Computer Algebra for Combinatorics

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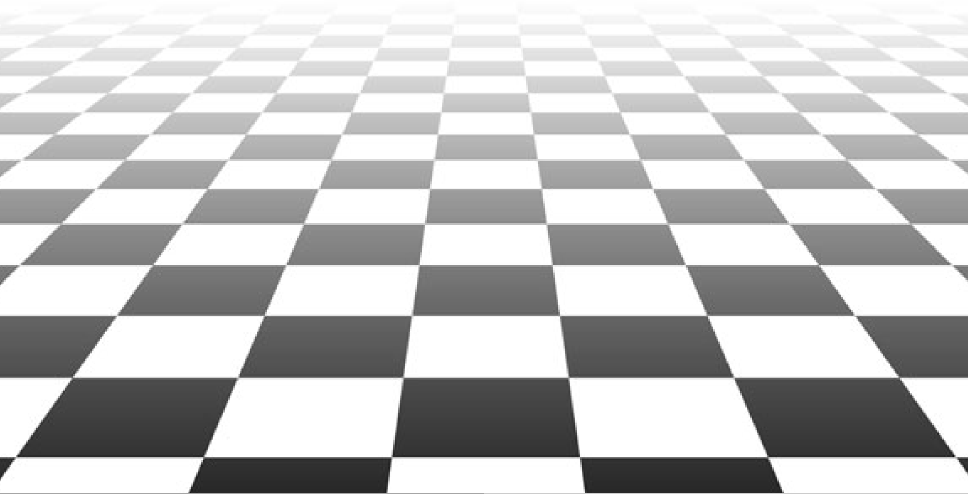
informatics mathematics

MPRI, C-2-22, January 31, 2019

Overview

Part 1:	General presentation
Part 2:	Guess'n'Prove
Part 3:	Creative telescoping

Part 3: Creative telescoping



DIAGONALS

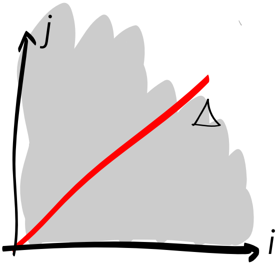
Definition

If F is a formal power series

$$F = \sum_{i_1,\ldots,i_n \geq 0} a_{i_1,\ldots,i_n} x_1^{i_1} \cdots x_n^{i_n},$$

its diagonal is

$$\mathrm{Diag}(F) \stackrel{def}{=} \sum_i a_{i,\ldots,i} t^i.$$



Theorem (Pólya, 1922)

Diagonals of bivariate rational functions are algebraic.

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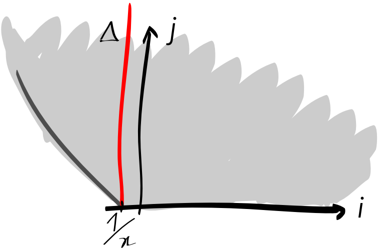


Theorem (Pólya, 1922)

Diagonals of bivariate rational functions are algebraic.

Proof:

$$\mathrm{Diag}\,(F) = [x^{-1}]\frac{1}{x}F\left(x, \frac{t}{x}\right) = \frac{1}{2\pi i}\oint_{|x|=\epsilon} F\left(x, \frac{t}{x}\right)\frac{dx}{x}$$

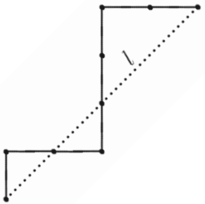
and evaluating the integral by residues concludes (residues are algebraic)

Example: Dyck walks

$$\mathfrak{S} = \{(1,1),(1,-1)\}$$

Let B_n be the number of Dyck bridges (i.e. $\mathfrak{S}$-walks in $\mathbb{Z}^2$ starting at $(0,0)$ and ending on the horizontal axis), of length n

Rotating a Dyck bridge



counterclockwise by $\pi/4$

B_n = number of $\{(1,0),(0,1)\}$-walks in $\mathbb{Z}^2$ from $(0,0)$ to (n,n)

$$\Longrightarrow \qquad B(t) = \sum_{n\geq 0} B_n t^n = \operatorname{Diag}\left(\frac{1}{1-x-y}\right)$$

$$B(t) = \frac{1}{2\pi i}\oint_{|x|=\epsilon} \frac{dx}{x-x^2-t} = \left.\frac{1}{1-2x}\right|_{x=\frac{1-\sqrt{1-4t}}{2}} = \frac{1}{\sqrt{1-4t}}$$

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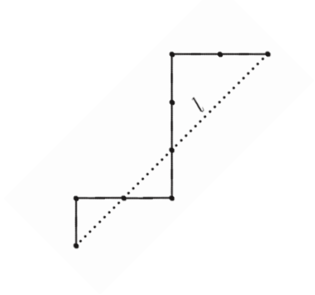


counterclockwise by $\pi/4$

B_n = number of $\{(1,0),(0,1)\}$-walks in $\mathbb{Z}^2$ from $(0,0)$ to (n,n) $= \binom{2n}{n}$

$$\Longrightarrow \qquad B(t) = \sum_{n\geq 0} B_n t^n = \operatorname{Diag}\left(\frac{1}{1-x-y}\right)$$

$$B(t) = \frac{1}{2\pi i}\oint_{|x|=\epsilon} \frac{dx}{x-x^2-t} = \frac{1}{\sqrt{1-4t}} = \sum_{n\geq 0}\binom{2n}{n} t^n$$

Rothstein-Trager resultant

Let $A, B \in \mathbb{K}[x]$ be such that $\deg(A) < \deg(B)$, with B squarefree.
The rational function $F = A/B$ has simple poles only.

