4+\epsilon$*, and west with probability* $1/4-\epsilon$*. The probability that the flea returns to* $(0,0)$ *sometime during its wanderings is* $1/2$*. What is* ϵ*?*





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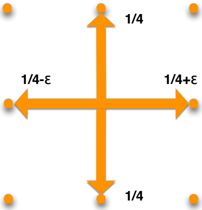
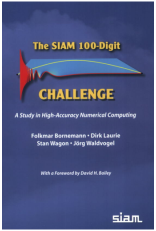
▷ Computer algebra conjectures and proves

$$p(\epsilon) = 1 - \sqrt{\frac{A}{2}} \cdot {}_2F_1\left(\begin{matrix} \frac{1}{2}, \frac{1}{2} \\ 1 \end{matrix} \;\middle|\; \frac{2\sqrt{1-16\epsilon^2}}{A} \right)^{-1}, \quad \text{with } A = 1 + 8\epsilon^2 + \sqrt{1-16\epsilon^2}.$$





Problem 6

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▷ Computer algebra conjectures and proves

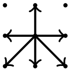
$$\epsilon \approx 0.0619139544739909428481752164732121769996387749983\;6207606146725885993101029759615845907105645752087861\ldots$$

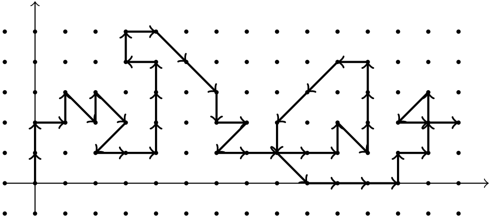
Lattice walks with small steps in the quarter plane

▷ From now on: we focus on nearest-neighbor walks in the quarter plane, i.e. walks in $\mathbb{N}^2$ starting at $(0,0)$ and using steps in a *fixed* subset $\mathscr{S}$ of

$$\{\swarrow, \leftarrow, \nwarrow, \uparrow, \nearrow, \rightarrow, \searrow, \downarrow\}.$$

▷ Example with $n = 45$, $i = 14$, $j = 2$ for:

$\mathscr{S} =$ 



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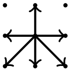
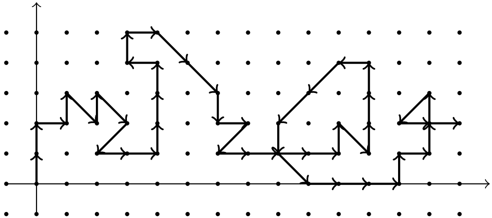
$\mathscr{S} =$ 



▷ Counting sequence: $f_{n;i,j}$ = number of walks of length n ending at (i,j).

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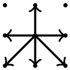
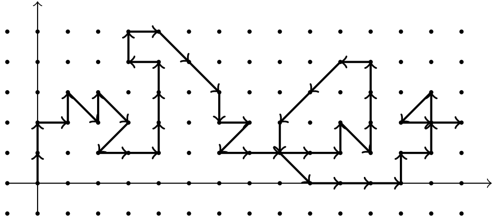
$\mathscr{S} =$ 



▷ Counting sequence: $f_{n;i,j}$ = number of walks of length n ending at (i,j).

▷ Specializations:

- $f_{n;0,0}$ = number of walks of length n returning to origin ("excursions");
- $f_n = \sum_{i,j\geq 0} f_{n;i,j}$ = number of walks with prescribed length n.

Generating functions and combinatorial problems

▷ Complete generating function:

$$F(t;x,y) = \sum_{n=0}^{\infty} \left(\sum_{i,j=0}^{\infty} f_{n;i,j} x^i y^j \right) t^n \ \in \mathbb{Q}[x,y][[t]].$$

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▷ Specializations:

- GF of excursions: $F(t;0,0)$;
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- GF of horizontal returns: $F(t;1,0)$;
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Combinatorial questions:
Given $\mathscr{S}$, what can be said about $F(t;x,y)$, resp. $f_{n;i,j}$, and their variants?

- Structure of F: algebraic? transcendental? solution of ODE?
- Explicit form: of F? of $f_{n;i,j}$?
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Our goal: Use computer algebra to give computational answers.

Among the 2^8 step sets $\mathscr{S} \subseteq \{-1, 0, 1\}^2 \setminus \{(0, 0)\}$, some are:

Small-step models of interest

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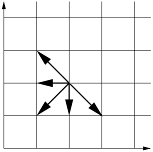

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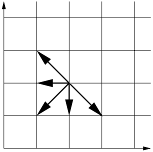

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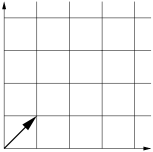

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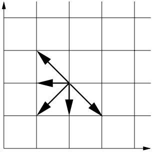

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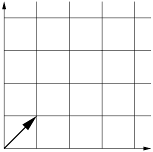

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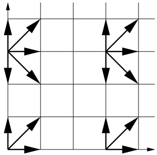

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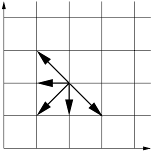

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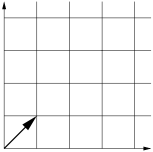

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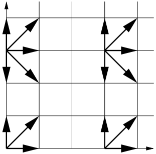

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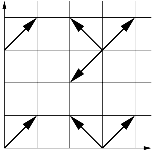

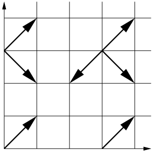

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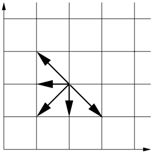

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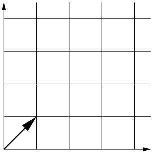

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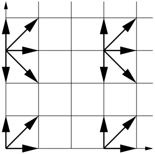

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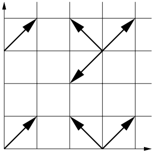

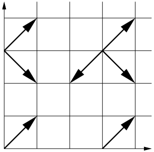

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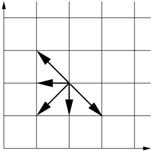
One is left with 79 interesting distinct models.

Among the 2^8 step sets $\mathscr{S} \subseteq \{-1,0,1\}^2 \setminus \{(0,0)\}$, some are:



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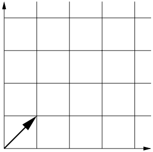

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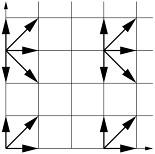

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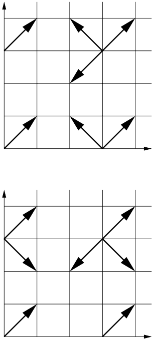

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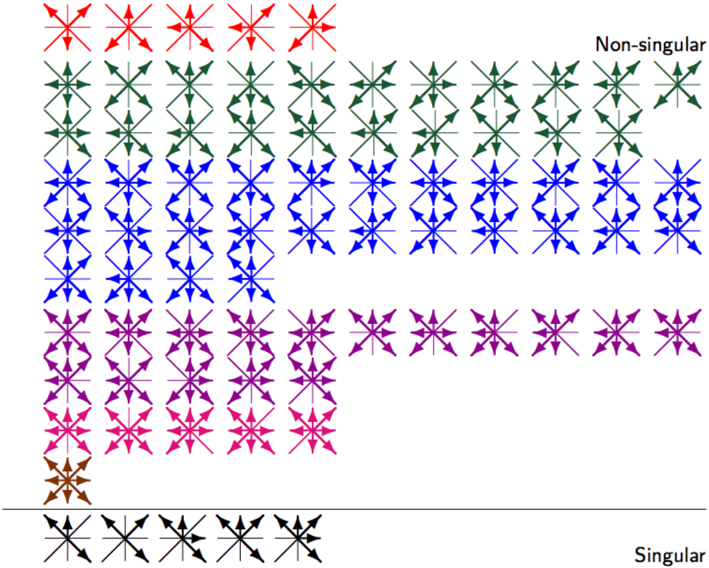
One is left with 79 interesting distinct models.

