

# Quasi-optimal multiplication of linear differential operators

Alexandre Benoit  
UPMC  
France

Email: [Alexandre.Benoit@lip6.fr](mailto:Alexandre.Benoit@lip6.fr)

Alin Bostan<sup>†</sup>  
INRIA  
France

Email: [Alin.Bostan@inria.fr](mailto:Alin.Bostan@inria.fr)

Joris van der Hoeven<sup>‡</sup>  
CNRS & École polytechnique  
France

Email: [vdhoeven@lix.polytechnique.fr](mailto:vdhoeven@lix.polytechnique.fr)

**Abstract**—We show that linear differential operators with polynomial coefficients over a field of characteristic zero can be multiplied in quasi-optimal time. This answers an open question raised by van der Hoeven.

**Keywords**—Linear differential operators; multiplication; algebraic algorithms; computational complexity.

## I. INTRODUCTION

The product of *polynomials* and the product of *matrices* are two of the most basic operations in mathematics; the study of their computational complexity is central in computer science. In this paper, we will be interested in the computational complexity of multiplying two *linear differential operators*. These algebraic objects encode linear differential equations, and form a non-commutative ring that shares many properties with the commutative ring of usual polynomials [21], [22]. The structural analogy between polynomials and linear differential equations was discovered long ago by Libri and Brassinne [18], [7], [13]. Yet, the algorithmic study of linear differential operators is currently much less advanced than in the polynomial case: the complexity of multiplication has been addressed only recently [16], [6], but not completely solved. The aim of the present work is to make a step towards filling this gap, and to solve an open question raised in [16].

Let  $\mathbb{K}$  be an effective field. That is, we assume data structures for representing the elements of  $\mathbb{K}$  and algorithms for performing the field operations. The aim of *algebraic complexity theory* is to study the cost of basic or more complex algebraic operations over  $\mathbb{K}$  (such as the cost of computing the greatest common divisor of two polynomials of degrees less than  $d$  in  $\mathbb{K}[x]$ , or the cost of Gaussian elimination on an  $r \times r$  matrix in  $\mathbb{K}^{r \times r}$ ) in terms of the number of operations in  $\mathbb{K}$ . The *algebraic complexity* usually does not coincide with the *bit complexity*, which also takes into account the potential growth of the actual coefficients in  $\mathbb{K}$ . Nevertheless, understanding the algebraic complexity usually

constitutes a first useful step towards understanding the bit complexity. Of course, in the special, very important case when the field  $\mathbb{K}$  is finite, both complexities coincide up to a constant factor.

The complexities of operations in the rings  $\mathbb{K}[x]$  and  $\mathbb{K}^{r \times r}$  have been intensively studied during the last decades. It is well established that polynomial multiplication is a *commutative complexity yardstick*, while matrix multiplication is a *non-commutative complexity yardstick*, in the sense that the complexity of operations in  $\mathbb{K}[x]$  (resp. in  $\mathbb{K}^{r \times r}$ ) can generally be expressed in terms of the cost of multiplication in  $\mathbb{K}[x]$  (resp. in  $\mathbb{K}^{r \times r}$ ), and for most of them, in a quasi-linear way [2], [4], [8], [24], [14].

Therefore, understanding the algebraic complexity of multiplication in  $\mathbb{K}[x]$  and  $\mathbb{K}^{r \times r}$  is a fundamental question. It is well known that polynomials of degrees  $< d$  can be multiplied in time  $M(d) = \mathcal{O}(d \log d \log \log d)$  using algorithms based on the Fast Fourier Transform (FFT) [11], [25], [9], and two  $r \times r$  matrices in  $\mathbb{K}^{r \times r}$  can be multiplied in time  $\mathcal{O}(r^\omega)$ , with  $2 \leq \omega \leq 3$  [27], [23], [12]. The current tightest upper bound, due to Vassilevska Williams [28], is  $\omega < 2.3727$ , following work of Coppersmith and Winograd [12] and Stothers [26]. Finding the best upper bound on  $\omega$  is one of the most important open problems in algebraic complexity theory.

In a similar vein, our thesis is that understanding the algebraic complexity of multiplication of linear differential operators is a very important question, since the complexity of more involved, higher-level operations on linear differential operators can be reduced to that of multiplication [17].