If $F = \sum_i \frac{\gamma_i}{x - \beta_i}$, then the residue γ_i of F at the pole β_i equals $\gamma_i = \frac{A(\beta_i)}{B'(\beta_i)}$.

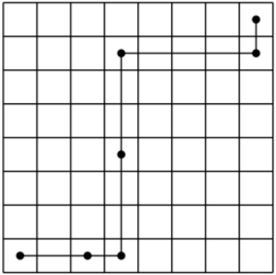
Theorem. The residues γ_i of F are roots of the resultant

$$R(t) = \mathsf{Res}_x\big(B(x),\, A(x) - t \cdot B'(x)\big).$$

Proof. Poisson's formula: $R(t) = \prod_i \Big(A(\beta_i) - t \cdot B'(\beta_i)\Big).$

▶ Introduced in computer algebra by [Rothstein-Trager 1976] for the symbolic (indefinite) integration of rational functions.
▶ Generalized by [Bronstein 1992] to multiple poles.

Question: A chess Rook can move any number of squares horizontally or vertically in one step. How many paths can a Rook take from the lower-left corner square to the upper-right corner square of an $N \times N$ chessboard? Assume that the Rook moves right or up at each step.



$$1,\ 2,\ 14,\ 106,\ 838,\ 6802,\ 56190,\ 470010,\ \dots$$

Generating function of the sequence

$$1,\ 2,\ 14,\ 106,\ 838,\ 6802,\ 56190,\ 470010,\ \ldots$$

is

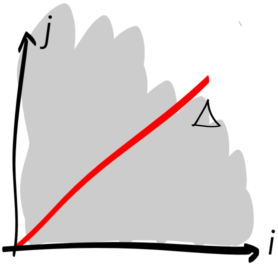
$$\mathsf{Diag}(F) = [x^0]\, F(x, t/x) = \frac{1}{2\pi i} \oint F(x, t/x)\, \frac{dx}{x}, \quad \text{where} \quad F = \frac{1}{1 - \frac{x}{1-x} - \frac{y}{1-y}}.$$

By the residue theorem, $\mathsf{Diag}(F)$ is a sum of roots $y(t)$ of the Rothstein-Trager resultant

```
> F:=1/(1-x/(1-x)-y/(1-y)):
> G:=normal(1/x*subs(y=t/x,F)):
> factor(resultant(denom(G),numer(G)-y*diff(denom(G),x),x));
```

$$t^2(1-t)(2y-1)(36ty^2 - 4y^2 + 1 - t)$$

Answer: Generating series of diagonal Rook paths is $\frac{1}{2}\left(1 + \sqrt{\frac{1-t}{1-9t}}\right)$.



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Theorem (Pólya, 1922)

Diagonals of bivariate rational functions are algebraic.[a]

[a] The converse is also true [Furstenberg, 1967]

Pólya's theorem

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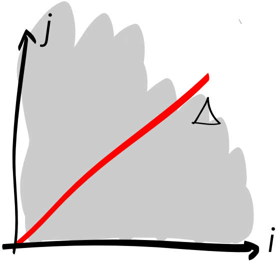

Theorem (Pólya, 1922)

Diagonals of bivariate rational functions are algebraic.

▷ This is false for more than 2 variables. E.g.

$$\mathrm{Diag}\left(\frac{1}{1-x-y-z}\right) = \sum_{n\geq 0} \frac{(3n)!}{n!^3} t^n = {}_2F_1\left(\left.\begin{matrix}\frac{1}{3} & \frac{2}{3} \\ & 1\end{matrix}\right| 27t\right) \quad \text{is transcendental}$$

Pólya's theorem

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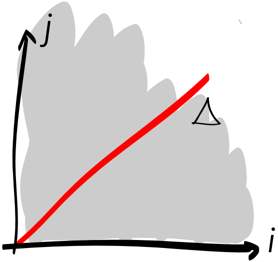

Theorem (Pólya, 1922)

Diagonals of bivariate rational functions are algebraic and thus D-finite.

▷ Algebraic equation has exponential size [B., Dumont, Salvy, 2015]
▷ Differential equation has polynomial size [B., Chen, Chyzak, Li, 2010]

Lipshitz's theorem

Definition

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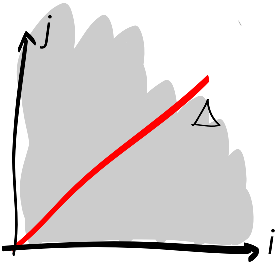

Theorem (Lipshitz, 1988)

Diagonals of rational functions are D-finite.

Example: Diagonal Rook paths on a 3D chessboard

Question [Erickson 2010]
How many ways can a Rook move from $(0,0,0)$ to (N,N,N), where each step is a positive integer multiple of $(1,0,0)$, $(0,1,0)$, or $(0,0,1)$?

1, 6, 222, 9918, 486924, 25267236, 1359631776, 75059524392, ...

Answer [B.-Chyzak-Hoeij-Pech 2011]: GF of 3D diagonal Rook paths is

$$G(t) = 1 + 6 \cdot \int_0^t \frac{{}_2F_1\left(\begin{matrix} 1/3 & 2/3 \\ & 2 \end{matrix} \,\middle|\, \frac{27x(2-3x)}{(1-4x)^3}\right)}{(1-4x)(1-64x)}\, dx$$

Proof of Lipshitz's theorem on the 3D Rooks example

Problem: Show that $\operatorname{Diag}(F)$ is D-finite, where $F(x,y,z)$ is

$$\left(1-\sum_{n\geq 1} x^n - \sum_{n\geq 1} y^n - \sum_{n\geq 1} z^n\right)^{-1} = \frac{(1-x)(1-y)(1-z)}{1-2(x+y+z)+3(xy+yz+zx)-4xyz}$$

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Idea: If one is able to find a nonzero differential operator of the form

$$L(t,\partial_t,\partial_x,\partial_y) = P(t,\partial_t) + (\text{ higher-order terms in } \partial_x \text{ and } \partial_y\ )$$

that annihilates $G = \dfrac{1}{xy} \cdot F\left(x, \dfrac{y}{x}, \dfrac{t}{y}\right)$, then $P(t,\partial_t)$ annihilates $\mathrm{Diag}(F)$.

