Computer Algebra for Combinatorics

Alin Bostan

MPRI, C-2-22, January 31, 2019

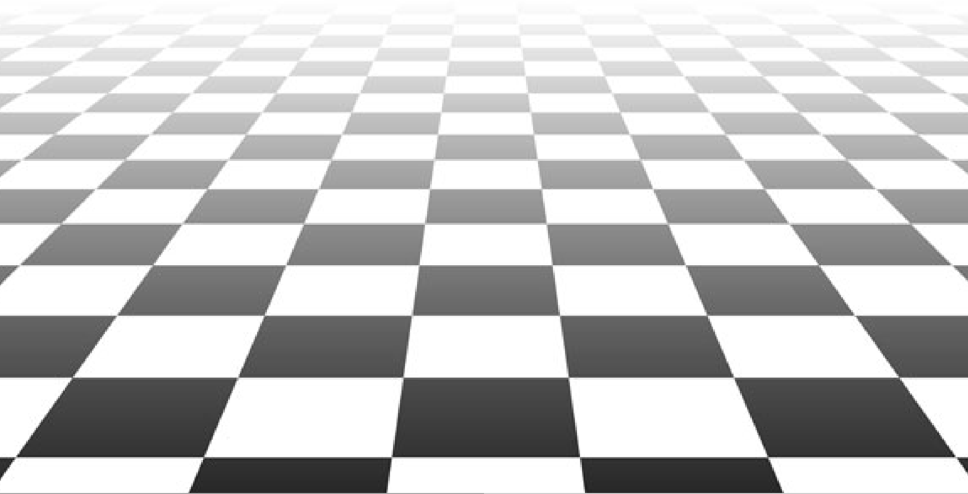
Overview

Part 1: General presentation

Part 2: Guess'n'Prove

Part 3: Creative telescoping

Part 2: Guess’n’Prove



Guess'n'Prove

–The easiest example–

$\{\rightarrow, \uparrow\}$-walks with prescribed endpoint

Question: Find $B_{i,j}$, the number of $\{\rightarrow, \uparrow\}$-walks in $\mathbb{Z}^2$ from $(0,0)$ to (i,j)

$\{\rightarrow, \uparrow\}$-walks with prescribed endpoint

Question: Find $B_{i,j}$, the number of $\{\rightarrow, \uparrow\}$-walks in $\mathbb{Z}^2$ from $(0,0)$ to (i,j)

▷ There are two ways to get to (i,j), either from $(i-1,j)$, or from $(i,j-1)$:

$$B_{i,j} = B_{i-1,j} + B_{i,j-1}.$$

▷ There is only one way to get to a point on an axis:

$$B_{i,0} = B_{0,j} = 1$$

▷ These two rules completely determine all the numbers $B_{i,j}$

$\{\rightarrow, \uparrow\}$-walks with prescribed endpoint

Question: Find $B_{i,j}$, the number of $\{\rightarrow, \uparrow\}$-walks in $\mathbb{Z}^2$ from $(0,0)$ to (i,j)

▷ There are two ways to get to (i,j), either from $(i-1,j)$, or from $(i,j-1)$:

$$B_{i,j} = B_{i-1,j} + B_{i,j-1}.$$

▷ There is only one way to get to a point on an axis:

$$B_{i,0} = B_{0,j} = 1$$

▷ These two rules completely determine all the numbers $B_{i,j}$

⋮							
1	7	28	84	210	462	924	
1	6	21	56	126	252	462	
1	5	15	35	70	126	210	
1	4	10	20	35	56	84	
1	3	6	10	15	21	28	
1	2	3	4	5	6	7	
1	1	1	1	1	1	1	...

$\{\rightarrow,\uparrow\}$-walks with prescribed endpoint

Question: Find $B_{i,j}$, the number of $\{\rightarrow,\uparrow\}$-walks in $\mathbb{Z}^2$ from $(0,0)$ to (i,j)

▷ There are two ways to get to (i,j), either from $(i-1,j)$, or from $(i,j-1)$:

$$B_{i,j} = B_{i-1,j} + B_{i,j-1}.$$

▷ There is only one way to get to a point on an axis:

$$B_{i,0} = B_{0,j} = 1$$

▷ These two rules completely determine all the numbers $B_{i,j}$

Pascal's triangle

$\vdots$							
1	7	28	84	210	462	924	
1	6	21	56	126	252	462	
1	5	15	35	70	126	210	
1	4	10	20	35	56	84	
1	3	6	10	15	21	28	
1	2	3	4	5	6	7	
1	1	1	1	1	1	1	$\cdots$

$\{\rightarrow, \uparrow\}$-walks with prescribed endpoint

Question: Find $B_{i,j}$, the number of $\{\rightarrow, \uparrow\}$-walks in $\mathbb{Z}^2$ from $(0,0)$ to (i,j)

▷ There are two ways to get to (i,j), either from $(i-1,j)$, or from $(i,j-1)$:

$$B_{i,j} = B_{i-1,j} + B_{i,j-1}.$$

▷ There is only one way to get to a point on an axis:

$$B_{i,0} = B_{0,j} = 1$$

▷ These two rules completely determine all the numbers $B_{i,j}$

Pascal's triangle

$\vdots$

1	7	28	84	210	462	924	
1	6	21	56	126	252	462	
1	5	15	35	70	126	210	
1	4	10	20	35	56	84	$\longrightarrow \cdots$
1	3	6	10	15	21	28	$\longrightarrow \frac{(i+1)(i+2)}{2}$
1	2	3	4	5	6	7	$\longrightarrow i+1$
1	1	1	1	1	1	1	$\longrightarrow 1$

The answer can be guessed from the triangle:

$$B_{i,j} = \frac{(i+j)!}{i!j!}$$

$\{\rightarrow,\uparrow\}$-walks with prescribed endpoint

Question: Find $B_{i,j}$, the number of $\{\rightarrow,\uparrow\}$-walks in $\mathbb{Z}^2$ from $(0,0)$ to (i,j)