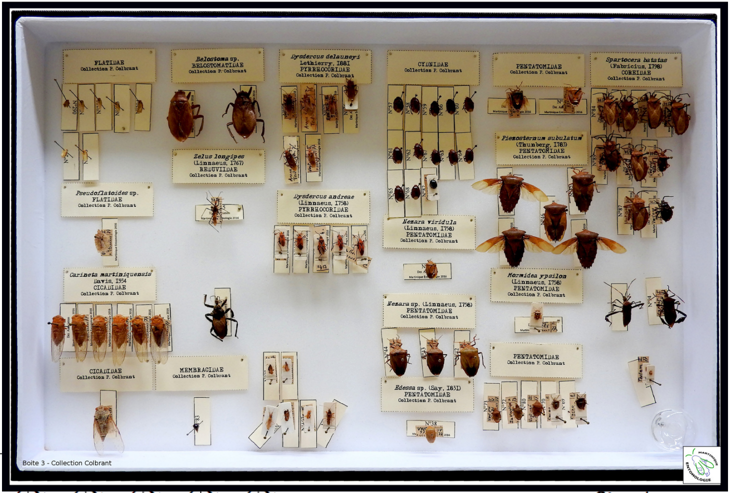
Is any further classification possible?

The 79 models



Task: classify their generating functions!

Non-singular

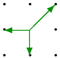


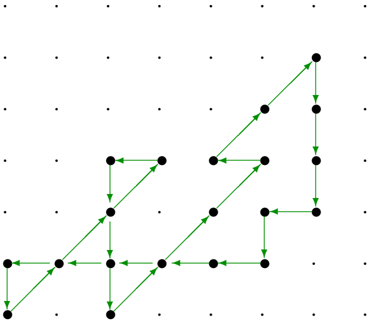
Singular

Two important models: Kreweras and Gessel walks

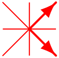
$\mathscr{S} = \{\downarrow, \leftarrow, \nearrow\}$ $\qquad F_{\mathscr{S}}(t; x, y) \equiv K(t; x, y)$

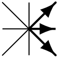
$\mathscr{S} = \{\nearrow, \swarrow, \leftarrow, \rightarrow\}$ $\qquad F_{\mathscr{S}}(t; x, y) \equiv G(t; x, y)$





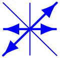
Example: A Kreweras excursion.

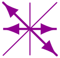
Dyck: 

Motzkin: 

Pólya: 

Kreweras: 

Gessel: 

Gouyou-Beauchamps: 

King walks: 

Tandem walks: 

Gessel's walks

$$\mathscr{S} = \{\nearrow, \swarrow, \leftarrow, \rightarrow\}$$

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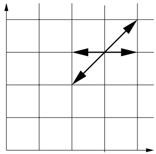
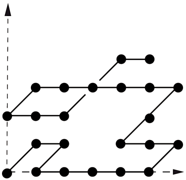
Displaying 1-1 of 1 result found. page 1

Sort: relevance | references | number | modified | created Format: long | short | data

A135404 Gessel sequence: the number of paths of length 2m in the plane, starting and ending at (0,1), with unit steps in the four directions (north, east, south, west) and staying in the region y>0, x>-y. +20 6

1, 2, 11, 85, 782, 8004, 88044, 1020162, 12294260, 152787976, 1946310467, 25302036071, 334560525538, 4488007049900, 60955295750460, 836838395382645, 11597595644244186, 162074575606984788, 2281839419729917410, 32340239369121304038, 461109219391987625316, 6610306991283738684600 (list; graph; refs; listen; history; text; internal format)





Conjecture 1 The generating function of Gessel excursions is equal to

$$G(t;0,0) = {}_3F_2\left(\begin{matrix} 5/6 & 1/2 & 1 \\ & 5/3 & 2 \end{matrix}\,\middle|\, 16t^2\right)$$

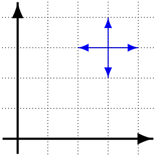
$$= \sum_{n=0}^{\infty} \frac{(5/6)_n (1/2)_n}{(5/3)_n (2)_n} (4t)^{2n}$$

$$= 1 + 2t^2 + 11t^4 + 85t^6 + 782t^8 + \cdots$$

Conjecture 2

The full generating function $G(t;x,y)$ is not D-finite.

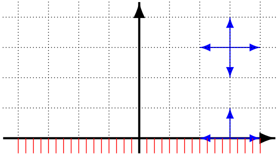
The simple walk in the plane

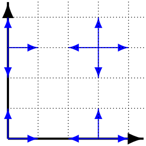


[Pólya, 1921]:

▷ Formula $\binom{2n}{n}^2$ for $2n$-excursions
▷ Rational generating function

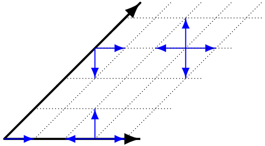
The simple walk in the half-plane and in the quarter-plane

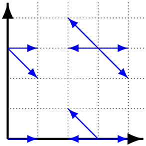




▷ Formulas $\binom{2n+1}{n}C_n$, resp. $C_n C_{n+1}$, for $2n$-excursions [Arquès, 1986]
▷ Full generating functions: algebraic [Bousquet-Mélou, Petkovšek, 2000], resp. D-finite [Bousquet-Mélou, 2002]

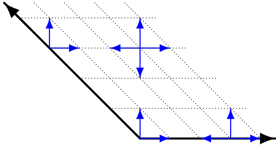
The simple walk in the cone with angle 45°

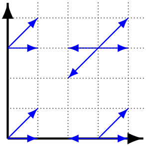




▷ Formula $C_n C_{n+2} - C_{n+1}^2$ for $2n$-excursions [Gouyou-Beauchamps, 1986]
▷ D-finite generating function [Gessel, Zeilberger, 1992]

What about the simple walk in the cone with angle 135°?



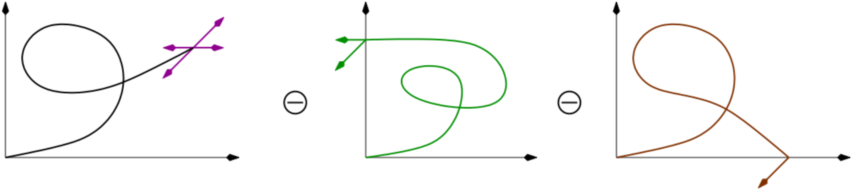


Algebraic reformulation: solving a functional equation

Generating function: $G(t;x,y) = \sum_{n=0}^{\infty}\sum_{i=0}^{n}\sum_{j=0}^{n} g_{n;i,j}x^i y^j t^n \in \mathbb{Q}[x,y][[t]]$

"Kernel equation":

$$G(t;x,y) = 1 + t\left(xy + x + \frac{1}{xy} + \frac{1}{x}\right)G(t;x,y) - t\left(\frac{1}{x} + \frac{1}{x}\frac{1}{y}\right)G(t;0,y) - t\frac{1}{xy}\left(G(t;x,0) - G(t;0,0)\right)$$

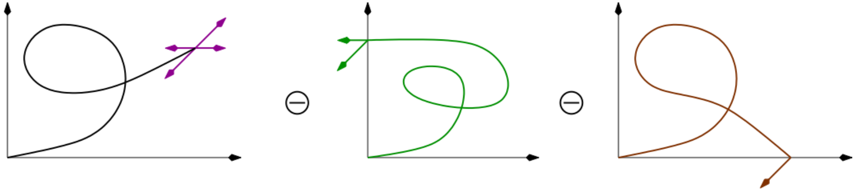


Algebraic reformulation: solving a functional equation

Generating function: $G(t;x,y) = \sum_{n=0}^{\infty}\sum_{i=0}^{n}\sum_{j=0}^{n} g_{n;i,j}x^i y^j t^n \in \mathbb{Q}[x,y][[t]]$

"Kernel equation":

$$G(t;x,y) = 1 + t\left(xy + x + \frac{1}{xy} + \frac{1}{x}\right)G(t;x,y) - t\left(\frac{1}{x} + \frac{1}{x}\frac{1}{y}\right)G(t;0,y) - t\frac{1}{xy}\left(G(t;x,0) - G(t;0,0)\right)$$



Task: Solve this functional equation!