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Proof:

1. $\mathrm{Diag}(F) = [x^0 y^0]\, F\left(x, \frac{y}{x}, \frac{t}{y}\right)$
2. $0 = L(G) = P(G) + \partial_x(\cdot) + \partial_y(\cdot)$
3. $0 = [x^{-1}y^{-1}]L(G) = [x^{-1}y^{-1}]P(G) = P([x^{-1}y^{-1}]G) = P(\mathrm{Diag}(F))$

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▷ Remaining task: Show that such an L does exist.

Proof of Lipshitz's theorem on the 3D Rooks example

Counting argument: By Leibniz's rule, the $\binom{N+4}{4}$ rational functions

$$t^i \partial_t^j \partial_x^k \partial_y^\ell (G), \quad 0 \le i+j+k+\ell \le N$$

are contained in the $\mathbb{Q}$-vector space of dimension $\le 18(N+1)^3$ spanned by

$$\frac{t^i x^j y^k}{\operatorname{denom}(G)^{N+1}}, \quad 0 \le i \le 2N+1,\ 0 \le j \le 3N+2,\ 0 \le k \le 3N+2.$$

▷ If $\binom{N+4}{4} > 18(N+1)^3$, then there exists $L(t, \partial_t, \partial_x, \partial_y)$ (resp. $P(t, \partial_t)$) of total degree at most N, such that $LG = 0$ (resp. $P(\operatorname{Diag}(F)) = 0$).

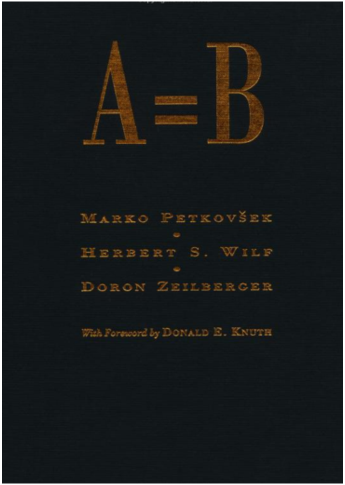
▷ $N = 425$ is the smallest integer satisfying $\binom{N+4}{4} > 18(N+1)^3$

▷ Finding the operator P by Lipshitz' argument would require solving a linear system with 1,391,641,251 unknowns and 1,391,557,968 equations!

▷ A better solution is provided by creative telescoping.

CREATIVE TELESCOPING

General framework in computer algebra –initiated by Zeilberger in the ’90s– for proving identities on multiple integrals and sums with parameters.



Examples I: hypergeometric summation

- $\displaystyle\sum_{k\in\mathbb{Z}} (-1)^k \binom{a+b}{a+k}\binom{a+c}{c+k}\binom{b+c}{b+k} = \frac{(a+b+c)!}{a!b!c!}$ [Dixon 1891]

- $\displaystyle A_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2$ satisfies the recurrence [Apéry 1978]:

$$(n+1)^3 A_{n+1} = (34n^3+51n^2+27n+5)A_n - n^3 A_{n-1}.$$

(*Neither Cohen nor I had been able to prove this in the intervening two months* [Van der Poorten 1979])

- $\displaystyle\sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 = \sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k}\sum_{j=0}^{k}\binom{k}{j}^3$ [Strehl 1992]

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$$(n+1)^3 A_{n+1} = (34n^3 + 51n^2 + 27n + 5) A_n - n^3 A_{n-1}.$$

(*The specific problem was mentioned to Don Zagier, who solved it with irritating speed* [Van der Poorten 1979])

- $\displaystyle\sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 = \sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k} \sum_{j=0}^{k} \binom{k}{j}^3$ [Strehl 1992]

Examples II: Integrals and Diagonals

- $\displaystyle\int_0^1 \frac{\cos(zu)}{\sqrt{1-u^2}}\,du = \int_1^{+\infty} \frac{\sin(zu)}{\sqrt{u^2-1}}\,du = \frac{\pi}{2} J_0(z);$

- $\displaystyle\int_0^{+\infty} x J_1(ax) I_1(ax) Y_0(x) K_0(x)\,dx =$ $\displaystyle -\frac{\ln(1-a^4)}{2\pi a^2}$ [Glasser-Montaldi 1994];

- $\displaystyle\frac{1}{2\pi i}\oint \frac{(1+2xy+4y^2)\exp\left(\frac{4x^2y^2}{1+4y^2}\right)}{y^{n+1}(1+4y^2)^{\frac{3}{2}}}\,dy = \frac{H_n(x)}{\lfloor n/2 \rfloor !}$ [Doetsch 1930];

- $\displaystyle\mathrm{Diag}\,\frac{1}{(1-x-y)(1-z-u)-xyzu} = \sum_{n\geq 0} A_n t^n$ [Straub 2014].