▷ There are two ways to get to (i,j), either from $(i-1,j)$, or from $(i,j-1)$:

$$B_{i,j} = B_{i-1,j} + B_{i,j-1}.$$

▷ There is only one way to get to a point on an axis:

$$B_{i,0} = B_{0,j} = 1$$

▷ These two rules completely determine all the numbers $B_{i,j}$

Pascal's triangle

$\vdots$							
1	7	28	84	210	462	924	
1	6	21	56	126	252	462	
1	5	15	35	70	126	210	
1	4	10	20	35	56	84	
1	3	6	10	15	21	28	
1	2	3	4	5	6	7	
1	1	1	1	1	1	1	$\cdots$

Answer: binomials

$$B_{i,j} = \frac{(i+j)!}{i!j!} =: \binom{i+j}{i}$$

$\{\to,\uparrow\}$-walks with prescribed endpoint

Question: Find $B_{i,j}$, the number of $\{\to,\uparrow\}$-walks in $\mathbb{Z}^2$ from $(0,0)$ to (i,j)

▷ There are two ways to get to (i,j), either from $(i-1,j)$, or from $(i,j-1)$:

$$B_{i,j} = B_{i-1,j} + B_{i,j-1}.$$

▷ There is only one way to get to a point on an axis:

$$B_{i,0} = B_{0,j} = 1$$

▷ These two rules completely determine all the numbers $B_{i,j}$

Pascal's triangle

$\vdots$							
1	7	28	84	210	462	924	
1	6	21	56	126	252	462	
1	5	15	35	70	126	210	
1	4	10	20	35	56	84	
1	3	6	10	15	21	28	
1	2	3	4	5	6	7	
1	1	1	1	1	1	1	$\cdots$

Answer: binomials

$$B_{i,j} = \frac{(i+j)!}{i!j!} =: \binom{i+j}{i}$$

Proof: If $C_{i,j} \stackrel{\text{def}}{=} \frac{(i+j)!}{i!j!}$,

$$\frac{C_{i-1,j}}{C_{i,j}} + \frac{C_{i,j-1}}{C_{i,j}} = \frac{i}{i+j} + \frac{j}{i+j} = 1$$

$$C_{i,0} = C_{0,j} = 1$$

Thus $B_{i,j} = C_{i,j}$

$\{\rightarrow,\uparrow\}$-walks with prescribed endpoint

Question: Find $B_{i,j}$, the number of $\{\rightarrow,\uparrow\}$-walks in $\mathbb{Z}^2$ from $(0,0)$ to (i,j)

- ▷ There are two ways to get to (i,j), either from $(i-1,j)$, or from $(i,j-1)$:

$$B_{i,j} = B_{i-1,j} + B_{i,j-1}.$$

- ▷ There is only one way to get to a point on an axis:

$$B_{i,0} = B_{0,j} = 1$$

- ▷ These two rules completely determine all the numbers $B_{i,j}$

Pascal's triangle

⋮							
1	7	28	84	210	462	924	
1	6	21	56	126	252	462	
1	5	15	35	70	126	210	
1	4	10	20	35	56	84	
1	3	6	10	15	21	28	
1	2	3	4	5	6	7	
1	1	1	1	1	1	1	…

Answer: binomials

$$B_{i,j} = \frac{(i+j)!}{i!j!} =: \binom{i+j}{i}$$

This is a typical *guess-and-prove* proof!

$\{\to,\uparrow\}$-walks with prescribed endpoint

Question: Find $B_{i,j}$, the number of $\{\to,\uparrow\}$-walks in $\mathbb{Z}^2$ from $(0,0)$ to (i,j)

▷ There are two ways to get to (i,j), either from $(i-1,j)$, or from $(i,j-1)$:

$$B_{i,j} = B_{i-1,j} + B_{i,j-1}.$$

▷ There is only one way to get to a point on an axis:

$$B_{i,0} = B_{0,j} = 1$$

▷ These two rules completely determine all the numbers $B_{i,j}$

Pascal's triangle

⋮							
1	7	28	84	210	462	924	
1	6	21	56	126	252	462	
1	5	15	35	70	126	210	
1	4	10	20	35	56	84	
1	3	6	10	15	21	28	
1	2	3	4	5	6	7	
1	1	1	1	1	1	1	…

Answer: binomials

$$B_{i,j} = \frac{(i+j)!}{i!j!} =: \binom{i+j}{i}$$

Exercise 3

$$B_{n,0} + B_{n-1,1} + \cdots + B_{0,n} = 2^n$$

$$B_{n,0}^2 + B_{n-1,1}^2 + \cdots + B_{0,n}^2 = B_{n,n}$$

$\{\rightarrow,\uparrow\}$-walks with prescribed endpoint

Question: Find $B_{i,j}$, the number of $\{\rightarrow,\uparrow\}$-walks in $\mathbb{Z}^2$ from $(0,0)$ to (i,j)

▷ There are two ways to get to (i,j), either from $(i-1,j)$, or from $(i,j-1)$:

$$B_{i,j} = B_{i-1,j} + B_{i,j-1}.$$

▷ There is only one way to get to a point on an axis:

$$B_{i,0} = B_{0,j} = 1$$

▷ These two rules completely determine all the numbers $B_{i,j}$

Pascal's triangle

⋮							
1	7	28	84	210	462	924	
1	6	21	56	126	252	462	
1	5	15	35	70	126	210	
1	4	10	20	35	56	84	
1	3	6	10	15	21	28	
1	2	3	4	5	6	7	
1	1	1	1	1	1	1	…

Answer: binomials

$$B_{i,j} = \frac{(i+j)!}{i!j!} =: \binom{i+j}{i}$$

Exercise 3

$$\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} = 2^n$$

$$\binom{n}{0}^2 + \binom{n}{1}^2 + \cdots + \binom{n}{n}^2 = \binom{2n}{n}$$

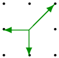
Summary of Part 1: Walks with unit steps in $\mathbb{N}^2$