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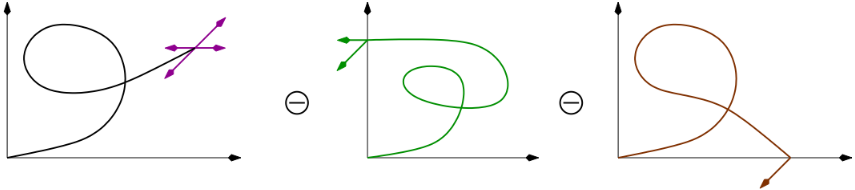
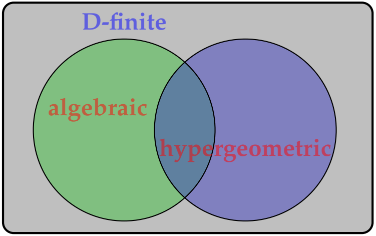


Task: For the other models – solve 78 similar equations!

Important classes of univariate generating functions



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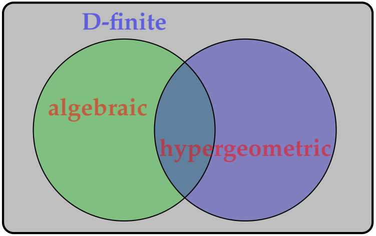

$$S(t) = \sum_{n=0}^{\infty} s_n t^n \in \mathbb{Q}[[t]] \quad \text{is}$$

▷ *algebraic* if $P(t, S(t)) = 0$ for some $P(x,y) \in \mathbb{Z}[x,y] \setminus \{0\}$;

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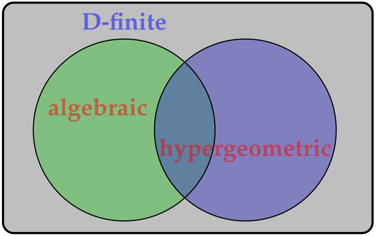

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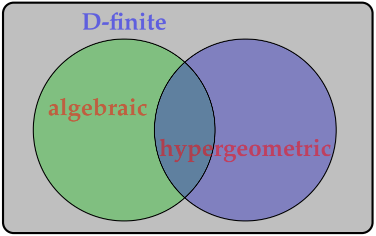

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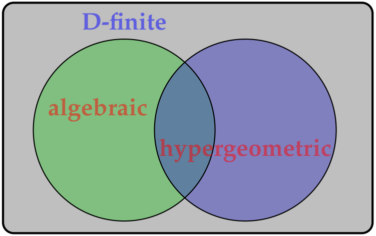

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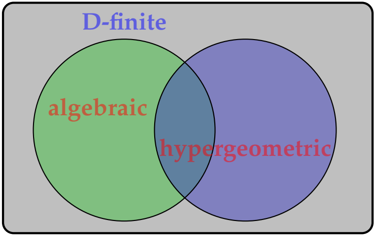
$$\ln(1-t); \quad \frac{\arcsin(\sqrt{t})}{\sqrt{t}}; \quad (1-t)^{\alpha}, \alpha \in \mathbb{Q}$$

Important classes of univariate generating functions



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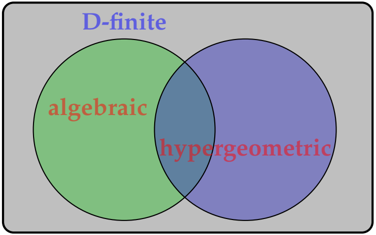
$${}_2F_1\left(\begin{matrix} a \ b \\ c \end{matrix} \middle| \, t\right) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{t^n}{n!}, \quad (a)_n = a(a+1)\cdots(a+n-1).$$

Important classes of univariate generating functions



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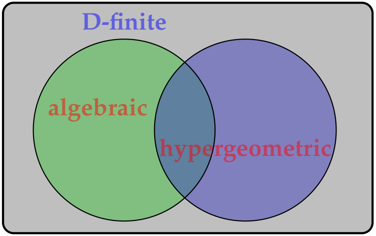
$${}_2F_1\left(\begin{matrix}\frac{1}{2}\ \frac{1}{2}\\ 1\end{matrix}\,\middle|\, t\right) = \frac{2}{\pi}\int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-tx^2)}}.$$

Important classes of univariate generating functions



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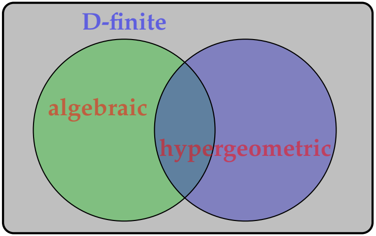

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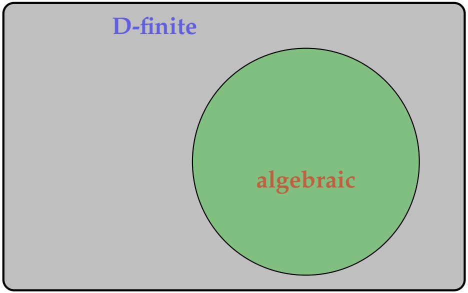
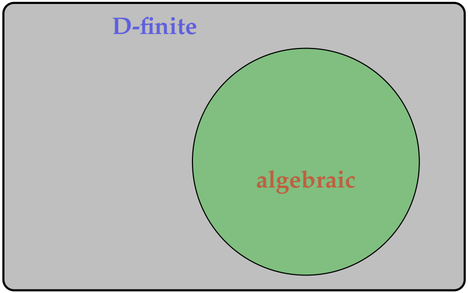
Theorem [Schwarz, 1873; Beukers, Heckman, 1989]

Characterization of { *hypergeometric* } $\cap$ { *algebraic* }.



▷ $S \in \mathbb{Q}[[x, y, t]]$ is *algebraic* if it is the root of a polynomial $P \in \mathbb{Q}[x, y, t, T]$;

Important classes of multivariate generating functions



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▷ $S \in \mathbb{Q}[[x,y,t]]$ is *D-finite* if it satisfies a system of linear partial differential equations with polynomial coefficients

$$\sum_i a_i(t,x,y)\frac{\partial^i S}{\partial x^i} = 0, \quad \sum_i b_i(t,x,y)\frac{\partial^i S}{\partial y^i} = 0, \quad \sum_i c_i(t,x,y)\frac{\partial^i S}{\partial t^i} = 0.$$

Main results (I): algebraicity of Gessel walks

Theorem [Kreweras, 1965; 100 pages long combinatorial proof!]

$$K(t;0,0) = {}_3F_2\left(\begin{matrix}1/3 & 2/3 & 1\\ & 3/2 & 2\end{matrix}\,\middle|\, 27\,t^3\right) = \sum_{n=0}^{\infty} \frac{4^n\binom{3n}{n}}{(n+1)(2n+1)} t^{3n}.$$

Theorem [Kauers, Koutschan, Zeilberger, 2009: former Gessel's conj. 1]

$$G(t;0,0) = {}_3F_2\left(\begin{matrix}5/6 & 1/2 & 1\\ & 5/3 & 2\end{matrix}\,\middle|\, 16t^2\right) = \sum_{n=0}^{\infty} \frac{(5/6)_n(1/2)_n}{(5/3)_n(2)_n} (4t)^{2n}.$$

Question: What about the structure of $K(t;x,y)$ and $G(t;x,y)$?