Summation by Creative Telescoping

$$I_n := \sum_{k=0}^{n} \binom{n}{k} = 2^n.$$

IF one knows Pascal's triangle:

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1} = 2\binom{n}{k} + \binom{n}{k-1} - \binom{n}{k},$$

then summing over k gives

$$I_{n+1} = 2I_n.$$

The initial condition $I_0 = 1$ concludes the proof.

$$F_n = \sum_k u_{n,k} =?$$

IF one knows $P(n, S_n)$ and $R(n, k, S_n, S_k)$ such that

$$(P(n, S_n) + \Delta_k R(n, k, S_n, S_k)) \cdot u_{n,k} = 0$$

(where Δ_k is the difference operator, $\Delta_k \cdot v_{n,k} = v_{n,k+1} - v_{n,k}$), then the sum "telescopes", leading to

$$P(n, S_n) \cdot F_n = 0.$$

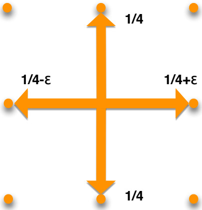
Zeilberger's Algorithm [1990]

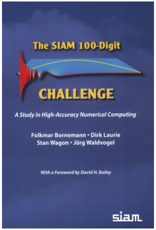
Input: a hypergeometric term $u_{n,k}$, i.e., $u_{n+1,k}/u_{n,k}$ and $u_{n,k+1}/u_{n,k}$ rational functions in n and k;

Output:

- a linear recurrence (P) satisfied by $F_n = \sum_k u_{n,k}$
- a certificate (Q), such that checking the result is easy from $P(n, S_n) \cdot u_{n,k} = \Delta_k Q \cdot u_{n,k}$.

Example: SIAM flea





$$U_{n,k} := \binom{2n}{2k}\binom{2k}{k}\binom{2n-2k}{n-k}\left(\frac{1}{4}+c\right)^k\left(\frac{1}{4}-c\right)^k\frac{1}{4^{2n-2k}},$$

$$p_n = \sum_{k=0}^{n} U_{n,k} \quad = \text{probability of return to } (0,0) \text{ at step } 2n.$$

```
> p:=SumTools[Hypergeometric][Zeilberger](U,n,k,Sn);
```

$$\left[\left(4\,n^2+16\,n+16\right)Sn^2+\left(-4\,n^2+32\,c^2n^2+96\,c^2n-12\,n+72\,c^2-9\right)Sn + 128\,c^4n+64\,c^4n^2+48\,c^4,\ \text{...(BIG certificate)...}\right]$$

$$I(t) = \oint_{\gamma} H(t,x)\, dx = ?$$

IF one knows $P(t, \partial_t)$ and $R(t, x, \partial_t, \partial_x)$ such that

$$(P(t,\partial_t) + \partial_x R(t,x,\partial_t,\partial_x)) \cdot H(t,x) = 0,$$

then the integral "telescopes", leading to

$$P(t,\partial_t) \cdot I(t) = 0.$$

Example: diagonal Rook paths again

Generating function of the sequence

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By creative telescoping, $\mathsf{Diag}(F)$ satisfies the differential equation

```
> F:=1/(1-x/(1-x)-y/(1-y)):
> G:=normal(1/x*subs(y=t/x,F)):
> Zeilberger(G, t, x, Dt)[1];
```

$$(9t^2-10t+1)\partial_t^2 + (18t-14)\partial_t$$

Answer: Generating series of diagonal Rook paths is $\frac{1}{2}\left(1+\sqrt{\frac{1-t}{1-9t}}\right)$.

Problem:

$$\left.\begin{array}{rl} \mathbf{x} = x_1, \ldots, x_n & \text{— integration variables} \\ t & \text{— parameter} \\ H(t, \mathbf{x}) & \text{— rational function} \\ \gamma & \text{— } n\text{-cycle in } \mathbb{C}^n \end{array}\right\} \oint_\gamma H(t, \mathbf{x}) \mathrm{d}\mathbf{x}$$

Principle of creative telescoping

$$\underbrace{\sum_{k=0}^{r} c_k(t) \frac{\partial^k H}{\partial t^k} = \overbrace{\sum_{i=1}^{n} \frac{\partial A_i}{\partial x_i}}^{\text{certificate}}}_{\text{telescopic relation}} \quad \Longrightarrow \quad \overbrace{\left(\sum_{k=0}^{r} c_k(t) \partial_t^k \right)}^{\text{telescoper}} \cdot \oint_\gamma H \mathrm{d}\mathbf{x} = 0$$

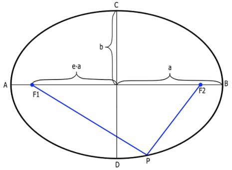
Task:

1. find the $c_k(t)$ which satisfy a telescopic relation,
2. ideally, without computing the certificate (A_i).

Example: Perimeter of an ellipse

Perimeter of an ellipse of eccentricity e, semi-major axis 1 [Euler, 1733]

$$p(e) = 4\int_0^1 \sqrt{\frac{1-e^2x^2}{1-x^2}}\,\mathrm{d}x = 2\pi - \frac{\pi}{2}e^2 - \frac{3\pi}{32}e^4 - \cdots$$



Creative telescoping finds the telescopic relation

$$\left((e-e^3)\partial_e^2 + (1-e^2)\partial_e + e\right)\cdot\left(\frac{1}{1-\frac{1-e^2x^2}{(1-x^2)y^2}}\right) =$$

$$\partial_x\left(-\frac{e\left(-1-x+x^2+x^3\right)y^2\left(-3+2x+y^2+x^2\left(-2+3e^2-y^2\right)\right)}{\left(-1+y^2+x^2(e^2-y^2)\right)^2}\right)$$

$$+\,\partial_y\left(\frac{2e\left(-1+e^2\right)x\left(1+x^3\right)y^3}{\left(-1+y^2+x^2(e^2-y^2)\right)^2}\right)$$

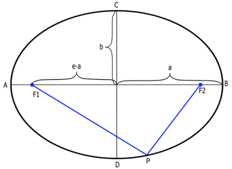
Thus $(e-e^3)p'' + (1-e^2)p' + ep = 0$.