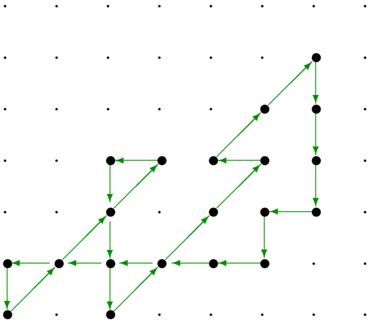
- quadrant models: 79
  - $|G| < \infty$: 23
    - nonzero orbit sum: 19
      - Kernel method + CT
        - **D-finite**
    - zero orbit sum: 4
      - Guess'n'Prove
        - **algebraic**
  - $|G| = \infty$: 56
    - asymptotics + GB
      - **non-D-finite**

Two important models: Kreweras and Gessel walks

$\mathscr{S} = \{\downarrow, \leftarrow, \nearrow\}$ $\qquad F_{\mathscr{S}}(t;x,y) \equiv K(t;x,y)$

$\mathscr{S} = \{\nearrow, \swarrow, \leftarrow, \rightarrow\}$ $\qquad F_{\mathscr{S}}(t;x,y) \equiv G(t;x,y)$



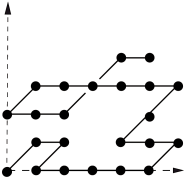


Example: A Kreweras excursion.

- Gessel walks: walks in $\mathbb{N}^2$ using only steps in $\mathscr{S} = \{\nearrow, \swarrow, \leftarrow, \rightarrow\}$
- $g(n;i,j)$ = number of walks from $(0,0)$ to (i,j) with n steps in $\mathscr{S}$

Question: Find the nature of the generating function

$$G(t;x,y) = \sum_{i,j,n=0}^{\infty} g(n;i,j)\, x^i y^j t^n \ \in \mathbb{Q}[[x,y,t]]$$

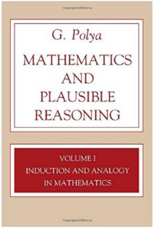


Theorem (B.-Kauers, 2010) $G(t;x,y)$ is an algebraic function†.

$\rightarrow$ Effective, computer-driven discovery and proof

† Minimal polynomial $P(x,y,t,G(t;x,y)) = 0$ has $> 10^{11}$ terms; $\approx 30\,\text{Gb}$ (!)

First guess, then prove [Pólya, 1954]



Guessing and Proving

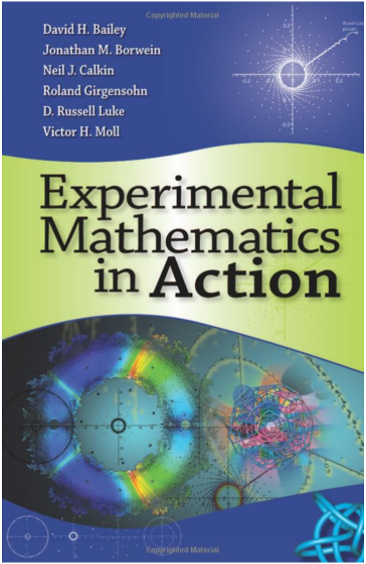
George Pólya

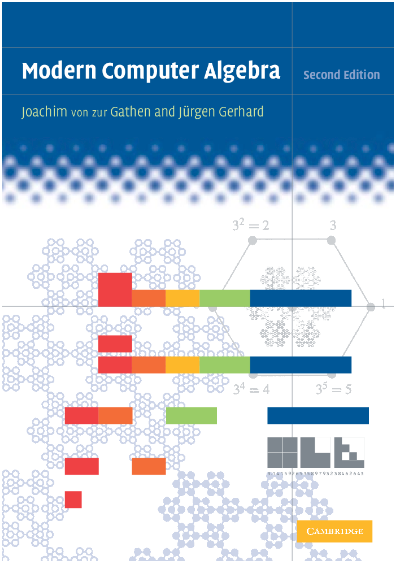
What is "scientific method"? Philosophers and non-philosophers have discussed this question and have not yet finished discussing it. Yet as a first introduction it can be described in three syllables:

Guess and test.

Mathematicians too follow this advice in their research although they sometimes refuse to confess it. They have, however, something which the other scientists cannot really have. For mathematicians the advice is

First guess, then prove.


David H. Bailey
Jonathan M. Borwein
Neil J. Calkin
Roland Girgensohn
D. Russell Luke
Victor H. Moll
Experimental Mathematics in Action
Copyrighted Material


Modern Computer Algebra
Second Edition
Joachim von zur Gathen and Jürgen Gerhard
CAMBRIDGE

Guess'n'Prove for

–PROVING ALGEBRAICITY–

Methodology for proving algebraicity

Experimental mathematics –Guess'n'Prove– approach:

(S1) Generate data

(S2) Conjecture

(S3) Prove

Methodology for proving algebraicity

Experimental mathematics –Guess'n'Prove– approach:

(S1) Generate data
compute a high order expansion of the series $F_{\mathscr{S}}(t;x,y)$;

(S2) Conjecture
guess a candidate for the minimal polynomial of $F_{\mathscr{S}}(t;x,y)$, using Hermite-Padé approximation;

(S3) Prove
rigorously certify the minimal polynomials, using (exact) polynomial computations.

Methodology for proving algebraicity

Experimental mathematics –Guess'n'Prove– approach:

(S1) Generate data
compute a high order expansion of the series $F_{\mathscr{S}}(t; x, y)$;

(S2) Conjecture
guess candidates for minimal polynomials of $F_{\mathscr{S}}(t; x, 0)$ and $F_{\mathscr{S}}(t; 0, y)$, using Hermite-Padé approximation;

(S3) Prove
rigorously certify the minimal polynomials, using (exact) polynomial computations.

Methodology for proving algebraicity

Experimental mathematics –Guess'n'Prove– approach:

(S1) Generate data
compute a high order expansion of the series $F_{\mathscr{S}}(t;x,y)$;

(S2) Conjecture
guess candidates for minimal polynomials of $F_{\mathscr{S}}(t;x,0)$ and $F_{\mathscr{S}}(t;0,y)$, using Hermite-Padé approximation;

(S3) Prove
rigorously certify the minimal polynomials, using (exact) polynomial computations.