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▷ Computer-driven discovery and proof.
▷ Guess'n'Prove method, using Hermite-Padé approximants† $\longrightarrow$ Part 2

† Minimal polynomial $P(x,y,t,G(t;x,y)) = 0$ has $> 10^{11}$ terms; $\approx 30\,$Gb (!)

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Main results (II): Explicit form for $G(t;x,y)$

Theorem [B., Kauers, van Hoeij, 2010]

Let $V = 1 + 4t^2 + 36t^4 + 396t^6 + \cdots$ be a root of

$$(V-1)(1+3/V)^3 = (16t)^2,$$

let $U = 1 + 2t^2 + 16t^4 + 2xt^5 + 2(x^2+83)t^6 + \cdots$ be a root of

$$x(V-1)(V+1)U^3 - 2V(3x+5xV-8Vt)U^2$$
$$-xV(V^2-24V-9)U + 2V^2(xV-9x-8Vt) = 0,$$

let $W = t^2 + (y+8)t^4 + 2(y^2+8y+41)t^6 + \cdots$ be a root of

$$y(1-V)W^3 + y(V+3)W^2 - (V+3)W + V - 1 = 0.$$

Then $G(t;x,y)$ is equal to

$$\frac{\dfrac{64(U(V+1)-2V)V^{3/2}}{x(U^2-V(U^2-8U+9-V))^2} - \dfrac{y(W-1)^4(1-Wy)V^{-3/2}}{t(y+1)(1-W)(W^2y+1)^2}}{(1+y+x^2y+x^2y^2)t - xy} - \frac{1}{tx(y+1)}.$$

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Main results (III): Models with D-Finite $F(t;1,1)$

	OEIS	$\mathscr{S}$	Pol size	LDE size	Rec size
1	A005566		—	(3, 4)	(2, 2)
2	A018224		—	(3, 5)	(2, 3)
3	A151312		—	(3, 8)	(4, 5)
4	A151331		—	(3, 6)	(3, 4)
5	A151266		—	(5, 16)	(7, 10)
6	A151307		—	(5, 20)	(8, 15)
7	A151291		—	(5, 15)	(6, 10)
8	A151326		—	(5, 18)	(7, 14)
9	A151302		—	(5, 24)	(9, 18)
10	A151329		—	(5, 24)	(9, 18)
11	A151261		—	(4, 15)	(5, 8)
12	A151297		—	(5, 18)	(7, 11)
13	A151275		—	(5, 24)	(9, 18)
14	A151314		—	(5, 24)	(9, 18)
15	A151255		—	(4, 16)	(6, 8)
16	A151287		—	(5, 19)	(7, 11)
17	A001006		(2, 2)	(2, 3)	(2, 1)
18	A129400		(2, 2)	(2, 3)	(2, 1)
19	A005558		—	(3, 5)	(2, 3)
20	A151265		(6, 8)	(4, 9)	(6, 4)
21	A151278		(6, 8)	(4, 12)	(7, 4)
22	A151323		(4, 4)	(2, 3)	(2, 1)
23	A060900		(8, 9)	(3, 5)	(2, 3)

Equation sizes = (order, degree)

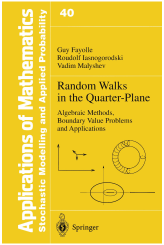
- ▷ Computerized discovery: enumeration + guessing [B., Kauers, 2009]
- ▷ 1–22: Confirmed by human proofs in [Bousquet-Mélou, Mishna, 2010]
- ▷ 23: Confirmed by a human proof in [B., Kurkova, Raschel, 2013]

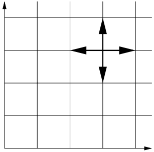
Main results (III): Models with D-Finite $F(t;1,1)$

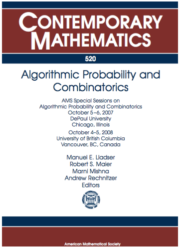
	OEIS	$\mathscr{S}$	algebraic?	asymptotics
1	A005566		N	$\frac{4}{\pi}\frac{4^n}{n}$
2	A018224		N	$\frac{2}{\pi}\frac{4^n}{n}$
3	A151312		N	$\frac{\sqrt{6}}{\pi}\frac{6^n}{n}$
4	A151331		N	$\frac{8}{3\pi}\frac{8^n}{n}$
5	A151266		N	$\frac{1}{2}\sqrt{\frac{3}{\pi}}\frac{3^n}{n^{1/2}}$
6	A151307		N	$\frac{1}{2}\sqrt{\frac{5}{2\pi}}\frac{5^n}{n^{1/2}}$
7	A151291		N	$\frac{4}{3\sqrt{\pi}}\frac{4^n}{n^{1/2}}$
8	A151326		N	$\frac{2}{\sqrt{3\pi}}\frac{6^n}{n^{1/2}}$
9	A151302		N	$\frac{1}{3}\sqrt{\frac{5}{2\pi}}\frac{5^n}{n^{1/2}}$
10	A151329		N	$\frac{1}{3}\sqrt{\frac{7}{3\pi}}\frac{7^n}{n^{1/2}}$
11	A151261		N	$\frac{12\sqrt{3}}{\pi}\frac{(2\sqrt{3})^n}{n^2}$
12	A151297		N	$\frac{\sqrt{3}B^{7/2}}{2\pi}\frac{(2B)^n}{n^2}$
13	A151275		N	$\frac{12\sqrt{30}}{\pi}\frac{(2\sqrt{6})^n}{n^2}$
14	A151314		N	$\frac{\sqrt{6}\lambda\mu C^{5/2}}{5\pi}\frac{(2C)^n}{n^2}$
15	A151255		N	$\frac{24\sqrt{2}}{\pi}\frac{(2\sqrt{2})^n}{n^2}$
16	A151287		N	$\frac{2\sqrt{2}A^{7/2}}{\pi}\frac{(2A)^n}{n^2}$
17	A001006		Y	$\frac{3}{2}\sqrt{\frac{3}{\pi}}\frac{3^n}{n^{3/2}}$
18	A129400		Y	$\frac{3}{2}\sqrt{\frac{3}{\pi}}\frac{6^n}{n^{3/2}}$
19	A005558		N	$\frac{8}{\pi}\frac{4^n}{n^2}$
20	A151265		Y	$\frac{2\sqrt{2}}{\Gamma(1/4)}\frac{3^n}{n^{3/4}}$
21	A151278		Y	$\frac{3\sqrt{3}}{\sqrt{2}\Gamma(1/4)}\frac{3^n}{n^{3/4}}$
22	A151323		Y	$\frac{\sqrt{2}3^{3/4}}{\Gamma(1/4)}\frac{6^n}{n^{3/4}}$
23	A060900		Y	$\frac{4\sqrt{3}}{3\Gamma(1/3)}\frac{4^n}{n^{2/3}}$

$$A = 1+\sqrt{2},\ B = 1+\sqrt{3},\ C = 1+\sqrt{6},\ \lambda = 7+3\sqrt{6},\ \mu = \sqrt{\tfrac{4\sqrt{6}-1}{19}}$$

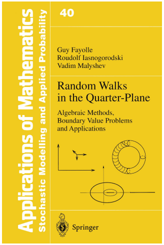
- ▷ Computerized discovery: conv. acc. + LLL/PSLQ [B., Kauers, 2009]
- ▷ Confirmed by human proofs using ACSV in [Melczer, Wilson, 2015]

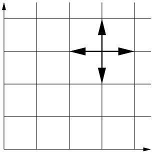


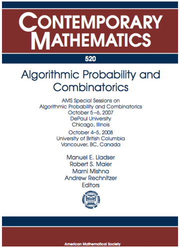




The characteristic polynomial $\chi_{\mathscr{S}} := x + \dfrac{1}{x} + y + \dfrac{1}{y}$