▷ Note the size of the certificate (compared to that of the telescoper).

Example: Perimeter of an ellipse

Perimeter of an ellipse of eccentricity e, semi-major axis 1 [Euler, 1733]

$$p(e) = 4\int_0^1 \sqrt{\frac{1-e^2x^2}{1-x^2}}\,\mathrm{d}x = 4\oint \frac{\mathrm{d}x\mathrm{d}y}{1-\frac{1-e^2x^2}{(1-x^2)y^2}},$$



Creative telescoping finds the telescopic relation

$$\left((e-e^3)\partial_e^2 + (1-e^2)\partial_e + e\right)\cdot\left(\frac{1}{1-\frac{1-e^2x^2}{(1-x^2)y^2}}\right) =$$

$$\partial_x\left(-\frac{e\left(-1-x+x^2+x^3\right)y^2\left(-3+2x+y^2+x^2\left(-2+3e^2-y^2\right)\right)}{\left(-1+y^2+x^2(e^2-y^2)\right)^2}\right)$$

$$+\,\partial_y\left(\frac{2e\left(-1+e^2\right)x\left(1+x^3\right)y^3}{\left(-1+y^2+x^2(e^2-y^2)\right)^2}\right)$$

Thus $(e-e^3)p'' + (1-e^2)p' + ep = 0.$

▷ Note the size of the certificate (compared to that of the telescoper).

Example: 3D rook paths [B.-Chyzak-Hoeij-Pech 2011]

Task: Given G in $\mathbb{Q}(t,x,y)$, construct a linear differential operator $P(t,\partial_t)$, and two rational functions R and S in $\mathbb{Q}(t,x,y)$ such that

$$P(G) = \frac{\partial R}{\partial x} + \frac{\partial S}{\partial y}.$$

Solution: Creative telescoping!

```
> G:=subs(y=y/x,z=t/y,1/(1-x/(1-x)-y/(1-y)-z/(1-z)))/y/x:
> P,R,S:=op(op(Mgfun:-creative_telescoping(G,t::diff,[x::diff,y::diff]))):
> P;
```

$$\begin{aligned} P = \quad & t(t-1)(64t-1)(3t-2)(6t+1)\partial_t^3 \\ & +(4608t^4-6372t^3+813t^2+514t-4)\partial_t^2 \\ & +4(576t^3-801t^2-108t+74)\partial_t \end{aligned}$$

▷ The whole computation takes < 10 seconds on a personal laptop.

▷ Proves a recurrence conjectured by [Erickson 2010]

General-purpose creative telescoping algorithms:

- using linear algebra [Lipshitz, 1988];
- using non-commutative Gröbner bases:
  - and elimination [Takayama, 1990; Zeilberger, 1990];
  - and rational resolution of differential equations [Chyzak, 2000];
  - and very efficient heuristics [Koutschan, 2009, 2010].

▷ Drawbacks: Bad or unknown complexity; all compute certificates.

Rational case:

- univariate integrals [B., Chen, Chyzak, Li, 2010];
- double integrals [Chen, Kauers, Singer, 2012];
- multiple integrals [B., Lairez, Salvy, 2013], [Lairez 2015].

Hermite's method for indefinite integration

Pb. Given coprime $f, g \in \mathbb{K}[x]$, "compute" $\int \frac{f}{g}$ in the following sense:
write $\frac{f}{g} = \partial_x \left(\frac{c}{d}\right) + \frac{a}{b}$ with $a, b, c, d \in \mathbb{K}[x]$, $\deg a < \deg b$ and b squarefree.

Def. c/d is the *rational part* and $\int a/b$ is the *logarithmic part* of $\int f/g$.

Idea [Ostrogradsky 1833–1845, Hermite 1872]
1. Compute the (rational) partial fraction decomposition of f/g, that is

$$\frac{f}{g} = P + \sum_{i=1}^{m} \sum_{j=1}^{i} \frac{f_{i,j}}{g_i^j}$$

where $P, f_{i,j}, g_i \in \mathbb{K}[x]$ with $\deg f_{i,j} < \deg g_i$ and $g = g_1 g_2^2 \cdots g_m^m$ a squarefree factorization: g_i's are squarefree and mutually coprime, $g_m \neq 1$.
2. Then put into c/d all that comes from P and $j > 1$, and into a/b all that comes from $j = 1$. To do this, use integration by parts and extended gcds.

Hermite's method for indefinite integration

Task: Given coprime $f, g \in \mathbb{K}[x]$, compute $a, b, c, d \in \mathbb{K}[x]$, with

$\deg a < \deg b$ and b squarefree, such that $\frac{f}{g} = \partial_x \left(\frac{c}{d}\right) + \frac{a}{b}$.

Algorithm:

1. Compute the partial fraction decomposition $\frac{f}{g} = P + \sum_{i=1}^{m} \sum_{j=1}^{i} \frac{f_{i,j}}{g_i^j}$

2. Perform Hermite reduction:
   Let $i \in \{1, 2, \ldots, m\}$. As $\gcd(g_i, g_i') = 1$, there exist $u, v \in \mathbb{K}[x]$ such that

$$u g_i + v g_i' = f_{i,j}.$$

Exercise: Show that one can assume $\deg u < \deg g_i$ and $\deg v < \deg g_i$.

