Fast algorithms for polynomials and matrices

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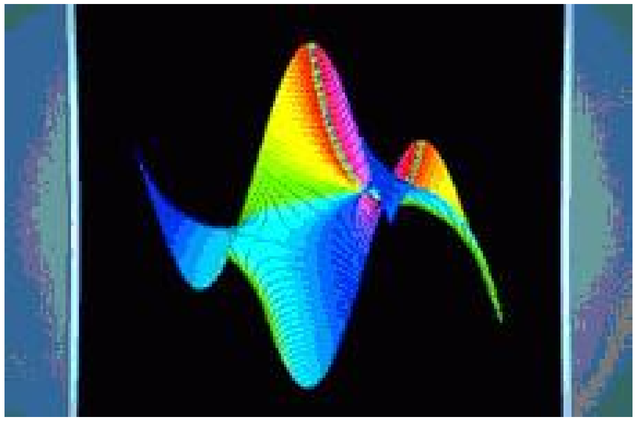

Algorithms Project, INRIA

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This course

- Main objects
  - polynomials $\mathbb{K}[x]$
  - rational fractions $\mathbb{K}(x)$
  - power series $\mathbb{K}[[x]]$
  - matrices $\mathcal{M}_r(\mathbb{K})$
  - polynomial matrices $\mathcal{M}_r(\mathbb{K}[x])$
  - power series matrices $\mathcal{M}_r(\mathbb{K}[[x]])$

  where $\mathbb{K}$ is a field (generally supposed of characteristic 0 or large)

- Secondary/auxiliary objects
  - linear recurrences with constant, or polynomial, coefficients
  - linear differential equations with polynomial, or power series, coefficients

This course

- Aims
  - design and analysis of fast algorithms for various algebraic problems
  - Fast = using asymptotically few operations $(+, \times, \div)$ in the basefield $\mathbb{K}$
  - Holy Grail: quasi-optimal algorithms = (time) complexity almost linear in the input/output size

- Specific algorithms depending on the kind of the input
  - dense (i.e., arbitrary)
  - structured (i.e., special relations between coefficients)
  - sparse (i.e., few elements)

- In this lecture, we focus on dense objects

A word about structure and sparsity

- sparse means
  - for degree n polynomials: $s \ll n$ coefficients
  - for $r \times r$ matrices: $s \ll r^2$ entries

- structured means
  - for $r \times r$ matrices: special form, e.g., Toeplitz, Hankel, Vandermonde, Cauchy, Sylvester, etc) $\longrightarrow$ encoded by $O(r)$ elements
  - for polynomials/power series: satisfying an equation (algebraic or differential) $\longrightarrow$ encoded by degree/order of size $O(1)$

- In this lecture, we focus on dense objects

Typical problems

- On all objects
  - sum, product
  - inversion, division
- On power series
  - logarithm, exponential
  - composition
  - Padé and Hermite-Padé approximation
- On polynomials
  - (multipoint) evaluation, interpolation
  - (extended) greatest commun divisor, resultant
  - shift
  - composed sum and product
- On matrices
  - system solving
  - determinant, characteristic polynomial

Complexity yardsticks

Important issues:

- addition is easy: naive algorithm already optimal
- multiplication is the most basic (non-trivial) problem
- almost all problems can be reduced to multiplication

Are there quasi-optimal algorithms for:

- polynomial/power series multiplication? Yes!
- matrix multiplication? Big open problem!

Complexity yardsticks

$$\begin{aligned}
\mathsf{M}(n) &= \text{complexity of polynomial multiplication in } \mathbb{K}[x]_{<n} \\
&= O(n^2) \text{ by the naive algorithm} \\
&= O\big(n^{1.58}\big) \text{ by Karatsuba's algorithm} \\
&= O\big(n^{\log_\alpha (2\alpha-1)}\big) \text{ by the Toom-Cook algorithm } (\alpha \geq 3) \\
&= O\big(n \log n \operatorname{loglog} n\big) \text{ by the Schönhage-Strassen algorithm}
\end{aligned}$$

$$\begin{aligned}
\mathsf{MM}(r) &= \text{complexity of matrix product in } \mathcal{M}_r(\mathbb{K}) \\
&= O(r^3) \text{ by the naive algorithm} \\
&= O(r^{2.81}) \text{ by Strassen's algorithm} \\
&= O(r^{2.38}) \text{ by the Coppersmith-Winograd algorithm}
\end{aligned}$$

$$\begin{aligned}
\mathsf{MM}(r,n) &= \text{complexity of polynomial matrix product in } \mathcal{M}_r(\mathbb{K}[x]_{<n}) \\
&= O(r^3\, \mathsf{M}(n)) \text{ by the naive algorithm} \\
&= O(\mathsf{MM}(r)\, n \log(n) + r^2 n \log n \operatorname{loglog} n) \text{ by the Cantor-Kaltofen algo} \\
&= O(\mathsf{MM}(r)\, n + r^2\, \mathsf{M}(n)) \text{ by the B-Schost algorithm}
\end{aligned}$$