+ Efficient Computer Algebra

Step (S1): high order series expansions

$f_{\mathscr{S}}(n;i,j)$ satisfies the recurrence with constant coefficients

$$f_{\mathscr{S}}(n+1;i,j) = \sum_{(u,v)\in\mathscr{S}} f_{\mathscr{S}}(n;i-u,j-v) \quad \text{for} \quad n,i,j \geq 0$$

+ initial conditions $f_{\mathscr{S}}(0;i,j) = \delta_{0,i,j}$ and $f_{\mathscr{S}}(n;-1,j) = f_{\mathscr{S}}(n;i,-1) = 0$.

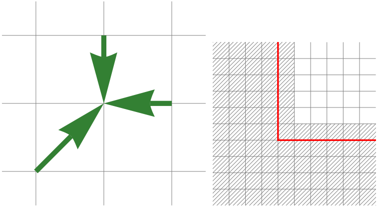
$f_{\mathscr{S}}(n; i, j)$ satisfies the recurrence with constant coefficients

$$f_{\mathscr{S}}(n+1; i, j) = \sum_{(u,v)\in\mathscr{S}} f_{\mathscr{S}}(n; i-u, j-v) \quad \text{for} \quad n, i, j \geq 0$$

+ initial conditions $f_{\mathscr{S}}(0; i, j) = \delta_{0,i,j}$ and $f_{\mathscr{S}}(n; -1, j) = f_{\mathscr{S}}(n; i, -1) = 0$.

Example: for the Kreweras walks,

$$\begin{aligned} k(n+1; i, j) = k(n; i+1, j) \\ + k(n; i, j+1) \\ + k(n; i-1, j-1) \end{aligned}$$



Step (S1): high order series expansions

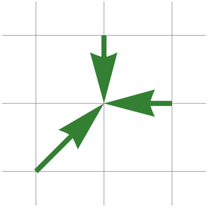
$f_{\mathscr{S}}(n;i,j)$ satisfies the recurrence with constant coefficients

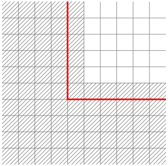
$$f_{\mathscr{S}}(n+1;i,j) = \sum_{(u,v)\in\mathscr{S}} f_{\mathscr{S}}(n;i-u,j-v) \quad \text{for} \quad n,i,j \geq 0$$

+ initial conditions $f_{\mathscr{S}}(0;i,j) = \delta_{0,i,j}$ and $f_{\mathscr{S}}(n;-1,j) = f_{\mathscr{S}}(n;i,-1) = 0$.

Example: for the Kreweras walks,

$$\begin{aligned} k(n+1;i,j) = {} & k(n;i+1,j) \\ & + k(n;i,j+1) \\ & + k(n;i-1,j-1) \end{aligned}$$





▷ Recurrence is used to compute $F_{\mathscr{S}}(t;x,y) \bmod t^N$ for large N.

$$\begin{aligned} K(t;x,y) = {} & 1 + xyt + (x^2y^2 + y + x)t^2 + (x^3y^3 + 2xy^2 + 2x^2y + 2)t^3 \\ & + (x^4y^4 + 3x^2y^3 + 3x^3y^2 + 2y^2 + 6xy + 2x^2)t^4 \\ & + (x^5y^5 + 4x^3y^4 + 4x^4y^3 + 5xy^3 + 12x^2y^2 + 5x^3y + 8y + 8x)t^5 + \cdots \end{aligned}$$

Step (S2): guessing equations for $F_{\mathscr{S}}(t;x,y)$, a first idea

In terms of generating functions, the recurrence on $k(n;i,j)$ reads

$$\boxed{\begin{aligned}&(xy-(x+y+x^2y^2)t)K(t;x,y)\\&\qquad = xy - xt\,K(t;x,0) - yt\,K(t;0,y)\end{aligned}} \qquad \text{(KerEq)}$$

▷ A similar kernel equation holds for $F_{\mathscr{S}}(t;x,y)$, for any $\mathscr{S}$-walk.

Corollary. $F_{\mathscr{S}}(t;x,y)$ is algebraic (resp. D-finite) if and only if $F_{\mathscr{S}}(t;x,0)$ and $F_{\mathscr{S}}(t;0,y)$ are both algebraic (resp. D-finite).

▷ Crucial simplification: equations for $G(t;x,y)$ are huge ($\approx$ 30 Gb)

Step (S2): guessing equations for $F_{\mathscr{S}}(t;x,0)$ and $F_{\mathscr{S}}(t;0,y)$

Task 1: Given the first N terms of $S = F_{\mathscr{S}}(t;x,0) \in \mathbb{Q}[x][[t]]$, search for a differential equation satisfied by S at precision N:

$$c_r(x,t) \cdot \frac{\partial^r S}{\partial t^r} + \cdots + c_1(x,t) \cdot \frac{\partial S}{\partial t} + c_0(x,t) \cdot S = 0 \bmod t^N.$$

Step (S2): guessing equations for $F_{\mathscr{S}}(t;x,0)$ and $F_{\mathscr{S}}(t;0,y)$

Task 1: Given the first N terms of $S = F_{\mathscr{S}}(t;x,0) \in \mathbb{Q}[x][[t]]$, search for a differential equation satisfied by S at precision N:

$$c_r(x,t) \cdot \frac{\partial^r S}{\partial t^r} + \cdots + c_1(x,t) \cdot \frac{\partial S}{\partial t} + c_0(x,t) \cdot S = 0 \bmod t^N.$$

Task 2: Search for an algebraic equation $\mathcal{P}_{x,0}(S) = 0 \bmod t^N$.

Step (S2): guessing equations for $F_{\mathscr{S}}(t;x,0)$ and $F_{\mathscr{S}}(t;0,y)$

Task 1: Given the first N terms of $S = F_{\mathscr{S}}(t;x,0) \in \mathbb{Q}[x][[t]]$, search for a differential equation satisfied by S at precision N:

$$c_r(x,t) \cdot \frac{\partial^r S}{\partial t^r} + \cdots + c_1(x,t) \cdot \frac{\partial S}{\partial t} + c_0(x,t) \cdot S = 0 \bmod t^N.$$

Task 2: Search for an algebraic equation $\mathcal{P}_{x,0}(S) = 0 \bmod t^N$.