Then for all $j > 1$, integration by parts gives

$$\int \frac{f_{i,j}}{g_i^j} = \int \frac{u}{g_i^{j-1}} + \int \frac{v g_i'}{g_i^j} = \frac{-v}{(j-1) g_i^{j-1}} + \int \frac{u + \frac{v'}{j-1}}{g_i^{j-1}}.$$

Repeating the process at most j times results in

$$\int \frac{f_{i,j}}{g_i^j} = \frac{k_{i,j}}{g_i^{j-1}} + \int \frac{\ell_{i,j}}{g_i}, \quad \text{with } \deg k_{i,j} < \deg g_i \text{ and } \deg \ell_{i,j} < \deg g_i.$$

Problem: Given $H = P/Q \in \mathbb{K}(t,x)$ compute $\oint_\gamma H(t,x)\mathrm{d}x$

Hermite reduction: H can be written in reduced form

$$H = \partial_x(g) + \frac{a}{Q^\star},$$

where $Q^\star$ is the squarefree part of Q and $\deg_x(a) < d^\star := \deg_x(Q^\star)$.

Algorithm [B., Chen, Chyzak, Li, 2010]

(1) For $i = 0, 1, \ldots, d^\star$ compute Hermite reduction of $\partial_t^i(H)$:

$$\partial_t^i(H) = \partial_x(g_i) + \frac{a_i}{Q^\star}, \quad \deg_x(a_i) < d^\star$$

(2) Find the first linear relation over $\mathbb{K}(t)$ of the form $\sum_{k=0}^r c_k a_k = 0$.

▷ $L = \sum_{k=0}^r c_k \partial_t^k$ is a telescoper (and $\sum_{k=0}^r c_k g_k$ the corresponding certificate).

4G Creative Telescoping: multivariate case

$$H = \tfrac{P}{Q} \text{ — a rational function in } t \text{ and } \mathbf{x} = x_1, \dots, x_n$$
$$d_{\mathbf{x}} \text{ — the degree of } Q \text{ w.r.t. } \mathbf{x}$$
$$d_t \text{ — } \max(\deg_t P, \deg_t Q)$$

Theorem ([B., Lairez, Salvy, 2013])

A telescoper for H can be computed using $\tilde{\mathcal{O}}(e^{3n} d_{\mathbf{x}}^{8n} d_t)$ operations.
The minimal telescoper has order $\leq d_{\mathbf{x}}^n$ and degree $\mathcal{O}(e^n d_{\mathbf{x}}^{3n} d_t)$.
These size bounds are generically reached.

- ▷ First polynomial time algorithm for rational creative telescoping.
- ▷ It avoids the costly computation of certificates.
- ▷ Generically, certificates have size $\Omega(d_{\mathbf{x}}^{n^2/2})$.
- ▷ General-purpose algorithms have doubly-exponential complexity.
- ▷ Applies to diagonals: $\mathrm{Diag}(F)(t) = \frac{1}{(2\pi i)^n} \oint F\left(\frac{t}{x_1 \dots x_n}, x_1, \dots, x_n\right)$.

Main ingredients of the integration algorithm

Griffiths–Dwork method for the generic case
Linear reduction classical in algebraic geometry;
Generalization of Hermite's reduction.

Fast linear algebra on polynomial matrices
Macaulay matrices encoding Gröbner bases computations;
Sophisticated algorithms due to Villard, Storjohann, Zhou, etc.

Deformation technique for the general case
Input perturbation using a new free variable.

▷ Recent, highly non-trivial, extension by [Lairez, 2015] tremendously improves the efficiency of the algorithm in [B., Lairez, Salvy, 2013]

BACK TO THE EXERCISE

–Solution–

The exercise

Let $\mathfrak{S} = \{\uparrow, \leftarrow, \searrow\}$. A $\mathfrak{S}$-walk is a path in $\mathbb{Z}^2$ using only steps from $\mathfrak{S}$. Show that, for any integer n, the following quantities are equal:

(i) the number a_n of $\mathfrak{S}$-walks of length n confined to the upper half plane $\mathbb{Z} \times \mathbb{N}$ that start and end at the origin $(0,0)$;

(ii) the number b_n of $\mathfrak{S}$-walks of length n confined to the quarter plane $\mathbb{N}^2$ that start at the origin $(0,0)$ and finish on the diagonal $x = y$.

The exercise

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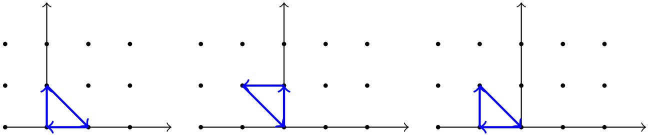
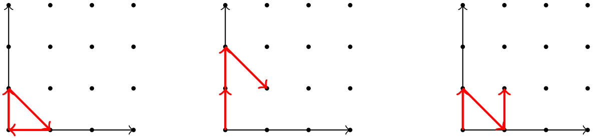
For instance, for $n = 3$, this common value is $a_3 = b_3 = 3$:

(i)



(ii)



A recurrence relation for $\{\uparrow, \leftarrow, \searrow\}$-walks in $\mathbb{Z} \times \mathbb{N}$

$h(n;i,j)$ = nb. of $\{\uparrow, \leftarrow, \searrow\}$-walks in $\mathbb{Z} \times \mathbb{N}$ of length n from $(0,0)$ to (i,j)