From now on, we will assume that *the base field  $\mathbb{K}$  has characteristic zero*. Let  $\mathbb{K}[x, \partial]$  denote the associative algebra  $\mathbb{K}\langle x, \partial; \partial x = x\partial + 1 \rangle$  of linear differential operators in  $\partial = \frac{d}{dx}$  with polynomial coefficients in  $x$ . Any element  $L$  of  $\mathbb{K}[x, \partial]$  can be written as a finite sum  $\sum_i L_i(x)\partial^i$  for uniquely determined polynomials  $L_i$  in  $\mathbb{K}[x]$ . We say that  $L$  has bidegree less than  $(d, r)$  in  $(x, \partial)$  if  $L$  has degree less than  $r$  in  $\partial$ , and if all  $L_i$ 's have degrees less than  $d$  in  $x$ . The degree in  $\partial$  of  $L$  is usually called the *order* of  $L$ .

The main difference with the commutative ring  $\mathbb{K}[x, y]$

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of usual bivariate polynomials is the commutation rule  $\partial x = x\partial + 1$  that simply encodes, in operator notation, Leibniz's differentiation rule  $\frac{d}{dx}(xf) = x\frac{d}{dx}(f) + f$ . This slight difference between  $\mathbb{K}[x, \partial]$  and  $\mathbb{K}[x, y]$  has a considerable impact on the complexity level. On the one hand, it is classical that multiplication in  $\mathbb{K}[x, y]$  can be reduced to that of polynomials in  $\mathbb{K}[x]$ , due to a technique commonly called *Kronecker's trick* [19], [14]. As a consequence, any two polynomials of degrees less than  $d$  in  $x$ , and less than  $r$  in  $y$ , can be multiplied in *quasi-optimal time*  $\mathcal{O}(M(dr))$ . On the other hand, under our hypothesis that  $\mathbb{K}$  has characteristic zero, it was shown by van der Hoeven [16] that the product of two elements from  $\mathbb{K}[x, \partial]$  of bidegree less than  $(n, n)$  can be computed in time  $\mathcal{O}(n^\omega)$ . Moreover, it has been proved in [6] that conversely, multiplication in  $\mathbb{K}^{n \times n}$  can be reduced to a constant number of multiplications in  $\mathbb{K}[x, \partial]$ , in bidegree less than  $(n, n)$ . In other words, multiplying operators of *well-balanced bidegree* is computationally equivalent to matrix multiplication.

However, contrary to the commutative case, higher-level operations in  $\mathbb{K}[x, \partial]$ , such as the *least common left multiple* (LCLM) and the *greatest common right divisor* (GCRD), do not preserve well-balanced bidegrees [15], [5]. For instance, the LCLM of two operators of bidegrees less than  $(n, n)$  is of bidegree less than  $(2n(n+1), 2n) = \mathcal{O}(n^2, n)$ , and this bound is generically reached. This is a typical phenomenon: operators obtained from computations in  $\mathbb{K}[x, \partial]$  tend to have much larger degrees in  $x$  than in  $\partial$ .

In the general case of operators with possibly unbalanced degrees  $d$  in  $x$  and  $r$  in  $\partial$ , the naive algorithm has cost  $\mathcal{O}(d^2 r^2 \min(d, r))$ ; a better algorithm, commonly attributed to Takayama, has complexity  $\tilde{\mathcal{O}}(dr \min(d, r))$ . We refer to [6, §2] for a review of these algorithms. When  $r \leq d \leq r^{4-\omega}$ , the best current upper bound for multiplication is  $\mathcal{O}(r^{\omega-2} d^2)$  [16], [17]. It was asked by van der Hoeven [16, §6] whether this complexity could be lowered to  $\tilde{\mathcal{O}}(r^{\omega-1} d)$ . Here, and hereafter, the soft- $\mathcal{O}$  notation  $\tilde{\mathcal{O}}(\cdot)$  indicates that polylogarithmic factors in  $d$  and in  $r$  are neglected. The purpose of the present work is to provide a positive answer to this open question. Our main result is encapsulated in the following theorem:

*Theorem 1:* Let  $\mathbb{K}$  be an effective field of characteristic zero. Operators in  $\mathbb{K}[x, \partial]$  of bidegree less than  $(d, r)$  in  $(x, \partial)$  can be multiplied using

$$\tilde{\mathcal{O}}(\min(d, r)^{\omega-2} dr)$$

operations in  $\mathbb{K}$ .

Actually, we will prove slightly more refined versions of this theorem (see Theorems 3 and 5 below), by making the hidden log-terms in the complexity explicit.

In the important case  $d \geq r$ , our complexity bound reads  $\tilde{\mathcal{O}}(r^{\omega-1} d)$ . This is quasi-linear (thus quasi-optimal) with respect to  $d$ . Moreover, by the equivalence result from [6, §3], the exponent of  $r$  is also the best possible. Besides, under the (plausible, still conjectural) assumption that  $\omega = 2$ , the complexity in Theorem 1 is almost linear with respect to the output size. For  $r = 1$  we retrieve the fact that multiplication in  $\mathbb{K}[x]$  in degree  $< d$  can be done in quasi-linear time  $\tilde{\mathcal{O}}(d)$ ; from this perspective, the result of Theorem 1 can be seen as a generalization of the fast multiplication for usual polynomials.