Typical problems, and their complexities

• Polynomials, power series and matrices	
– product	$\mathsf{M}(n)$, $\mathsf{MM}(r)$
– division/inversion	$O(\mathsf{M}(n))$, $O(\mathsf{MM}(r))$
• On power series	
– logarithm, exponential	$O(\mathsf{M}(n))$
– composition	$O(\sqrt{n \log n}\,\mathsf{M}(n))$
– Padé approximation	$O(\mathsf{M}(n) \log n)$
• On polynomials	
– (multipoint) evaluation, interpolation	$O(\mathsf{M}(n) \log n)$
– extended gcd, resultant	$O(\mathsf{M}(n) \log n)$
– shift	$O(\mathsf{M}(n))$
– composed sum and product	$O(\mathsf{M}(n))$
• On matrices	
– system solving, determinant	$O(\mathsf{MM}(r))$
– characteristic / minimal polynomial	$O(\mathsf{MM}(r))$

Typical problems, and the algorithms' designers

- Polynomials, power series and matrices
  - product
  - division/inversion — Sieveking-Kung 1972, Strassen 1969, 1973
- On power series
  - logarithm, exponential — Brent 1975
  - composition — Brent-Kung 1978
  - Padé approximation — Brent-Gustavson-Yun 1980
- On polynomials
  - (multipoint) evaluation, interpolation — Borodin-Moenck 1974
  - extended gcd, resultant — Knuth-Schönhage 1971, Schwartz 1980
  - shift — Aho-Steiglitz-Ullman 1975
  - composed sum and product — B-Flajolet-Salvy-Schost 2006
- On matrices
  - system solving, determinant — Strassen 1969
  - characteristic polynomial / minimal polynomial — Keller-Gehrig 1985

Typical problems, and their complexities

- On power series matrices
  - product $\mathsf{MM}(r,n)$
  - inversion $O(\mathsf{MM}(r,n))$
  - quasi-exponential (sol. of $Y' = AY$) $O(\mathsf{MM}(r,n))$
- On power series
  - Hermite-Padé approximation of r series $O(\mathsf{MM}(r,n)\log n)$
- On polynomial matrices
  - product $\mathsf{MM}(r,n)$
  - system solving $O(\mathsf{MM}(r,n)\log n)$
  - determinant $O(\mathsf{MM}(r,n)\log^2(n))$
  - inversion $\tilde{O}(r^3\, n)$, if $r = 2^k$
  - characteristic / minimal polynomial $\tilde{O}(r^{2.6972}\, n)$

Typical problems, and the algorithms' designers

• **On power series matrices**	
– product	
– inversion	Schulz 1933
– quasi-exponential	B-Chyzak-Ollivier-Salvy-Schost-Sedoglavic 2007
• **On power series**	
– Hermite-Padé approximation	Beckermann-Labahn 1994
• **On polynomial matrices**	
– product	
– system solving	Storjohann 2002
– determinant	Storjohann 2002
– inversion	Jeannerod-Villard 2005
– characteristic / minimal polynomial	Kaltofen-Villard 2004

Other problems, and their complexities

- On structured (D-finite, algebraic) power series
  - sum, product, Hadamard product $O(n)$
  - inversion $O(\mathsf{M}(n))$, $O(n)$
- On structured matrices
  - Toeplitz-like: system solving, determinant $O(\mathsf{M}(r)\log r)$
  - Vandermonde-like: system solving, determinant $O(\mathsf{M}(r)\log^2(r))$
  - Cauchy-like: system solving, determinant $O(\mathsf{M}(r)\log^2(r))$
- On sparse matrices
  - system solving $O(r^2)$
  - determinant $O(r^2)$
  - rank $O(r^2)$
  - minimal polynomial $O(r^2)$

Other problems, and their complexities

- On structured (D-finite, algebraic) power series
  - sum, product, Hadamard product — folklore, but not sufficiently known!
  - inversion
- On structured matrices
  - Toeplitz-like: system solving, determinant — Bitmead-Anderson-Morf 1980
  - Vandermonde-like: system solving, determinant — Pan 1990
  - Cauchy-like: system solving, determinant — Pan 2000
- On sparse matrices
  - system solving — Wiedemann 1986
  - determinant — Wiedemann 1986
  - rank — Kaltofen-Saunders 1991
  - minimal polynomial — Wiedemann 1986

Algorithmic paradigms

Given a problem, how to find an efficient algorithm for its solution?