The numbers $h(n;i,j)$ satisfy

$$h(n;i,j) = \begin{cases} 0 & \text{if } j < 0 \text{ or } n < 0, \\ \mathbb{1}_{i=j=0} & \text{if } n = 0, \\ \sum\limits_{(i',j') \in \mathfrak{S}} h(n-1; i-i', j-j') & \text{otherwise.} \end{cases}$$

```
> h:=proc(n,i,j)
  option remember;
    if j<0 or n<0 then 0
    elif n=0 then if i=0 and j=0 then 1 else 0 fi
    else h(n-1,i,j-1)+h(n-1,i+1,j)+h(n-1,i-1,j+1) fi
  end:

> A:=series(add(h(n,0,0)*t^n,n=0..12),t,12);
```

$$A = 1 + 3t^3 + 30t^6 + 420t^9 + O(t^{12})$$

$q(n;i,j)$ = nb. of $\{\uparrow, \leftarrow, \searrow\}$-walks in $\mathbb{N}^2$ of length n from $(0,0)$ to (i,j)

The numbers $q(n;i,j)$ satisfy

$$q(n;i,j) = \begin{cases} 0 & \text{if } i<0 \text{ or } j<0 \text{ or } n<0, \\ \mathbb{1}_{i=j=0} & \text{if } n=0, \\ \sum_{(i',j')\in\mathfrak{S}} q(n-1;i-i',j-j') & \text{otherwise.} \end{cases}$$

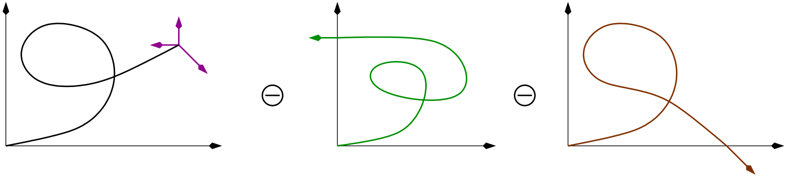
```
> q:=proc(n,i,j)
  option remember;
    if i<0 or j<0 or n<0 then 0
    elif n=0 then if i=0 and j=0 then 1 else 0 fi
    else q(n-1,i,j-1)+q(n-1,i+1,j)+q(n-1,i-1,j+1) fi
  end:

> B:=series(add(add(q(n,k,k),k=0..n)*t^n,n=0..12),t,12);
```

$$B = 1 + 3t^3 + 30t^6 + 420t^9 + O(t^{12})$$

Generating function: $Q(t;x,y) = \sum_{n=0}^{\infty} \sum_{i=0}^{n} \sum_{j=0}^{n} q(n;i,j) t^n x^i y^j \in \mathbb{Q}[x,y][[t]]$

Kernel equation ($\bar{x} = 1/x$, $\bar{y} = 1/y$):

$$Q(t;x,y) \equiv Q(x,y) = 1 + t(y + \bar{x} + x\bar{y})Q(x,y) - t\bar{x}Q(0,y) - tx\bar{y}Q(x,0)$$



A functional equation for $\{\uparrow, \leftarrow, \searrow\}$-walks in $\mathbb{N}^2$

Generating function: $Q(t;x,y) = \sum_{n=0}^{\infty} \sum_{i=0}^{n} \sum_{j=0}^{n} q(n;i,j) t^n x^i y^j \in \mathbb{Q}[x,y][[t]]$

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or

$$\big(1 - t(y + \bar{x} + x\bar{y})\big)Q(x,y) = 1 - t\bar{x}Q(0,y) - tx\bar{y}Q(x,0),$$

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or

$$\big(1 - t(y + \bar{x} + x\bar{y})\big)Q(x,y) = 1 - t\bar{x}Q(0,y) - tx\bar{y}Q(x,0),$$

or

$$\boxed{\big(1 - t(y + \bar{x} + x\bar{y})\big)xyQ(x,y) = xy - tyQ(0,y) - tx^2Q(x,0)}$$

Task (Q): Find $B(t) = [x^0]\, Q(x, \bar{x})$

Generating function: $H(t;x,y) = \sum_{n=0}^{\infty} \sum_{i=-n}^{n} \sum_{j=0}^{\infty} h(n;i,j) t^n x^i y^j \in \mathbb{Q}[x,\bar{x},y][[t]]$

Kernel equation ($\bar{x} = 1/x$, $\bar{y} = 1/y$):

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or

$$\big(1 - t(y + \bar{x} + x\bar{y})\big)H(x,y) = 1 - tx\bar{y}H(x,0),$$

or

$$\boxed{\big(1 - t(y + \bar{x} + x\bar{y})\big)yH(x,y) = y - txH(x,0)}$$

Task (H): Find $A(t) = [x^0]\, H(x,0)$

The kernel method for $\mathbb{Z} \times \mathbb{N}$

- The kernel equation reads (with $K(x,y) = 1 - t(y + \bar{x} + x\bar{y})$):

$$K(x,y)yH(x,y) = y - txH(x,0)$$

The kernel method for $\mathbb{Z} \times \mathbb{N}$

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- Let

$$y_0 = \frac{x - t - \sqrt{(t-x)^2 - 4t^2x^3}}{2tx} \qquad = xt + t^2 + (x^2 + \bar{x})t^3 + (3x + \bar{x}^2)t^4 + \cdots$$

be the (unique) root in $\mathbb{Q}[x, \bar{x}][[t]]$ of $K(x, y_0) = 0$.