In an expanded version [3] of this paper, we will show that analogues of Theorem 1 also hold for other types of *skew polynomials*. More precisely, we will prove similar complexity bounds when the skew indeterminate  $\partial : f(x) \mapsto f'(x)$  is replaced by the Euler derivative  $\delta : f(x) \mapsto xf'(x)$ , or a shift operator  $\sigma^c : f(x) \mapsto f(x+c)$ , or a dilatation  $\chi_q : f(x) \mapsto f(qx)$ . Most of these other cases are treated by showing that rewritings such as  $\delta \leftrightarrow x\partial$  or  $\sigma^c \leftrightarrow \exp(c\partial)$  can be performed efficiently. We will also prove complexity bounds for a few other interesting operations on skew polynomials.

**Main ideas.** The fastest known algorithms for multiplication of usual polynomials in  $\mathbb{K}[x]$  rely on an evaluation-interpolation strategy at special points in the base field  $\mathbb{K}$  [11], [25], [9]. This reduces polynomial multiplication to the “inner product” in  $\mathbb{K}$ . We adapt this strategy to the case of linear differential operators in  $\mathbb{K}[x, \partial]$ : the evaluation “points” are *exponential polynomials* of the form  $x^n e^{\alpha x}$  on which differential operators act nicely. With this choice, the evaluation and interpolation of operators is encoded by *Hermite evaluation and interpolation* for usual polynomials (generalizing the classical Lagrange interpolation), for which quasi-optimal algorithms exist. For operators of bidegree less than  $(d, r)$  in  $(x, \partial)$ , with  $r \geq d$ , we use  $p = \mathcal{O}(r/d)$  evaluation points, and encode the inner multiplication step by  $p$  matrix multiplications in size  $d$ . All in all, this gives an FFT-type multiplication algorithm for differential operators of complexity  $\tilde{\mathcal{O}}(d^{\omega-1} r)$ . Finally, we reduce the case  $r < d$  to the case  $r \geq d$ . To do this efficiently, we design a fast algorithm for the computation of the so-called *reflection* of a differential operator, a useful ring morphism that swaps the indeterminates  $x$  and  $\partial$ , and whose effect is exchanging degrees and orders.

## II. PRELIMINARIES

Recall that  $\mathbb{K}$  denotes an effective field of characteristic zero. Throughout the paper,  $\mathbb{K}[x]_d$  will denote the set of polynomials of degree less than  $d$  with coefficients in the field  $\mathbb{K}$ , and  $\mathbb{K}[x, \partial]_{d,r}$  will denote the set of linear differential operators in  $\mathbb{K}[x, \partial]$  with degree less than  $r$  in  $\partial$ , and polynomial coefficients in  $\mathbb{K}[x]_d$ .

The cost of our algorithms will be measured by the number of field operations in  $\mathbb{K}$  they use. We recall that polynomials in  $\mathbb{K}[x]_d$  can be multiplied within  $M(d) = \mathcal{O}(d \log d \log \log d) = \tilde{\mathcal{O}}(d)$  operations in  $\mathbb{K}$ , using the FFT-based algorithms in [25], [9], and that  $\omega$  denotes a feasible exponent for matrix multiplication over  $\mathbb{K}$ , that is, a real constant  $2 \leq \omega \leq 3$  such that two  $r \times r$  matrices with coefficients in  $\mathbb{K}$  can be multiplied in time  $\mathcal{O}(r^\omega)$ . Throughout this paper, we will make the classical assumption that  $M(d)/d$  is an increasing function in  $d$ .

Most basic polynomial operations in  $\mathbb{K}[x]_d$  (division, Taylor shift, extended gcd, multipoint evaluation, interpolation, etc.) have cost  $\tilde{\mathcal{O}}(d)$  [2], [4], [8], [24], [14]. Our algorithms will make a crucial use of the following result due to Chin [10], see also [20] for a formulation in terms of structured matrices.