- Both tasks amount to linear algebra in size N over $\mathbb{Q}(x)$.
- In practice, we use modular Hermite-Padé approximation (Beckermann-Labahn algorithm) combined with (rational) evaluation-interpolation and rational number reconstruction.
- Fast (FFT-based) arithmetic in $\mathbb{F}_p[t]$ and $\mathbb{F}_p[t]\langle \frac{t}{\partial t} \rangle$.

Step (S2): guessing equations for $K(t;x,0)$

Using $N=80$ terms of $K(t;x,0)$, one can guess

▷ a linear differential equation of order 4, degrees $(14,11)$ in (t,x), such that

$$t^3\cdot(3t-1)\cdot(9t^2+3t+1)\cdot(3t^2+24t^2x^3-3xt-2x^2)\cdot$$
$$\cdot(16t^2x^5+4x^4-72t^4x^3-18x^3t+5t^2x^2+18xt^3-9t^4)\cdot$$
$$\cdot(4t^2x^3-t^2+2xt-x^2)\cdot\frac{\partial^4 K(t;x,0)}{\partial t^4}+\cdots$$
$$=0 \bmod t^{80}$$

▷ a polynomial of tridegree $(6,10,6)$ in (T,t,x)

$$\mathcal{P}_{x,0}=x^6t^{10}T^6-3x^4t^8(x-2t)T^5+$$
$$+x^2t^6\left(12t^2+3t^2x^3-12xt+\frac{7}{2}x^2\right)T^4+\cdots$$

such that $\mathcal{P}_{x,0}(K(t;x,0),t,x)=0 \bmod t^{80}$.

Step (S2): guessing equations for $G(t;x,0)$ and $G(t;0,y)$

Using $N = 1200$ terms of $G(t;x,y)$, our guesser found candidates

- $\mathcal{P}_{x,0}$ in $\mathbb{Z}[T,t,x]$ of degree $(24,43,32)$, coefficients of 21 digits
- $\mathcal{P}_{0,y}$ in $\mathbb{Z}[T,t,y]$ of degree $(24,44,40)$, coefficients of 23 digits

such that

$$\mathcal{P}_{x,0}(G(t;x,0),t,x) = 0 \bmod t^{1200}, \quad \mathcal{P}_{0,y}(G(t;0,y),t,y) = 0 \bmod t^{1200}.$$

Step (S2): guessing equations for $G(t;x,0)$ and $G(t;0,y)$

Using $N = 1200$ terms of $G(t;x,y)$, our guesser found candidates

- $\mathcal{P}_{x,0}$ in $\mathbb{Z}[T,t,x]$ of degree $(24,43,32)$, coefficients of 21 digits
- $\mathcal{P}_{0,y}$ in $\mathbb{Z}[T,t,y]$ of degree $(24,44,40)$, coefficients of 23 digits

such that

$$\mathcal{P}_{x,0}(G(t;x,0),t,x) = 0 \bmod t^{1200}, \quad \mathcal{P}_{0,y}(G(t;0,y),t,y) = 0 \bmod t^{1200}.$$

$\triangleright$ Guessing $\mathcal{P}_{x,0}$ by undetermined coefficients would have required to solve a dense linear system of size $\approx 100\,000$, and ≈ 1000 digits entries!

Step (S2): guessing equations for $G(t;x,0)$ and $G(t;0,y)$

Using $N = 1200$ terms of $G(t;x,y)$, our guesser found candidates

- $\mathcal{P}_{x,0}$ in $\mathbb{Z}[T,t,x]$ of degree $(24,43,32)$, coefficients of 21 digits
- $\mathcal{P}_{0,y}$ in $\mathbb{Z}[T,t,y]$ of degree $(24,44,40)$, coefficients of 23 digits

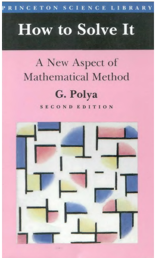
such that

$$\mathcal{P}_{x,0}(G(t;x,0),t,x) = 0 \bmod t^{1200}, \quad \mathcal{P}_{0,y}(G(t;0,y),t,y) = 0 \bmod t^{1200}.$$

▷ Guessing $\mathcal{P}_{x,0}$ by undetermined coefficients would have required to solve a dense linear system of size $\approx 100\,000$, and ≈ 1000 digits entries!

▷ [B., Kauers '09] actually first guessed differential equations†, then computed their p-curvatures to empirically certify them. This led them suspect the algebraicity of $G(t;x,0)$ and $G(t;0,y)$, using a conjecture of Grothendieck's (on differential equations modulo p) as an oracle.

† of order 11, and bidegree $(96,78)$ for $G(t;x,0)$, and $(68,28)$ for $G(t;0,y)$



Guessing and Proving

George Pólya

Guessing is good, proving is better.

Step (S3): warm-up – Gessel excursions are algebraic

Theorem

$$g(t) := G(\sqrt{t}; 0, 0) = \sum_{n=0}^{\infty} \frac{(5/6)_n (1/2)_n}{(5/3)_n (2)_n} (16t)^n \quad \text{is algebraic.}$$

Step (S3): warm-up – Gessel excursions are algebraic

Theorem

$$g(t) := G(\sqrt{t}; 0, 0) = \sum_{n=0}^{\infty} \frac{(5/6)_n (1/2)_n}{(5/3)_n (2)_n} (16t)^n \quad \text{is algebraic.}$$

Proof: First guess a polynomial $P(t, T)$ in $\mathbb{Q}[t, T]$, then prove that P admits the power series $g(t) = \sum_{n=0}^{\infty} g_n t^n$ as a root.