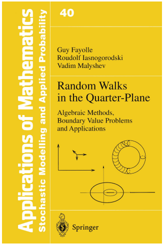


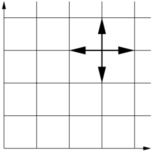


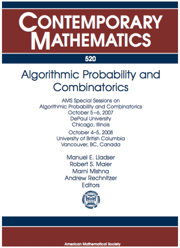


The characteristic polynomial $\chi_{\mathscr{S}} := x + \frac{1}{x} + y + \frac{1}{y}$ is left invariant under

$$\psi(x,y) = \left(x, \frac{1}{y}\right), \qquad \phi(x,y) = \left(\frac{1}{x}, y\right),$$







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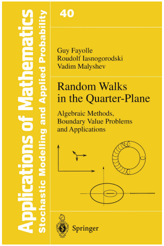
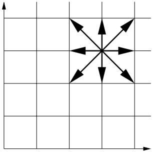
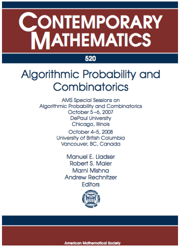
and thus under any element of the group

$$\langle \psi, \phi \rangle = \left\{ (x,y), \left(x, \frac{1}{y}\right), \left(\frac{1}{x}, \frac{1}{y}\right), \left(\frac{1}{x}, y\right) \right\}.$$

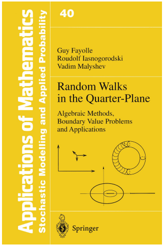


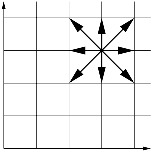


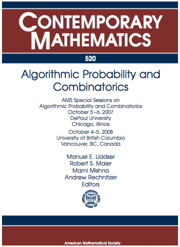


The generating polynomial $\chi_{\mathfrak{S}} := \sum_{(i,j)\in\mathscr{S}} x^i y^j = \sum_{i=-1}^{1} B_i(y) x^i = \sum_{j=-1}^{1} A_j(x) y^j$

The group of a model: the general case







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$$\psi(x,y) = \left(x, \frac{A_{-1}(x)}{A_{+1}(x)} \frac{1}{y}\right), \quad \phi(x,y) = \left(\frac{B_{-1}(y)}{B_{+1}(y)} \frac{1}{x}, y\right),$$

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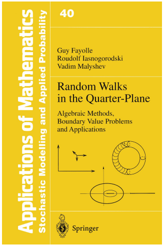

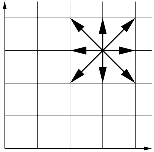

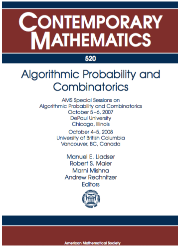

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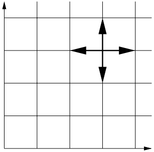
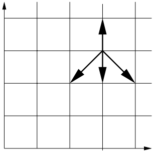
and thus under any element of the (dihedral) group

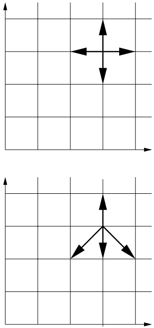
$$\mathcal{G}_{\mathscr{S}} := \langle \psi, \phi \rangle.$$

Examples of groups





Order 4,



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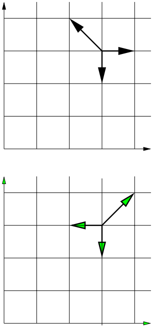
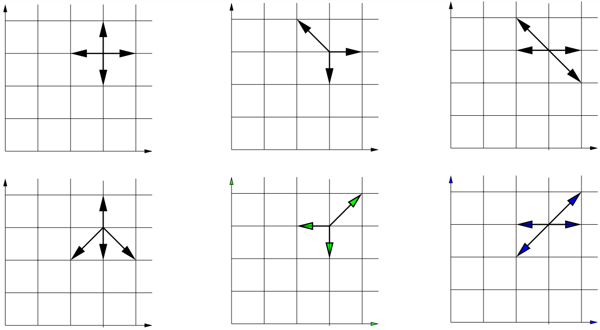


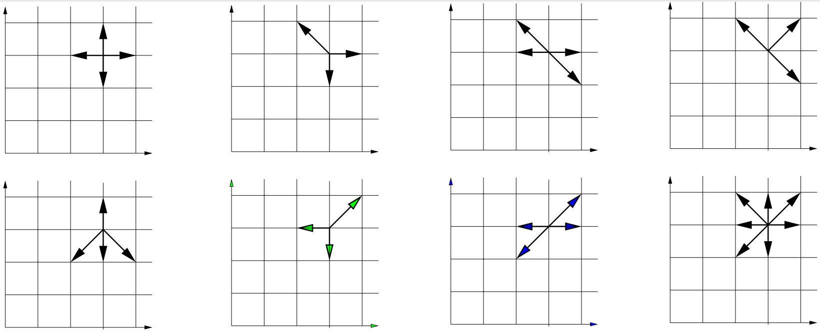
order 6,

Examples of groups

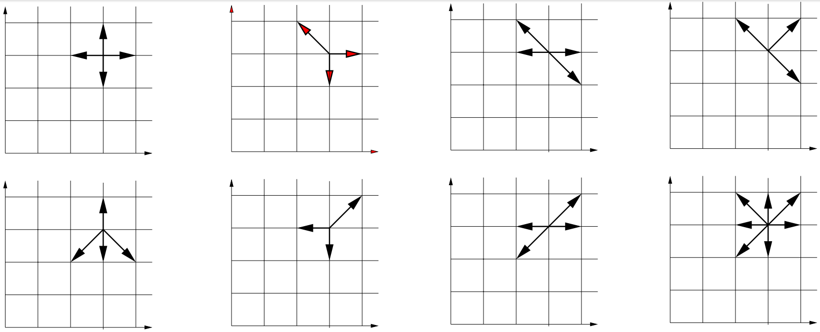


Order 4, order 6, order 8,

Examples of groups



Order 4, order 6, order 8, order ∞.



Order 4, order 6, order 8, order ∞.

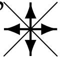
$$\begin{tikzcd}
 & \left(\frac{y}{x}, y\right) \arrow[r, dash, "\Psi"] & \left(\frac{y}{x}, \frac{1}{x}\right) \arrow[dr, dash, "\Phi"] & \\
(x, y) \arrow[ur, dash, "\Phi"] \arrow[dr, dash, "\Psi"'] & & & \left(\frac{1}{y}, \frac{1}{x}\right) \\
 & \left(x, \frac{x}{y}\right) \arrow[r, dash, "\Phi"'] & \left(\frac{1}{y}, \frac{x}{y}\right) \arrow[ur, dash, "\Psi"'] &
\end{tikzcd}$$

Another important concept: the orbit sum (OS)

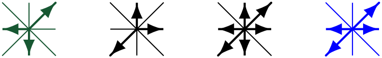
When $\mathcal{G}_{\mathscr{S}}$ is finite, the orbit sum of $\mathscr{S}$ is the polynomial in $\mathbb{Q}[x, x^{-1}, y, y^{-1}]$:

$$\mathrm{OS}_{\mathscr{S}} := \sum_{\theta \in \mathcal{G}_{\mathscr{S}}} (-1)^{\theta} \theta(xy)$$

▷ E.g., for the simple walk, with $\mathcal{G}_{\mathscr{S}} = \left\{ (x,y), \left(x, \frac{1}{y}\right), \left(\frac{1}{x}, \frac{1}{y}\right), \left(\frac{1}{x}, y\right) \right\}$:

OS  $= x \cdot y - \frac{1}{x} \cdot y + \frac{1}{x} \cdot \frac{1}{y} - x \cdot \frac{1}{y}$

▷ For 4 models, the orbit sum is zero:



E.g., for the Kreweras model:

OS  $= x \cdot y - \frac{1}{xy} \cdot y + \frac{1}{xy} \cdot x - y \cdot x + y \cdot \frac{1}{xy} - x \cdot \frac{1}{xy} \quad = 0$

The 79 models: finite and infinite groups

79 models

23 admit a finite group

[Mishna'07]

79 models

56 have an infinite group

[Bousquet-Mélou, Mishna'10]

The 79 models: finite and infinite groups

all $F(t; x, y)$ D-finite

19 transcendental

(OS $\neq 0$)

[Gessel, Zeilberger'92]

[Bousquet-Mélou'02]

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4 algebraic (OS $= 0$)

(3 Kreweras-type + Gessel)

[BMM'10] + [B., Kauers'10]

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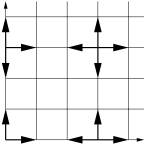
[Bousquet-Mélou, Mishna'10]

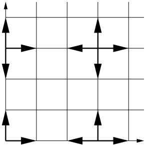
$\longrightarrow$ all non-D-finite

- [Mishna, Rechnitzer'07] and [Melczer, Mishna'13] for 5 singular models
- [Kurkova, Raschel'13] and [B., Raschel, Salvy'13] for all others



The kernel $J = 1 - t \cdot \sum_{(i,j)\in\mathscr{S}} x^i y^j = 1 - t\left(x + \frac{1}{x} + y + \frac{1}{y}\right)$ is invariant under the change of (x, y) into, respectively:

$$\left(\tfrac{1}{x}, y\right), \left(\tfrac{1}{x}, \tfrac{1}{y}\right), \left(x, \tfrac{1}{y}\right).$$



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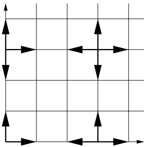
Kernel equation:

$$J(t; x, y)xyF(t; x, y) = xy - txF(t; x, 0) - tyF(t; 0, y)$$

D-Finiteness via the finite group [Bousquet-Mélou, Mishna, 2010]



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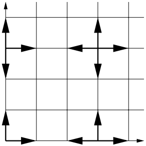

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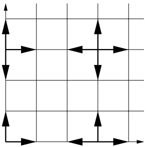
D-Finiteness via the finite group [Bousquet-Mélou, Mishna, 2010]



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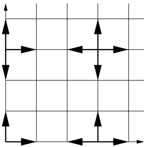
-J(t;x,y)x\tfrac{1}{y}F(t;x,\tfrac{1}{y}) &= -x\tfrac{1}{y} + txF(t;x,0) + t\tfrac{1}{y}F(t;0,\tfrac{1}{y})
\end{aligned}$$

D-Finiteness via the finite group [Bousquet-Mélou, Mishna, 2010]



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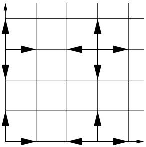
\end{aligned}$$

Summing up yields the orbit equation:

$$\sum_{\theta\in\mathcal{G}} (-1)^\theta \theta\big(xy\,F(t;x,y)\big) = \frac{xy - \frac{1}{x}y + \frac{1}{x}\frac{1}{y} - x\frac{1}{y}}{J(t;x,y)}$$



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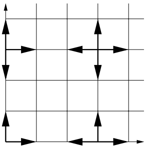
\end{aligned}$$

Taking positive parts yields:

$$[x^> y^>] \sum_{\theta\in\mathcal{G}} (-1)^\theta \theta\big(xy\,F(t;x,y)\big) = [x^> y^>] \frac{xy - \frac{1}{x}y + \frac{1}{x}\frac{1}{y} - x\frac{1}{y}}{J(t;x,y)}$$



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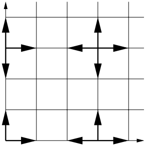
\end{aligned}$$

Summing up and taking positive parts yields:

$$xy\,F(t;x,y) = [x^{>}y^{>}]\frac{xy - \frac{1}{x}y + \frac{1}{x}\frac{1}{y} - x\frac{1}{y}}{J(t;x,y)}$$



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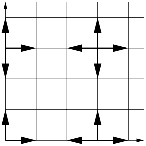
$$\mathsf{GF} \;=\; \mathsf{PosPart}\left(\frac{\mathsf{OS}}{\mathsf{kernel}}\right)$$



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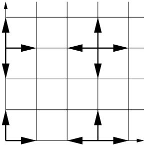

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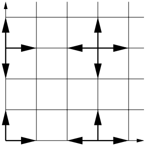
▷ Argument works if $\mathsf{OS} \neq 0$: algebraic version of the reflection principle



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▷ Creative Telescoping finds a differential equation for PosPart(OS/ker)

Theorem [B., Chyzak, van Hoeij, Kauers, Pech, 2017]

Let $\mathscr{S}$ be one of the 19 models with finite group $\mathcal{G}_{\mathscr{S}}$, and orbit sum OS$\neq 0$. Then

- $F_{\mathscr{S}}$ is expressible using iterated integrals of ${}_2F_1$ expressions.
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Main results (IV): explicit expressions for models 1–19

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Example (King walks in the quarter plane, A025595)

F$$(t;1,1) = \frac{1}{t}\int_0^t \frac{1}{(1+4x)^3} \cdot {}_2F_1\left(\begin{matrix}\frac{3}{2} & \frac{3}{2} \\ & 2\end{matrix} \,\middle|\, \frac{16x(1+x)}{(1+4x)^2}\right) dx$$

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▷ Computer-driven discovery and proof; no human proof yet.

▷ Proof uses creative telescoping, ODE factorization, ODE solving.