*Theorem 2 (Fast Hermite evaluation–interpolation):*

Let  $\mathbb{K}$  be an effective field of characteristic zero, let  $c_0, \dots, c_{k-1}$  be  $k$  positive integers, and let  $d = \sum_i c_i$ . Given  $k$  mutually distinct points  $\alpha_0, \dots, \alpha_{k-1}$  in  $\mathbb{K}$  and a polynomial  $P \in \mathbb{K}[x]_d$ , one can compute the vector of  $d$  values

$$\mathcal{H} = (P(\alpha_0), P'(\alpha_0), \dots, P^{(c_0-1)}(\alpha_0), \dots, P(\alpha_{k-1}), P'(\alpha_{k-1}), \dots, P^{(c_{k-1}-1)}(\alpha_{k-1}))$$

in  $\mathcal{O}(M(d) \log k) = \tilde{\mathcal{O}}(d)$  arithmetic operations in  $\mathbb{K}$ . Conversely,  $P$  is uniquely determined by  $\mathcal{H}$ , and its coefficients can be recovered from  $\mathcal{H}$  in  $\mathcal{O}(M(d) \log k) = \tilde{\mathcal{O}}(d)$  arithmetic operations in  $\mathbb{K}$ .

### III. THE NEW ALGORITHM IN THE CASE $r \geq d$

#### A. Multiplication by evaluation and interpolation

Most fast algorithms for multiplying two polynomials  $P, Q \in \mathbb{K}[x]_d$  are based on the evaluation–interpolation strategy. The idea is to pick  $2d - 1$  distinct points  $\alpha_0, \dots, \alpha_{2d-2}$  in  $\mathbb{K}$ , and to perform the following three steps:

- 1) (Evaluation) Evaluate  $P$  and  $Q$  at  $\alpha_0, \dots, \alpha_{2d-2}$ .
- 2) (Inner multiplication) Compute the  $2d - 1$  values  $(PQ)(\alpha_i) = P(\alpha_i)Q(\alpha_i)$  for  $i < 2d - 1$ .
- 3) (Interpolation) Recover the product  $PQ$  from the values  $(PQ)(\alpha_0), \dots, (PQ)(\alpha_{2d-2})$ .

The inner multiplication step requires only  $\mathcal{O}(d)$  operations. Consequently, if both the evaluation and interpolation steps can be performed fast, then we obtain a fast algorithm for multiplying  $P$  and  $Q$ . For instance, if  $\mathbb{K}$  contains a  $2^p$ -th primitive root of unity with  $2^{p-1} \leq 2d - 1 < 2^p$ , then both evaluation and interpolation can be performed in time  $\mathcal{O}(d \log d)$  using the Fast Fourier Transform [11].

For a linear differential operator  $L \in \mathbb{K}[x, \partial]_{d,r}$  it is natural to consider evaluations at powers of  $x$  instead

of roots of unity. It is also natural to represent the evaluation of  $L$  at a suitable number of such powers by a matrix. More precisely, given  $k \in \mathbb{N}$ , we may regard  $L$  as an operator from  $\mathbb{K}[x]_k$  to  $\mathbb{K}[x]_{k+d}$ . We may also regard elements of  $\mathbb{K}[x]_k$  and  $\mathbb{K}[x]_{k+d}$  as column vectors, written in the canonical bases with powers of  $x$ . We will denote by

$$\Phi_L^{k+d,k} = \begin{pmatrix} L(1)_0 & \cdots & L(x^{k-1})_0 \\ \vdots & & \vdots \\ L(1)_{k+d-1} & \cdots & L(x^{k-1})_{k+d-1} \end{pmatrix}$$

the matrix of the  $\mathbb{K}$ -linear map  $L : \mathbb{K}[x]_k \rightarrow \mathbb{K}[x]_{k+d}$  with respect to these bases. Given two operators  $K, L$  in  $\mathbb{K}[x, \partial]_{d,r}$ , we clearly have

$$\Phi_{KL}^{k+2d,k} = \Phi_K^{k+2d,k+d} \Phi_L^{k+d,k}, \quad \text{for all } k \geq 0.$$

For  $k = 2r$  (or larger), the operator  $KL$  can be recovered from the matrix  $\Phi_{KL}^{2r+2d,2r}$ , whence the formula

$$\Phi_{KL}^{2r+2d,2r} = \Phi_K^{2r+2d,2r+d} \Phi_L^{2r+d,2r} \quad (1)$$

yields a way to multiply  $K$  and  $L$ . For the complexity analysis, we thus have to consider the three steps:

- 1) (Evaluation) Computation of  $\Phi_K^{2r+2d,2r+d}$  and of  $\Phi_L^{2r+d,2r}$  from  $K$  and  $L$ .
- 2) (Inner multiplication) Computation of the matrix product (1).
- 3) (Interpolation) Recovery of  $KL$  from  $\Phi_{KL}^{2r+2d,2r}$ .

In [16], [6], this multiplication method was applied with success to the case when  $d = r$ . In this “square case”, the following result was proved in [6, §4.2].