Step (S3): warm-up – Gessel excursions are algebraic

Theorem

$$g(t) := G(\sqrt{t}; 0, 0) = \sum_{n=0}^{\infty} \frac{(5/6)_n (1/2)_n}{(5/3)_n (2)_n} (16t)^n \quad \text{is algebraic}.$$

Proof: First guess a polynomial $P(t, T)$ in $\mathbb{Q}[t, T]$, then prove that P admits the power series $g(t) = \sum_{n=0}^{\infty} g_n t^n$ as a root.

1. Find P such that $P(t, g(t)) = 0 \bmod t^{100}$ by (structured) linear algebra.

Step (S3): warm-up – Gessel excursions are algebraic

Theorem

$$g(t) := G(\sqrt{t}; 0, 0) = \sum_{n=0}^{\infty} \frac{(5/6)_n (1/2)_n}{(5/3)_n (2)_n} (16t)^n \quad \text{is algebraic.}$$

Proof: First guess a polynomial $P(t, T)$ in $\mathbb{Q}[t, T]$, then prove that P admits the power series $g(t) = \sum_{n=0}^{\infty} g_n t^n$ as a root.

1. Find P such that $P(t, g(t)) = 0 \bmod t^{100}$ by (structured) linear algebra.
2. Implicit function theorem: $\exists!$ root $r(t) \in \mathbb{Q}[[t]]$ of P.

Step (S3): warm-up – Gessel excursions are algebraic

Theorem

$$g(t) := G(\sqrt{t};0,0) = \sum_{n=0}^{\infty} \frac{(5/6)_n(1/2)_n}{(5/3)_n(2)_n}(16t)^n \quad \text{is algebraic.}$$

Proof: First guess a polynomial $P(t,T)$ in $\mathbb{Q}[t,T]$, then prove that P admits the power series $g(t) = \sum_{n=0}^{\infty} g_n t^n$ as a root.

1. Find P such that $P(t, g(t)) = 0 \bmod t^{100}$ by (structured) linear algebra.
2. Implicit function theorem: $\exists!$ root $r(t) \in \mathbb{Q}[[t]]$ of P.
3. $r(t) = \sum_{n=0}^{\infty} r_n t^n$ being algebraic, it is D-finite, and so is (r_n):

$$(n+2)(3n+5)r_{n+1} - 4(6n+5)(2n+1)r_n = 0, \qquad r_0 = 1$$

$\Rightarrow$ solution $r_n = \frac{(5/6)_n(1/2)_n}{(5/3)_n(2)_n} 16^n = g_n$, thus $g(t) = r(t)$ is algebraic.

Step (S3): rigorous proof for Kreweras walks 

1. Setting $y_0 = \frac{x-t-\sqrt{x^2-2tx+t^2(1-4x^3)}}{2tx^2} = t + \frac{1}{x}t^2 + \frac{x^3+1}{x^2}t^3 + \cdots$ in the kernel equation

$$\underbrace{(xy-(x+y+x^2y^2)t)}_{\overset{!}{=}0}K(t;x,y) = xy - xtK(t;x,0) - ytK(t;0,y)$$

Step (S3): rigorous proof for Kreweras walks

1. Setting $y_0 = \frac{x-t-\sqrt{x^2-2tx+t^2(1-4x^3)}}{2tx^2} = t + \frac{1}{x}t^2 + \frac{x^3+1}{x^2}t^3 + \cdots$ in the kernel equation (diagonal symmetry implies $K(t;y,x) = K(t;x,y)$)

$$\underbrace{(xy - (x+y+x^2y^2)t)}_{\stackrel{!}{=}\,0} K(t;x,y) = xy - xtK(t;x,0) - ytK(t;y,0)$$

Step (S3): rigorous proof for Kreweras walks 

1. Setting $y_0 = \frac{x-t-\sqrt{x^2-2tx+t^2(1-4x^3)}}{2tx^2} = t + \frac{1}{x}t^2 + \frac{x^3+1}{x^2}t^3 + \cdots$ in the kernel equation

$$\underbrace{(xy-(x+y+x^2y^2)t)}_{\stackrel{!}{=}0}K(t;x,y) = xy - xtK(t;x,0) - ytK(t;y,0)$$

shows that $U = K(t;x,0)$ satisfies the reduced kernel equation

$$\boxed{0 = x\cdot y_0 - x\cdot t\cdot U(t,x) - y_0\cdot t\cdot U(t,y_0)} \qquad \text{(RKerEq)}$$

Step (S3): rigorous proof for Kreweras walks 

1. Setting $y_0 = \frac{x-t-\sqrt{x^2-2tx+t^2(1-4x^3)}}{2tx^2} = t + \frac{1}{x}t^2 + \frac{x^3+1}{x^2}t^3 + \cdots$ in the kernel equation

$$\underbrace{(xy-(x+y+x^2y^2)t)}_{\overset{!}{=}\,0}K(t;x,y) = xy - xtK(t;x,0) - ytK(t;y,0)$$

shows that $U = K(t;x,0)$ satisfies the reduced kernel equation

$$\boxed{0 = x\cdot y_0 - x\cdot t\cdot U(t,x) - y_0\cdot t\cdot U(t,y_0)} \qquad \text{(RKerEq)}$$

2. (RKerEq) admits a unique solution in $\mathbb{Q}[[x,t]]$, namely $U = K(t;x,0)$.

Step (S3): rigorous proof for Kreweras walks

1. Setting $y_0 = \frac{x-t-\sqrt{x^2-2tx+t^2(1-4x^3)}}{2tx^2} = t + \frac{1}{x}t^2 + \frac{x^3+1}{x^2}t^3 + \cdots$ in the kernel equation

$$\underbrace{(xy-(x+y+x^2y^2)t)}_{\stackrel{!}{=}0}K(t;x,y) = xy - xtK(t;x,0) - ytK(t;y,0)$$

shows that $U = K(t;x,0)$ satisfies the reduced kernel equation

$$\boxed{0 = x \cdot y_0 - x \cdot t \cdot U(t,x) - y_0 \cdot t \cdot U(t,y_0)} \qquad \text{(RKerEq)}$$

2. (RKerEq) admits a unique solution in $\mathbb{Q}[[x,t]]$, namely $U = K(t;x,0)$.