Five paradigms for algorithmic design

- divide and conquer (DAC)
- decrease and conquer (dac)
- baby steps / giant steps (BS-GS)
- change of representation (CR)
- Tellegen's transposition principle (Tellegen)

Algorithmic paradigms, and main techniques

Given a problem, how to find an efficient algorithm for its solution?

Five paradigms for algorithmic design

- divide and conquer
- decrease and conquer
  - binary powering
  - Newton iteration
  - Keller-Gehrig iteration
- baby steps / giant steps
- change of representation
  - evaluation-interpolation
  - expansion-reconstruction
- Tellegen's transposition principle

Divide and conquer

Idea: recursively break down a problem into two or more similar subproblems, solve them, and combine their solutions

Origin: unknown, probably very ancient.
Modern form: merge sort algorithm — von Neumann 1945

Our main examples:

• Karatsuba algorithm	polynomial multiplication
• Strassen algorithm	matrix product
• Strassen algorithm	matrix inversion
• Borodin-Moenck algorithm	polynomial evaluation-interpolation
• Beckermann-Labahn algorithm	Hermite-Padé approximation
• Bitmead-Anderson-Morf algorithm	solving Toeplitz-like linear systems
• Lehmer-Knuth-Schönhage-Moenck-Strassen algorithm	extended gcd

Decrease and conquer

Idea: reduce each problem to only one similar subproblem of half size

Origin: probably Pingala's Hindu classic Chandah-sutra, 200 BC

Modern form: binary search algorithm Mauchly 1946

Our main examples:

- binary powering exponentiation in rings
- modular exponentiation exponentiation in quotient rings
  - N-th term of a recurrence with constant coefficients
- Newton iteration power series root-finding
  - polynomial division
  - composed sum and product
  - polynomial shift
- Kehler-Gehrig algorithm Krylov sequence computation
- Storjohann's high order lifting algorithm polynomial matrices
- B-Schost algorithm interpolation on geometric sequences

Baby steps / giant steps

Idea: reduce a problem of size N to two similar subproblem of size $\sqrt{N}$

Origin: computational number theory, ≈ 1960

Modern form: discrete logarithm problem — Shanks 1969

Our main examples:

- Paterson-Stockmeyer 1973 — polynomial evaluation in an algebra
- Strassen 1976 — deterministic integer factorization
- Brent-Kung 1978 — composition of power series
- Chudnovsky-Chudnovsky 1987 — N-th term of a P-recursive sequence
  - point counting on hyperelliptic curves
  - polynomial solutions of linear differential equations
  - p-curvature of linear differential operators
- Shoup 1995 — power projection $[\ell(1), \ell(u), \ldots, \ell(u^{N-1})]$

Change of representation

Idea: represent objects in a different way, mathematically equivalent, but better suited for the algorithmic treatment

Origin: unknown, probably Sun Zi ≈ 300 (Chinese remainder theorem)
Modern form: the Czech number system Svoboda-Valach 1955

Our main examples: One can represent

- a polynomial by
  - the list of its coefficients
  - the values it takes at sufficiently many points easy $\times$
  - its Newton sums (= powersums of roots) easy $\otimes, \oplus$
- a rational fraction by
  - the coefficient lists of its denominator and numerator
  - its values at sufficiently many points
  - its Taylor series expansion

Tellegen's transposition principle

Idea: to solve a linear problem, find an algorithm for its dual, and transpose it

Origin: electrical network theory: Tellegen, Bordewijk, ≈ 1950
Modern form: transposition of algorithms, complexity version Fiduccia 1972

Our main examples:

- improve algorithms by constant factors
  - Hanrot-Quercia-Zimmermann 2002 middle product for polynomials
  - B-Lecerf-Schost 2003 multipoint evaluation and interpolation
- prove computational equivalence between problems
  - B-Schost 2004 multipoint evaluation $\Leftrightarrow$ interpolation
- discover new algorithms
  - B-Salvy-Schost 2008 base conversions
- understand (connections between) existing algorithms
  - DFT: decimation in time vs. decimation in frequency
  - Strassen's polynomial division vs. Shoup's extension of recurrences

The Master Theorem

Suppose that the complexity $\mathsf{C}(n)$ of an algorithm satisfies

$$\mathsf{C}(n) \leq s \cdot \mathsf{C}\left(\frac{n}{2}\right) + \mathsf{T}(n),$$

where the function T is such that $q\mathsf{T}(n) \leq \mathsf{T}(2n)$. Then, for $n \to \infty$

$$\mathsf{C}(n) \; = \begin{cases} O(\mathsf{T}(n)), & \text{if } s < q, \\ O(\mathsf{T}(n) \log n), & \text{if } s = q, \\ O\left(n^{\log s} \frac{\mathsf{T}(n)}{n^{\log q}}\right), & \text{if } s > q. \end{cases}$$