*Lemma 1:* Let  $L \in \mathbb{K}[x, \partial]_{d,d}$ . Then

- 1) We may compute  $\Phi_L^{2d,d}$  as a function of  $L$  in time  $\mathcal{O}(dM(d))$ ;
- 2) We may recover  $L$  from  $\Phi_L^{2d,d}$  in time  $\mathcal{O}(dM(d))$ .

#### B. Evaluation–interpolation at exponential polynomials

Assume now that  $r \geq d$ . Then a straightforward application of the above evaluation–interpolation strategy yields an algorithm of sub-optimal complexity. Indeed, the matrix  $\Phi_{KL}^{2r+2d,2r}$  contains a lot of redundant information and, since its mere total number of elements exceeds  $r^2$ , one cannot expect a direct multiplication algorithm of quasi-optimal complexity  $\tilde{\mathcal{O}}(d^{\omega-1}r)$ .

In order to maintain quasi-optimal complexity in this case as well, the idea is to evaluate at so called *exponential polynomials* instead of ordinary polynomials. More specifically, given  $L \in \mathbb{K}[x, \partial]_{d,r}$  and  $\alpha \in \mathbb{K}$ , we will use the fact that  $L$  also operates nicely on the vector space  $\mathbb{K}[x]e^{\alpha x}$ . Moreover, for any  $P \in \mathbb{K}[x]$ , we have

$$L(Pe^{\alpha x}) = L_{\times \alpha}(P)e^{\alpha x},$$

where

$$L_{\times\alpha} = \sum_i L_i(x)(\partial + \alpha)^i$$

is the operator obtained by substituting  $\partial + \alpha$  for  $\partial$  in  $L = \sum_i L_i(x)\partial^i$ . Indeed, this is a consequence of the fact that, by Leibniz's rule:

$$\partial^i (Pe^{\alpha x}) = \left( \sum_{j \leq i} \binom{i}{j} \alpha^j \partial^{i-j} P \right) e^{\alpha x} = (\partial + \alpha)^i (P) e^{\alpha x}.$$

Now let  $p = \lceil r/d \rceil$  and let  $\alpha_0, \dots, \alpha_{p-1}$  be  $p$  pairwise distinct points in  $\mathbb{K}$ . For each integer  $k \geq 1$ , we define the vector space

$$\mathbb{V}_k = \mathbb{K}[x]_k e^{\alpha_0 x} \oplus \dots \oplus \mathbb{K}[x]_k e^{\alpha_{p-1} x}$$

with canonical basis

$$(e^{\alpha_0 x}, \dots, x^{k-1} e^{\alpha_0 x}, \dots, e^{\alpha_{p-1} x}, \dots, x^{k-1} e^{\alpha_{p-1} x}).$$

Then we may regard  $L$  as an operator from  $\mathbb{V}_k$  into  $\mathbb{V}_{k+d}$  and we will denote by  $\Phi_L^{[k+d, k]}$  the matrix of this operator with respect to the canonical bases. By what precedes, this matrix is block diagonal, with  $p$  blocks of size  $d$ :

$$\Phi_L^{[k+d, k]} = \begin{pmatrix} \Phi_{L_{\times\alpha_0}}^{k+d, k} & & \\ & \ddots & \\ & & \Phi_{L_{\times\alpha_{p-1}}}^{k+d, k} \end{pmatrix}.$$

Let us now show that the operator  $L$  is uniquely determined by the matrix  $\Phi_L^{[2d, d]}$ , and that this gives rise to an efficient algorithm for multiplying two operators in  $\mathbb{K}[x, \partial]_{d, r}$ .

*Lemma 2:* Let  $L \in \mathbb{K}[x, \partial]_{d, r}$  with  $r \geq d$ . Then

- 1) We may compute  $\Phi_L^{[2d, d]}$  as a function of  $L$  in time  $\mathcal{O}(dM(r) \log r)$ ;
- 2) We may recover  $L$  from the matrix  $\Phi_L^{[2d, d]}$  in time  $\mathcal{O}(dM(r) \log r)$ .

*Proof:* For any operator  $L = \sum_{i < d, j < r} L_{i, j} x^i \partial^j$  in  $\mathbb{K}[x, \partial]_{d, r}$ , we define its truncation  $L^*$  at order  $\mathcal{O}(\partial^d)$  by

$$L^* = \sum_{i, j < d} L_{i, j} x^i \partial^j.$$

Since  $L - L^*$  vanishes on  $\mathbb{K}[x]_d$ , we notice that  $\Phi_L^{2d, d} = \Phi_{L^*}^{2d, d}$ .