Bonus: hypergeometric functions occurring in $F(t;x,y)$

	$\mathscr{S}$	occurring ${}_2F_1$	w		$\mathscr{S}$	occurring ${}_2F_1$	w
1		${}_2F_1\left(\left.\begin{matrix}\frac{1}{2}\ \frac{1}{2}\\ 1\end{matrix}\right\| w\right)$	$16t^2$	11		${}_2F_1\left(\left.\begin{matrix}\frac{1}{2}\ \frac{1}{2}\\ 1\end{matrix}\right\| w\right)$	$\frac{16t^2}{4t^2+1}$
2		${}_2F_1\left(\left.\begin{matrix}\frac{1}{2}\ \frac{1}{2}\\ 1\end{matrix}\right\| w\right)$	$16t^2$	12		${}_2F_1\left(\left.\begin{matrix}\frac{1}{4}\ \frac{3}{4}\\ 1\end{matrix}\right\| w\right)$	$\frac{64t^3(2t+1)}{(8t^2-1)^2}$
3		${}_2F_1\left(\left.\begin{matrix}\frac{1}{4}\ \frac{3}{4}\\ 1\end{matrix}\right\| w\right)$	$\frac{64t^2}{(12t^2+1)^2}$	13		${}_2F_1\left(\left.\begin{matrix}\frac{1}{4}\ \frac{3}{4}\\ 1\end{matrix}\right\| w\right)$	$\frac{64t^2(t^2+1)}{(16t^2+1)^2}$
4		${}_2F_1\left(\left.\begin{matrix}\frac{1}{2}\ \frac{1}{2}\\ 1\end{matrix}\right\| w\right)$	$\frac{16t(t+1)}{(4t+1)^2}$	14		${}_2F_1\left(\left.\begin{matrix}\frac{1}{4}\ \frac{3}{4}\\ 1\end{matrix}\right\| w\right)$	$\frac{64t^2(t^2+t+1)}{(12t^2+1)^2}$
5		${}_2F_1\left(\left.\begin{matrix}\frac{1}{4}\ \frac{3}{4}\\ 1\end{matrix}\right\| w\right)$	$64t^4$	15		${}_2F_1\left(\left.\begin{matrix}\frac{1}{4}\ \frac{3}{4}\\ 1\end{matrix}\right\| w\right)$	$64t^4$
6		${}_2F_1\left(\left.\begin{matrix}\frac{1}{4}\ \frac{3}{4}\\ 1\end{matrix}\right\| w\right)$	$\frac{64t^3(t+1)}{(1-4t^2)^2}$	16		${}_2F_1\left(\left.\begin{matrix}\frac{1}{4}\ \frac{3}{4}\\ 1\end{matrix}\right\| w\right)$	$\frac{64t^3(t+1)}{(1-4t^2)^2}$
7		${}_2F_1\left(\left.\begin{matrix}\frac{1}{2}\ \frac{1}{2}\\ 1\end{matrix}\right\| w\right)$	$\frac{16t^2}{4t^2+1}$	17		${}_2F_1\left(\left.\begin{matrix}\frac{1}{3}\ \frac{2}{3}\\ 1\end{matrix}\right\| w\right)$	$27t^3$
8		${}_2F_1\left(\left.\begin{matrix}\frac{1}{4}\ \frac{3}{4}\\ 1\end{matrix}\right\| w\right)$	$\frac{64t^3(2t+1)}{(8t^2-1)^2}$	18		${}_2F_1\left(\left.\begin{matrix}\frac{1}{3}\ \frac{2}{3}\\ 1\end{matrix}\right\| w\right)$	$27t^2(2t+1)$
9		${}_2F_1\left(\left.\begin{matrix}\frac{1}{4}\ \frac{3}{4}\\ 1\end{matrix}\right\| w\right)$	$\frac{64t^2(t^2+1)}{(16t^2+1)^2}$	19		${}_2F_1\left(\left.\begin{matrix}\frac{1}{2}\ \frac{1}{2}\\ 1\end{matrix}\right\| w\right)$	$16t^2$
10		${}_2F_1\left(\left.\begin{matrix}\frac{1}{4}\ \frac{3}{4}\\ 1\end{matrix}\right\| w\right)$	$\frac{64t^2(t^2+t+1)}{(12t^2+1)^2}$				

▷ All related to the complete elliptic integrals $\int_0^{\pi/2}(1-k^2\sin^2\theta)^{\pm\frac{1}{2}}\,d\theta$

Theorem [B., Raschel, Salvy, 2014]

Let $\mathscr{S}$ be one of the 51 non-singular models with infinite group $\mathcal{G}_{\mathscr{S}}$.
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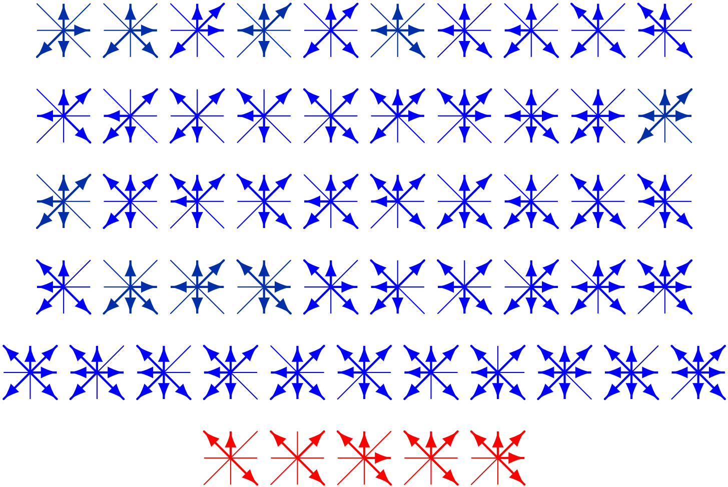
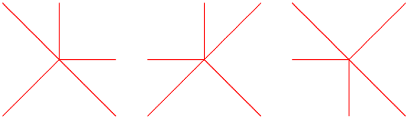
▷ [Bernardi, Bousquet-Mélou, Raschel, 2016] For 9 of these 51 models, $F_{\mathscr{S}}(t;x,y)$ is nevertheless D-algebraic!
▷ [Dreyfus, Hardouin, Roques, Singer, 2017]: hypertranscendence of the remaining 42 models.



In blue, non-singular models, solved by [B., Raschel, Salvy, 2014]
In red, singular models, solved by [Melczer, Mishna, 2013]

Example with infinite group: the scarecrows

[B., Raschel, Salvy, 2014]: $F_{\mathscr{S}}(t;0,0)$ is not D-finite for the models



▷ For the 1st and the 3rd, the excursions sequence $[t^n]\, F_{\mathscr{S}}(t;0,0)$

$$1, 0, 0, 2, 4, 8, 28, 108, 372, \ldots$$

is $\sim K \cdot 5^n \cdot n^{-\alpha}$, with $\alpha = 1 + \pi / \arccos(1/4) = 3.383396\ldots$

[Denisov, Wachtel, 2015]

▷ The irrationality of α prevents $F_{\mathscr{S}}(t;0,0)$ from being D-finite.

[Katz, 1970; Chudnovsky, 1985; André, 1989]

Main classification result

Theorem

Let $\mathscr{S}$ be one of the 74 non-singular models of small-step walks in $\mathbb{N}^2$. The following assertions are equivalent:

(1) the full generating function $F_{\mathscr{S}}(t; x, y)$ is D-finite

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▷ Many contributors (2010–2018): Bernardi, B., Bousquet-Mélou, Chyzak, Denisov, van Hoeij, Kauers, Kurkova, Mishna, Pech, Raschel, Salvy, Wachtel

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(3) the excursions sequence $[t^n]\, F_{\mathscr{S}}(t;0,0)$ is $\sim K \cdot \rho^n \cdot n^\alpha$, with $\alpha \in \mathbb{Q}$

(4) the group $\mathcal{G}_{\mathscr{S}}$ is finite (and $|\mathcal{G}_{\mathscr{S}}| = 2 \cdot \min\{\ell \in \mathbb{N}^\star \mid \frac{\ell}{\alpha+1} \in \mathbb{Z}\}$)

(5) $\exists I \in \mathbb{Q}(x,t), J \in \mathbb{Q}(y,t)$ s.t. $I(x) = J(y)$ on the curve $\chi_{\mathscr{S}}(x,y) = \frac{1}{t}$.

(6) the step set $\mathscr{S}$ has either an axial symmetry, or zero drift and $|\mathscr{S}| \neq 5$.

Moreover, under (1)–(6): $F_{\mathscr{S}}(t;x,y)$ is algebraic $\iff$ $\mathrm{OS}_{\mathscr{S}} = 0$ $\iff$ $\exists U \in \mathbb{Q}(x,t), V \in \mathbb{Q}(y,t)$ s.t. $U(x) + V(y) = xy$ on the curve $\chi_{\mathscr{S}}(x,y) = \frac{1}{t}$.

In this case, $F_{\mathscr{S}}(t;x,y)$ is expressible using nested radicals.

If not, $F_{\mathscr{S}}(t;x,y)$ is expressible using iterated integrals of ${}_2F_1$ expressions.

▷ Proof uses various tools: algebra, complex analysis, probability theory, computer algebra, etc.

quadrant models $\mathscr{S}$: 79

$|\mathcal{G}_{\mathscr{S}}| < \infty$: 23

$|\mathcal{G}_{\mathscr{S}}| = \infty$: 56

orbit sum $\neq 0$: 19

orbit sum $= 0$: 4

asymptotics + GB

kernel method + CT

Guess-and-Prove

non-D-finite

D-finite

algebraic

Theorem: differential finiteness $\Longleftrightarrow$ finiteness of the group !