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- Then

$$0 = K(x,y_0)yH(x,y_0) = y_0 - txH(x,0),$$

thus

$$H(x,0) = \frac{y_0}{tx} \quad \text{and} \quad A(t) = \left[x^0\right] \frac{y_0}{tx}.$$

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thus

$$H(x,0) = \frac{y_0}{tx} \quad \text{and} \quad A(t) = \left[x^0\right] \frac{y_0}{tx}.$$

- Creative telescoping then proves:

$$(27t^4 - t)A''(t) + (108t^3 - 4)A'(t) + 54t^2 A(t) = 0.$$

```
> Zeilberger(1/x * sqrt((t-x)^2 - 4*t^2*x^3)/(2*t^2*x^2), t, x, Dt);
```

The group of the model $\{\uparrow, \leftarrow, \searrow\}$

Step set $\mathfrak{S} = \{(-1,0),(0,1),(1,-1)\}$, with characteristic polynomial

$$\chi(x,y) = \frac{1}{x} + y + x \cdot \frac{1}{y} = \bar{x} + y + x\bar{y}$$

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$\chi(x,y)$ is left unchanged by the rational transformations

$$\Phi : (x,y) \mapsto (\bar{x}y, y) \quad \text{and} \quad \Psi : (x,y) \mapsto (x, x\bar{y}) \, .$$

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Φ and Ψ are involutions, and generate a finite dihedral group D_3 of order 6:

$$\begin{tikzcd}
& (\bar{x}y, y) \arrow[r, dash, "\Psi"] & (\bar{x}y, \bar{x}) \arrow[dr, dash, "\Phi"] & \\
(x,y) \arrow[ur, dash, "\Phi"] \arrow[dr, dash, "\Psi"'] & & & (\bar{y}, \bar{x}) \\
& (x, x\bar{y}) \arrow[r, dash, "\Phi"'] & (\bar{y}, x\bar{y}) \arrow[ur, dash, "\Psi"'] &
\end{tikzcd}$$

Solving the kernel equation for $\mathbb{N}^2$

- Orbit equation:

$$xyQ(x,y) - \bar{x}y^2Q(\bar{x}y,y) + \bar{x}^2yQ(\bar{x}y,\bar{x}) - \bar{x}\bar{y}Q(\bar{y},\bar{x}) + x\bar{y}^2Q(\bar{y},x\bar{y}) - x^2\bar{y}Q(x,x\bar{y}) = \frac{xy - \bar{x}y^2 + \bar{x}^2y - \bar{x}\bar{y} + x\bar{y}^2 - x^2\bar{y}}{1 - t(y + \bar{x} + x\bar{y})}$$

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- Corollary [Bousquet-Mélou & Mishna, 2010]:

$$xyQ(x,y) = [x^{>0}y^{>0}]\, \frac{xy - \bar{x}y^2 + \bar{x}^2y - \bar{x}\bar{y} + x\bar{y}^2 - x^2\bar{y}}{1 - t(y + \bar{x} + x\bar{y})}$$

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- Corollary [B.-Chyzak-van Hoeij-Kauers-Pech, 2015]:

$$B(t) = [z^0]Q(z,\bar{z}) = [u^{-1}v^{-1}z^{-1}]\,\frac{\bar{u}\bar{v} - u\bar{v}^2 + u^2\bar{v} - uv + \bar{u}v^2 - \bar{u}^2v}{z(1-zu)(1-v\bar{z})(1 - t(\bar{v} + u + \bar{u}v))}$$

Solving the kernel equation for $\mathbb{N}^2$

- Orbit equation:

$$xyQ(x,y) - \bar{x}y^2Q(\bar{x}y,y) + \bar{x}^2yQ(\bar{x}y,\bar{x}) - \bar{x}\bar{y}Q(\bar{y},\bar{x}) + x\bar{y}^2Q(\bar{y},x\bar{y}) - x^2\bar{y}Q(x,x\bar{y}) = \frac{xy - \bar{x}y^2 + \bar{x}^2y - \bar{x}\bar{y} + x\bar{y}^2 - x^2\bar{y}}{1 - t(y + \bar{x} + x\bar{y})}$$

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$$xyQ(x,y) = [x^{>0}y^{>0}]\, \frac{xy - \bar{x}y^2 + \bar{x}^2y - \bar{x}\bar{y} + x\bar{y}^2 - x^2\bar{y}}{1 - t(y + \bar{x} + x\bar{y})}$$

- Corollary [B.-Chyzak-van Hoeij-Kauers-Pech, 2015]:

$$B(t) = [z^0]Q(z,\bar{z}) = [u^{-1}v^{-1}z^{-1}]\, \frac{\bar{u}\bar{v} - u\bar{v}^2 + u^2\bar{v} - uv + \bar{u}v^2 - \bar{u}^2v}{z(1-zu)(1-v\bar{z})(1-t(\bar{v}+u+\bar{u}v))}$$

- Multivariate Creative Telescoping gives a differential equation for $B(t)$:

$$(27t^4 - t)B''(t) + (108t^3 - 4)B'(t) + 54t^2B(t) = 0.$$

Conclusion

We have proved that $A(t)$ and $B(t)$ are both solutions of

$$(27t^4 - t)y''(t) + (108t^3 - 4)y'(t) + 54t^2 y(t) = 0.$$

Solving this equation proves:

$$A(t) = B(t) = {}_2F_1\left(\left.\begin{matrix}1/3 & 2/3 \\ & 2\end{matrix}\,\right|\, 27\,t^3\right) = \sum_{n=0}^{\infty} \frac{(3n)!}{n!^3} \frac{t^{3n}}{n+1}.$$

Thus the two sequences are equal to

$$a_{3n} = b_{3n} = \frac{(3n)!}{n!^2 \cdot (n+1)!}, \qquad \text{and} \quad a_m = b_m = 0 \quad \text{if 3 does not divide } m.$$

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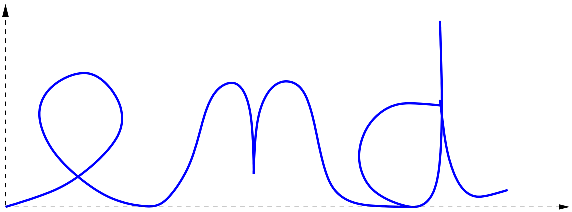

Thanks for your attention!