If  $L \in \mathbb{K}[\partial]_r$ , then  $L^*$  can be regarded as the power series expansion of  $L$  at  $\partial = 0$  and order  $d$ . More generally, for any  $i \in \{0, \dots, p-1\}$ , the operator  $L_{\times\alpha_i}^*(\partial) = L(\partial + \alpha_i)^*$  coincides with the Taylor series expansion at  $\partial = \alpha_i$  and order  $d$ :

$$L_{\times\alpha_i}^*(\partial) = L(\alpha_i) + L'(\alpha_i)\partial + \dots + \frac{1}{(d-1)!} L^{(d-1)}(\alpha_i)\partial^{d-1}.$$

In other words, the computation of the truncated operators  $L_{\times\alpha_0}^*, \dots, L_{\times\alpha_{p-1}}^*$  as a function of  $L$  corresponds

to a Hermite evaluation at the points  $\alpha_i$ , with multiplicity  $c_i = d$  at each point  $\alpha_i$ . By Theorem 2, this computation can be performed in time  $\mathcal{O}(M(pd) \log p) = \mathcal{O}(M(r) \log r)$ . Furthermore, Hermite interpolation allows us to recover  $L$  from  $L_{\times\alpha_0}^*, \dots, L_{\times\alpha_{p-1}}^*$  with the same time complexity  $\mathcal{O}(M(r) \log r)$ .

Now let  $L \in \mathbb{K}[x, \partial]_{d, r}$  and consider the expansion of  $L$  in  $x$

$$L(x, \partial) = L_0(\partial) + \dots + x^{d-1} L_{d-1}(\partial).$$

For each  $i$ , one Hermite evaluation of  $L_i$  allows us to compute the  $L_{\times\alpha_j, i}^*$  with  $j < p$  in time  $\mathcal{O}(M(r) \log r)$ . The operators  $L_{\times\alpha_j}^*$  with  $j < p$  can therefore be computed in time  $\mathcal{O}(dM(r) \log r)$ . By Lemma 1, we need  $\mathcal{O}(rM(d)) = \mathcal{O}(dM(r))$  additional operations in order to obtain  $\Phi_L^{[2d, d]}$ . Similarly, given  $\Phi_L^{[2d, d]}$ , Lemma 1 allows us to recover the operators  $L_{\times\alpha_j}^*$  with  $j < p$  in time  $\mathcal{O}(dM(r))$ . Using  $d$  Hermite interpolations, we also recover the coefficients  $L_i$  of  $L$  in time  $\mathcal{O}(dM(r) \log r)$ . ■

*Theorem 3:* Assume  $r \geq d$  and let  $K, L \in \mathbb{K}[x, \partial]_{d, r}$ . Then the product  $KL$  can be computed in time  $\mathcal{O}(d^{\omega-1}r + dM(r) \log r)$ .

*Proof:* Considering  $K$  and  $L$  as operators in  $\mathbb{K}[x, \partial]_{3d, 3r}$ , Lemma 2 implies that the computation of  $\Phi_K^{[4d, 3d]}$  and  $\Phi_L^{[3d, 2d]}$  as a function of  $K$  and  $L$  can be done in time  $\mathcal{O}(dM(r) \log r)$ . The multiplication

$$\Phi_{KL}^{[4d, 2d]} = \Phi_K^{[4d, 3d]} \Phi_L^{[3d, 2d]}$$

can be done in time  $\mathcal{O}(d^\omega p) = \mathcal{O}(d^{\omega-1}r)$ . Lemma 2 finally implies that we may recover  $KL$  from  $\Phi_{KL}^{[4d, 2d]}$  in time  $\mathcal{O}(dM(r) \log r)$ . ■

#### IV. THE NEW ALGORITHM IN THE CASE $d > r$

Any differential operator  $L \in \mathbb{K}[x, \partial]_{d, r}$  can be written in a unique form

$$L = \sum_{i < r, j < d} L_{i, j} x^j \partial^i, \quad \text{for some scalars } L_{i, j} \in \mathbb{K}.$$

This representation, with  $x$  on the left and  $\partial$  on the right, is called the *canonical form of  $L$* .