2D quadrant models: 527

$|\mathcal{G}_{\mathscr{S}}| < \infty$: 118

$|\mathcal{G}_{\mathscr{S}}| = \infty$: 409

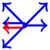
non-D-finite?

orbit sum $\neq 0$: 95

orbit sum $= 0$: 23

kernel method: 94

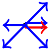
D-finite

CA: 

D-finite

22: reducible to Kreweras/Gessel

D-finite

CA: 

algebraic

[B., Bousquet-Mélou, Kauers, Melczer, 2016] + [Du, Hou, Wang, 2017]

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  - $|\mathcal{G}_{\mathscr{S}}| < \infty$: 118
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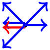
      - CA:  — **D-finite**
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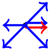
      - CA:  — **algebraic**
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    - **non-D-finite?**

[B., Bousquet-Mélou, Kauers, Melczer, 2016] + [Du, Hou, Wang, 2017]

Question: differential finiteness $\Longleftrightarrow$ finiteness of the group?

Answer: probably yes

$2^{3^3-1} \approx 67$ million models, of which ≈ 11 million inherently 3D

3D octant models $\mathscr{S}$ with ≤ 6 steps: 20804

$|\mathcal{G}_{\mathscr{S}}| < \infty$: 170

$|\mathcal{G}_{\mathscr{S}}| = \infty$: 20634

orbit sum $\neq 0$: 108

orbit sum $= 0$: 62

non-D-finite?

kernel method

2D-reducible: 43

not 2D-reducible: 19

D-finite

D-finite

non-D-finite?

[B., Bousquet-Mélou, Kauers, Melczer, 2016] + [Du, Hou, Wang, 2017];
completed by [Bacher, Kauers, Yatchak, 2016]

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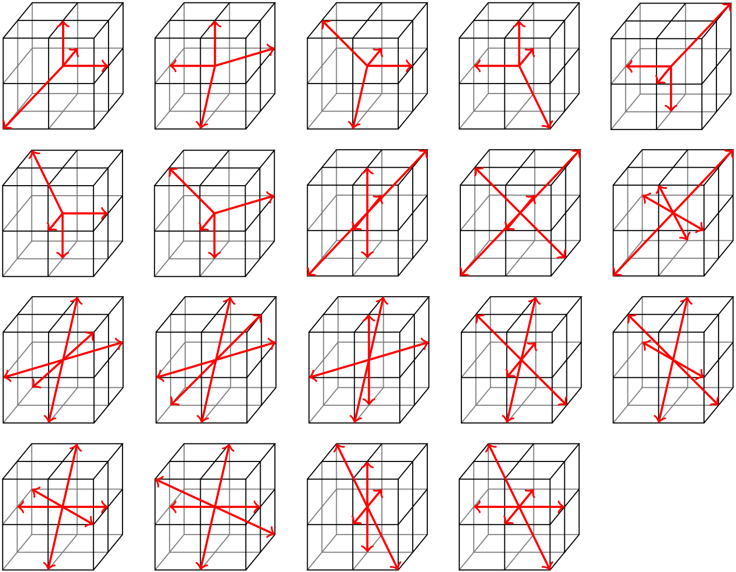
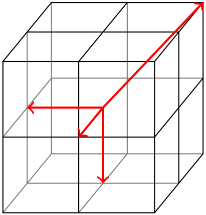
19 mysterious 3D-models





Two different computations suggest:

$$k_{4n} \approx C \cdot 256^n / n^{3.3257570041744\ldots},$$

so excursions are very probably transcendental (and even non-D-finite)

Extensions: Walks in $\mathbb{N}^2$ with large steps

quadrant models with steps in $\{-2,-1,0,1\}^2$: 13 110

$|\text{orbit}| < \infty$: 240

OS $\neq 0$: 431 → **D-finite**

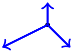
OS $= 0$: 9 → **D-finite?**

$|\text{orbit}| = \infty$: 12 870

α rational: 16 → **non-D-finite?**

α irrational: 12 854 → **non-D-finite**

[B., Bousquet-Mélou, Melczer, 2018]

- Example: For the model 

$$xyF(t;x,y) = [x^{>0}y^{>0}]\,\frac{(x-2x^{-2})(y-(x-x^{-2})y^{-1})}{1-t(xy^{-1}+y+x^{-2}y^{-1})}$$

quadrant models with steps in $\{-2,-1,0,1\}^2$: 13 110

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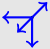
[B., Bousquet-Mélou, Melczer, 2018]

Question: differential finiteness $\iff$ finiteness of the orbit?

Answer: ?

Two challenging models with large steps

Conjecture 1 [B., Bousquet-Mélou, Melczer, 2018]

For the model  the excursions generating function $F(t^{1/2};0,0)$ equals

$$\frac{1}{3t} - \frac{1}{6t} \cdot \left(\frac{1-12t}{(1+36t)^{1/3}} \cdot {}_2F_1\left(\begin{matrix} \frac{1}{6} & \frac{2}{3} \\ & 1 \end{matrix} \middle| \frac{108t(1+4t)^2}{(1+36t)^2} \right) + \sqrt{1-12t} \cdot {}_2F_1\left(\begin{matrix} -\frac{1}{6} & \frac{2}{3} \\ 1 & \end{matrix} \middle| \frac{108t(1+4t)^2}{(1-12t)^2} \right) \right).$$

Conjecture 2 [B., Bousquet-Mélou, Melczer, 2018]

For the model  the excursions generating function $F(t;0,0)$ equals

$$\frac{(1-24\,U+120\,U^2-144\,U^3)\,(1-4\,U)}{(1-3\,U)\,(1-2\,U)^{3/2}\,(1-6\,U)^{9/2}},$$

where $U = t^4 + 53\,t^8 + 4363\,t^{12} + \cdots$ is the unique series in $\mathbb{Q}[[t]]$ satisfying

$$U\,(1-2\,U)^3\,(1-3\,U)^3\,(1-6\,U)^9 = t^4\,(1-4\,U)^4.$$

Conclusion

- Computer algebra may solve difficult combinatorial problems
- Classification of $F(t;x,y)$ fully completed for 2D small step walks
- Robust algorithmic methods, based on efficient algorithms:
  - Guess'n'Prove
  - Creative Telescoping
- Brute-force and/or use of naive algorithms = hopeless.
  E.g. size of algebraic equations for $G(t;x,y) \approx$ 30Gb.

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Lack of "purely human" proofs for some results.

Open: is $F(t;1,1)$ non-D-finite for all 56 models with infinite group?

Many beautiful open questions for 2D models with repeated or large steps, and in dimension > 2.

- Automatic classification of restricted lattice walks, with M. Kauers. Proceedings FPSAC, 2009.
- The complete generating function for Gessel walks is algebraic, with M. Kauers. Proceedings of the American Mathematical Society, 2010.
- Explicit formula for the generating series of diagonal 3D Rook paths, with F. Chyzak, M. van Hoeij and L. Pech. Séminaire Lotharingien de Combinatoire, 2011.
- Non-D-finite excursions in the quarter plane, with K. Raschel and B. Salvy. Journal of Combinatorial Theory A, 2014.
- On 3-dimensional lattice walks confined to the positive octant, with M. Bousquet-Mélou, M. Kauers and S. Melczer. Annals of Comb., 2016.
- A human proof of Gessel's lattice path conjecture, with I. Kurkova, K. Raschel, Transactions of the American Mathematical Society, 2017.
- Hypergeometric expressions for generating functions of walks with small steps in the quarter plane, with F. Chyzak, M. van Hoeij, M. Kauers and L. Pech, European Journal of Combinatorics, 2017.
- Counting walks with large steps in an orthant, with M. Bousquet-Mélou and S. Melczer, preprint, 2018.
- Computer Algebra for Lattice Path Combinatorics, preprint, 2018.