3. The guessed candidate $\mathcal{P}_{x,0}(T,t,x)$ has a root $H(t,x)$ in $\mathbb{Q}[[x,t]]$.

Step (S3): rigorous proof for Kreweras walks 

1. Setting $y_0 = \frac{x-t-\sqrt{x^2-2tx+t^2(1-4x^3)}}{2tx^2} = t + \frac{1}{x}t^2 + \frac{x^3+1}{x^2}t^3 + \cdots$ in the kernel equation

$$\underbrace{(xy-(x+y+x^2y^2)t)}_{\overset{!}{=}\,0}K(t;x,y) = xy - xtK(t;x,0) - ytK(t;y,0)$$

shows that $U = K(t;x,0)$ satisfies the reduced kernel equation

$$\boxed{0 = x \cdot y_0 - x \cdot t \cdot U(t,x) - y_0 \cdot t \cdot U(t,y_0)} \qquad \text{(RKerEq)}$$

2. (RKerEq) admits a unique solution in $\mathbb{Q}[[x,t]]$, namely $U = K(t;x,0)$.

3. The guessed candidate $\mathcal{P}_{x,0}(T,t,x)$ has a root $H(t,x)$ in $\mathbb{Q}[[x,t]]$.

4. $U = H(t,x)$ also satisfies (RKerEq) Resultant computations!

Step (S3): rigorous proof for Kreweras walks 

1. Setting $y_0 = \frac{x-t-\sqrt{x^2-2tx+t^2(1-4x^3)}}{2tx^2} = t + \frac{1}{x}t^2 + \frac{x^3+1}{x^2}t^3 + \cdots$ in the kernel equation

$$\underbrace{(xy-(x+y+x^2y^2)t)}_{\overset{!}{=}\,0}K(t;x,y) = xy - xtK(t;x,0) - ytK(t;y,0)$$

shows that $U = K(t;x,0)$ satisfies the reduced kernel equation

$$\boxed{0 = x \cdot y_0 - x \cdot t \cdot U(t,x) - y_0 \cdot t \cdot U(t,y_0)} \qquad \text{(RKerEq)}$$

2. (RKerEq) admits a unique solution in $\mathbb{Q}[[x,t]]$, namely $U = K(t;x,0)$.
3. The guessed candidate $\mathcal{P}_{x,0}(T,t,x)$ has a root $H(t,x)$ in $\mathbb{Q}[[x,t]]$.
4. $U = H(t,x)$ also satisfies (RKerEq) Resultant computations!
5. Uniqueness $\implies H(t,x) = K(t;x,0) \implies K(t;x,0)$ is algebraic!

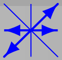
Algebraicity of Kreweras walks: a computer proof in a nutshell

```
# HIGH ORDER EXPANSION (S1)
> st,bu:=time(),kernelopts(bytesused):
> f:=proc(n,i,j) option remember;
    if i<0 or j<0 or n<0 then 0
    elif n=0 then if i=0 and j=0 then 1 else 0 fi
    else f(n-1,i-1,j-1)+f(n-1,i,j+1)+f(n-1,i+1,j) fi
  end:
> S:=series(add(add(f(k,i,0)*x^i,i=0..k)*t^k,k=0..80),t,80):

# GUESSING (S2)
> libname:=".",libname:gfun:-version();
                              3.76
> P:=subs(Fx0(t)=T,gfun:-seriestoalgeq(S,Fx0(t))[1]):

# RIGOROUS PROOF (S3)
> ker := (T,t,x) -> (x+T+x^2*T^2)*t-x*T:
> pol := unapply(P,T,t,x):
> p1 := resultant(pol(z-T,t,x),ker(t*z,t,x),z):
> p2 := subs(T=x*T,resultant(numer(pol(T/z,t,z)),ker(z,t,x),z)):
> normal(primpart(p1,T)/primpart(p2,T));
                                 1

# time (in sec) and memory consumption (in Mb)
> trunc(time()-st),trunc((kernelopts(bytesused)-bu)/1000^2);
                              8, 785
```

Step (S3): rigorous proof for Gessel walks 

Same strategy, but several complications:

- stepset diagonal symmetry is lost: $G(t;x,y) \neq G(t;y,x)$;
- $G(t;0,0)$ occurs in (KerEq) (because of the step $\swarrow$);
- equations are $\approx 5\,000$ times bigger.

$\longrightarrow$ replace equation (RKerEq) by a system of 2 reduced kernel equations.

$\longrightarrow$ fast algorithms needed (e.g., [B., Flajolet, Salvy, Schost, 2006] for computations with algebraic series).

ELSEVIER

Available online at www.sciencedirect.com

SCIENCE DIRECT

Journal of Symbolic Computation 41 (2006) 1–29

Journal of Symbolic Computation

www.elsevier.com/locate/jsc

Fast computation of special resultants

Alin Bostan[a,*], Philippe Flajolet[a], Bruno Salvy[a], Éric Schost[b]

[a] Algorithms Project, Inria Rocquencourt, 78153 Le Chesnay, France
[b] LIX, École polytechnique, 91128 Palaiseau, France

Received 3 September 2003; accepted 9 July 2005
Available online 25 October 2005

BACK TO THE EXERCISE

–A hint–

The exercise

Let $\mathfrak{S} = \{\uparrow, \leftarrow, \searrow\}$. A $\mathfrak{S}$-walk is a path in $\mathbb{Z}^2$ using only steps from $\mathfrak{S}$. Show that, for any integer n, the following quantities are equal:

(i) the number a_n of $\mathfrak{S}$-walks of length n confined to the upper half plane $\mathbb{Z} \times \mathbb{N}$ that start and end at the origin $(0, 0)$;

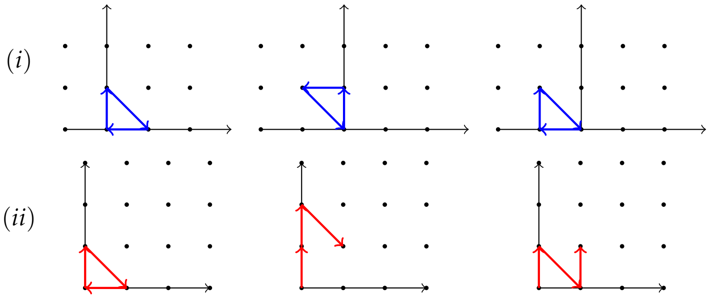
(ii) the number b_n of $\mathfrak{S}$-walks of length n confined to the quarter plane $\mathbb{N}^2$ that start at the origin $(0, 0)$ and finish on the diagonal $x = y$.