Let  $\varphi : \mathbb{K}[x, \partial] \rightarrow \mathbb{K}[x, \partial]$  denote the map defined by

$$\varphi \left( \sum_{i < r, j < d} L_{i, j} x^j \partial^i \right) = \sum_{i < r, j < d} L_{i, j} \partial^j (-x)^i.$$

In other words,  $\varphi$  is the unique  $\mathbb{K}$ -algebra automorphism of  $\mathbb{K}[x, \partial]$  that keeps the elements of  $\mathbb{K}$  fixed, and is defined on the generators of  $\mathbb{K}[x, \partial]$  by  $\varphi(x) = \partial$  and  $\varphi(\partial) = -x$ . We will call  $\varphi$  the *reflection morphism* of  $\mathbb{K}[x, \partial]$ . The map  $\varphi$  enjoys the nice property that it sends  $\mathbb{K}[x, \partial]_{d, r}$  onto  $\mathbb{K}[x, \partial]_{r, d}$ . In particular, to an operator whose degree is higher than its order,  $\varphi$  associates a “mirror operator” whose order is higher than its degree.

### A. Main idea of the algorithm in the case $d > r$

If  $d > r$ , then the reflection morphism  $\varphi$  is the key to our fast multiplication algorithm for operators in  $\mathbb{K}[x, \partial]_{d,r}$ , since it allows us to reduce this case to the previous case when  $r \geq d$ . More precisely, given  $K, L$  in  $\mathbb{K}[x, \partial]_{d,r}$  with  $d > r$ , the main steps of the algorithm are:

- (S1) compute the canonical forms of  $\varphi(K)$  and  $\varphi(L)$ ,
- (S2) compute the product  $M = \varphi(K)\varphi(L)$  of operators  $\varphi(K) \in \mathbb{K}[x, \partial]_{r,d}$  and  $\varphi(L) \in \mathbb{K}[x, \partial]_{r,d}$ , using the algorithm described in the previous section, and
- (S3) compute and return the (canonical form of the) operator  $KL = \varphi^{-1}(M)$ .

Since  $d > r$ , step (S2) can be performed in complexity  $\tilde{\mathcal{O}}(r^{\omega-1}d)$  using the results of Section III. In the next subsection, we will prove that both steps (S1) and (S3) can be performed in  $\tilde{\mathcal{O}}(rd)$  operations in  $\mathbb{K}$ . This will enable us to conclude the proof of Theorem 1.

### B. Quasi-optimal computation of reflections

We now show that the *reflection* and the *inverse reflection* of a differential operator can be computed quasi-optimally. The idea is that performing reflections can be interpreted in terms of Taylor shifts for polynomials, which can be computed in quasi-linear time using the algorithm from [1].

A first observation is that the composition  $\varphi \circ \varphi$  is equal to the involution  $\psi : \mathbb{K}[x, \partial] \rightarrow \mathbb{K}[x, \partial]$  defined by

$$\psi \left( \sum_{i < r, j < d} L_{i,j} x^j \partial^i \right) = \sum_{i < r, j < d} (-1)^{i+j} L_{i,j} x^j \partial^i.$$

As a direct consequence of this fact, it follows that the map  $\varphi^{-1}$  is equal to  $\varphi \circ \psi$ . Since  $\psi(L)$  is already in canonical form, computing  $\psi(L)$  only consists of sign changes, which can be done in linear time  $\mathcal{O}(dr)$ . Therefore, computing the inverse reflection  $\varphi^{-1}(L)$  can be performed within the same cost as computing the direct reflection  $\varphi(L)$ , up to a linear overhead  $\mathcal{O}(rd)$ .

In the remainder of this section, we focus on the fast computation of direct reflections. The key observation is encapsulated in the next lemma. Here, and in what follows, we use the convention that the entries of a matrix corresponding to indices beyond the matrix sizes are all zero.

*Lemma 3:* Assume that  $(p_{i,j})$  and  $(q_{i,j})$  are two matrices in  $\mathbb{K}^{r \times d}$  such that

$$\sum_{i,j} q_{i,j} x^i \partial^j = \sum_{i,j} p_{i,j} \partial^j x^i.$$

Then

$$i! q_{i,j} = \sum_{k \geq 0} \binom{j+k}{k} (i+k)! p_{i+k, j+k},$$

where we use the convention that  $p_{i,j} = 0$  as soon as  $i \geq r$  or  $j \geq d$ .

*Proof:* Leibniz's differentiation rule implies the commutation rule

$$\partial^j \frac{x^i}{i!} = \sum_{k=0}^j \binom{j}{k} \frac{x^{i-k}}{(i-k)!} \partial^{j-k}.$$

Together with the hypothesis, this implies the equality

$$\begin{aligned} \sum_{i,j} (i! q_{i,j}) \frac{x^i}{i!} \partial^j &= \sum_{i,j} (i! p_{i,j}) \partial^j \frac{x^i}{i!} \\ &= \sum_{k \geq 0} \left( \sum_{i,j} (i! p_{i,j}) \binom{j}{k} \frac{x^{i-k}}{(i-k)!} \partial^{j-k} \right). \end{aligned}$$

We conclude by extraction of coefficients.  $\blacksquare$

*Theorem 4:* Let  $L \in \mathbb{K}[x, \partial]_{d,r}$ . Then we may compute  $\varphi(L)$  and  $\varphi^{-1}(L)$  using  $\mathcal{O}(\min(dM(r), rM(d))) = \tilde{\mathcal{O}}(rd)$  operations in  $\mathbb{K}$ .

*Proof:* We first deal with the case  $r \geq d$ . If  $L = \sum_{i < r, j < d} p_{i,j} x^j \partial^i$ , then by the first equality of Lemma 3, the reflection  $\varphi(L)$  is equal to

$$\varphi(L) = \sum_{i < r, j < d} p_{i,j} \partial^j (-x)^i = \sum_{i < r, j < d} q_{i,j} (-x)^j \partial^i,$$

where

$$i! q_{i,j} = \sum_{\ell \geq 0} \binom{j+\ell}{j} (i+\ell)! p_{i+\ell, j+\ell}. \quad (2)$$

For any fixed  $k$  with  $1-r \leq k \leq d-1$ , let us introduce  $G_k = \sum_i i! q_{i, i+k} x^{i+k}$  and  $F_k = \sum_i i! p_{i, i+k} x^{i+k}$ . These polynomials belong to  $\mathbb{K}[x]_d$ , since  $p_{i,j} = q_{i,j} = 0$  for  $j \geq d$ . If  $k \leq 0$ , then Equation (2) translates into

$$G_k(x) = F_k(x+1).$$

Indeed, Equation (2) with  $j = i+k$  implies that  $G_k(x)$  is equal to

$$\begin{aligned} &\sum_{i,\ell} \binom{i+k+\ell}{i+k} (i+\ell)! p_{i+\ell, i+k+\ell} x^{i+k} \\ &= \sum_{j,s} j! p_{j, j+k} \binom{j+k}{s} x^s = F_k(x+1). \end{aligned}$$

Similarly, if  $k > 0$ , then the coefficients of  $x^i$  in  $G_k(x)$  and  $F_k(x+1)$  still coincide for all  $i \geq k$ . In particular, we may compute  $G_{1-r}, \dots, G_{d-1}$  from  $F_{1-r}, \dots, F_{d-1}$  by means of  $d+r \leq 2r$  Taylor shifts of polynomials in  $\mathbb{K}[x]_d$ . Using the fast algorithm for Taylor shift in [1], this can be done in time  $\mathcal{O}(rM(d))$ .

Once the coefficients of the  $G_k$ 's are available, the computation of the coefficients of  $\varphi(L)$  requires  $\mathcal{O}(dr)$  additional operations.

If  $d > r$ , then we notice that the equality (2) is equivalent to

$$j! q_{i,j} = \sum_{\ell \geq 0} \binom{i+\ell}{i} (j+\ell)! p_{i+\ell, j+\ell},$$

as can be seen by expanding the binomial coefficients. Redefining  $G_k := \sum_i i! q_{i+k, i} x^{i+k}$  and  $F_k := \sum_i i! p_{i+k, i} x^{i+k}$ , similar arguments as above show that  $\varphi(P)$  can be computed using  $\mathcal{O}(dM(r))$  operations in  $\mathbb{K}$ .

By what has been said at the beginning of this section, we finally conclude that the inverse reflection  $\varphi^{-1}(L) = \varphi(\psi(L))$  can be computed for the same cost as the direct reflection  $\varphi(L)$ . ■

### C. Proof of Theorem 1 in the case $d > r$

We will prove a slightly better result:

*Theorem 5:* Assume  $d > r$  and  $K, L \in \mathbb{K}[x, \partial]_{d,r}$ . Then the product  $KL$  can be computed using  $\mathcal{O}(r^{\omega-1}d + rM(d) \log d)$  operations in  $\mathbb{K}$ .

*Proof:* Assume that  $K$  and  $L$  are two operators in  $\mathbb{K}[x, \partial]_{d,r}$  with  $d > r$ . Then  $\varphi(K)$  and  $\varphi(L)$  belong to  $\mathbb{K}[x, \partial]_{r,d}$ , and their canonical forms can be computed in  $\mathcal{O}(dM(r))$  operations by Theorem 4. Using the algorithm from Section III, we may compute  $M = \varphi(K)\varphi(L)$  in  $\mathcal{O}(r^{\omega-1}d + rM(d) \log d)$  operations. Finally,  $KL = \varphi^{-1}(M)$  can be computed in  $\mathcal{O}(rM(d))$  operations by Theorem 4. We conclude by adding up the costs of these three steps. ■

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