Let $\mathfrak{S} = \{\uparrow, \leftarrow, \searrow\}$. A $\mathfrak{S}$-walk is a path in $\mathbb{Z}^2$ using only steps from $\mathfrak{S}$. Show that, for any integer n, the following quantities are equal:

(i) the number a_n of $\mathfrak{S}$-walks of length n confined to the upper half plane $\mathbb{Z} \times \mathbb{N}$ that start and end at the origin $(0, 0)$;

(ii) the number b_n of $\mathfrak{S}$-walks of length n confined to the quarter plane $\mathbb{N}^2$ that start at the origin $(0, 0)$ and finish on the diagonal $x = y$.

For instance, for $n = 3$, this common value is $a_3 = b_3 = 3$:



A recurrence relation for $\{\uparrow, \leftarrow, \searrow\}$-walks in $\mathbb{Z} \times \mathbb{N}$

$h(n;i,j)$ = nb. of $\{\uparrow, \leftarrow, \searrow\}$-walks in $\mathbb{Z} \times \mathbb{N}$ of length n from $(0,0)$ to (i,j)
The numbers $h(n;i,j)$ satisfy

$$h(n;i,j) = \begin{cases} 0 & \text{if } j < 0 \text{ or } n < 0, \\ \mathbb{1}_{i=j=0} & \text{if } n = 0, \\ \displaystyle\sum_{(i',j') \in \mathscr{S}} h(n-1; i-i', j-j') & \text{otherwise.} \end{cases}$$

```
> h:=proc(n,i,j)
  option remember;
    if j<0 or n<0 then 0
    elif n=0 then if i=0 and j=0 then 1 else 0 fi
    else h(n-1,i,j-1) + h(n-1,i+1,j) + h(n-1,i-1,j+1) fi
  end:

> A:=series(add(h(n,0,0)*t^n, n=0..12), t,12);
```

$$A = 1 + 3t^3 + 30t^6 + 420t^9 + O(t^{12})$$

$q(n;i,j)$ = nb. of $\{\uparrow, \leftarrow, \searrow\}$-walks in $\mathbb{N}^2$ of length n from $(0,0)$ to (i,j)
The numbers $q(n;i,j)$ satisfy

$$q(n;i,j) = \begin{cases} 0 & \text{if } i < 0 \text{ or } j < 0 \text{ or } n < 0, \\ \mathbb{1}_{i=j=0} & \text{if } n = 0, \\ \sum\limits_{(i',j') \in \mathscr{S}} q(n-1; i-i', j-j') & \text{otherwise.} \end{cases}$$

```
> q:=proc(n,i,j)
  option remember;
    if i<0 or j<0 or n<0 then 0
    elif n=0 then if i=0 and j=0 then 1 else 0 fi
    else q(n-1,i,j-1) + q(n-1,i+1,j) + q(n-1,i-1,j+1) fi
  end:

> B:=series(add(add(q(n,k,k), k=0..n)*t^n, n=0..12), t,12);
```

$$B = 1 + 3t^3 + 30t^6 + 420t^9 + O(t^{12})$$

```
> seriestorec(A, u(n))[1];
        2                        2
{(-27 n  - 81 n - 54) u(n) + (n  + 9 n + 18) u(n + 3),

                                    u(0) = 1, u(1) = 0, u(2) = 0}

> rsolve(%, u(n)):

> A:=sum(subs(n=3*n, op(2,%))*t^(3*n), n=0..infinity);
                                                       3
              A := hypergeom([1/3, 2/3], [2], 27 t )
```

▷ Thus, differential guessing predicts

$$A(t) = B(t) = {}_2F_1\left(\left. \begin{matrix} 1/3 \quad 2/3 \\ 2 \end{matrix} \right| 27\,t^3 \right) = \sum_{n=0}^{\infty} \frac{(3n)!}{n!^3} \frac{t^{3n}}{n+1}.$$

Guessing the answer

```
> seriestorec(A, u(n))[1];
       2                        2
{(-27 n  - 81 n - 54) u(n) + (n  + 9 n + 18) u(n + 3),

                             u(0) = 1, u(1) = 0, u(2) = 0}

> rsolve(%, u(n)):

> A:=sum(subs(n=3*n, op(2,%))*t^(3*n), n=0..infinity);
                                                  3
              A := hypergeom([1/3, 2/3], [2], 27 t )
```

▷ This can be algorithmically proved using creative telescoping

Bibliography

- Automatic classification of restricted lattice walks, with M. Kauers. Proceedings FPSAC, 2009.
- The complete generating function for Gessel walks is algebraic, with M. Kauers. Proceedings of the American Mathematical Society, 2010.
- Explicit formula for the generating series of diagonal 3D Rook paths, with F. Chyzak, M. van Hoeij and L. Pech. Séminaire Lotharingien de Combinatoire, 2011.
- Non-D-finite excursions in the quarter plane, with K. Raschel and B. Salvy. Journal of Combinatorial Theory A, 2013.
- On 3-dimensional lattice walks confined to the positive octant, with M. Bousquet-Mélou, M. Kauers and S. Melczer. Annals of Comb., 2016.
- A human proof of Gessel's lattice path conjecture, with I. Kurkova, K. Raschel, Transactions of the American Mathematical Society, 2017.
- Hypergeometric expressions for generating functions of walks with small steps in the quarter plane, with F. Chyzak, M. van Hoeij, M. Kauers and L. Pech, European Journal of Combinatorics, 2017.
- Computer Algebra for Lattice Path Combinatorics, preprint, 2017.

Thanks for your attention!