Proof:

$$\begin{aligned} \mathsf{C}(n) &\leq \mathsf{T}(n) + s \cdot \mathsf{C}\left(\frac{n}{2}\right) \\ &\leq \mathsf{T}(n) + s \cdot \mathsf{T}\left(\frac{n}{2}\right) + \cdots + s^{k-1} \cdot \mathsf{T}\left(\frac{n}{2^{k-1}}\right) + s^k \cdot \mathsf{C}\left(\frac{n}{2^k}\right) \\ &\leq \mathsf{T}(n) \cdot \left(1 + \frac{s}{q} + \cdots + \left(\frac{s}{q}\right)^{\log(n)-1}\right) + s^{\log n} \cdot \mathsf{C}(1) \end{aligned}$$

The Master Theorem, main consequences

Corollary DFT / Karatsuba

$$\mathsf{C}(n) \le s \cdot \mathsf{C}\left(\frac{n}{2}\right) + O(n) \qquad \Longrightarrow \qquad \mathsf{C}(n) = \begin{cases} O(n \log n), & \text{if } s = 2 \\ O(n^{\log s}), & \text{if } s \ge 3 \end{cases}$$

Corollary Newton / evaluation-interpolation

$$\mathsf{C}(n) \le s \cdot \mathsf{C}\left(\frac{n}{2}\right) + O(\mathsf{M}(n)) \qquad \Longrightarrow \qquad \mathsf{C}(n) = \begin{cases} O(\mathsf{M}(n)), & \text{if } s = 1 \\ O(\mathsf{M}(n) \log n), & \text{if } s = 2 \end{cases}$$

Corollary Strassen's matrix product

$$\mathsf{C}(n) \le s \cdot \mathsf{C}\left(\frac{n}{2}\right) + O(n^2), \quad (s \ge 5) \qquad \Longrightarrow \qquad \mathsf{C}(n) = O(n^{\log s})$$

Corollary Strassen's matrix inversion

$$\mathsf{C}(n) \le s \cdot \mathsf{C}\left(\frac{n}{2}\right) + O(\mathsf{MM}(n)), \quad (s \le 3) \qquad \Longrightarrow \qquad \mathsf{C}(n) = O(\mathsf{MM}(n))$$

Divide and conquer

Karatsuba's algorithm

Gauss's trick ($\approx$1800) The product of two complex numbers can be computed using only 3 real multiplications

$$(ai+b)(ci+d) = (ad+bc)i + (bd-ac) = ((a+b)(c+d) - bd - ac)i + (bd-ac)$$

Kolmogorov (1956) n^2 conjecture: n^2 ops. are needed to multiply n-digit integers

Karatsuba (1960) disproof of the Kolmogorov conjecture

$\longrightarrow$ first DAC algorithm in Computer algebra; it combines Gauss's trick (on polynomials) with the power of recursion

$$(ax^{n/2}+b)(cx^{n/2}+d) = acx^n + ((a+b)(c+d) - bd - ac)x^{n/2} + bd$$

Master Theorem: $\mathsf{K}(n) = 3\cdot\mathsf{K}(n/2) + O(n) \implies \mathsf{K}(n) = O(n^{\log(3)}) = O(n^{1.59})$

The idea behind the trick

Let $f = ax + b$, $g = cx + d$. Compute $h = fg$ by evaluation-interpolation:

Evaluation:

$$\begin{aligned} b &= f(0) & \qquad d &= g(0) \\ a+b &= f(1) & \qquad c+d &= g(1) \\ a &= f(\infty) & \qquad c &= g(\infty) \end{aligned}$$

Multiplication:

$$\begin{aligned} h(0) &= f(0)\cdot g(0) \\ h(1) &= f(1)\cdot g(1) \\ h(\infty) &= f(\infty)\cdot g(\infty) \end{aligned}$$

Interpolation:

$$h = h(0) + (h(1) - h(0) - h(\infty))\,x + h(\infty)\,x^2$$

Toom's algorithm

Let

$$f = f_0 + f_1 x + f_2 x^2, \quad g = g_0 + g_1 x + g_2 x^2$$

and

$$h = fg = h_0 + h_1 x + h_2 x^2 + h_3 x^3 + h_4 x^4.$$

To get h, do again:

- evaluation,
- multiplication,
- interpolation.

Now, 5 values are needed, because h has 5 unknown coefficients:

- $0, 1, -1, 2, \infty$ other choices are possible
- would not work with coefficients in $\mathbb{F}_2$.

The evaluation / interpolation phase

Evaluation:

$$\begin{array}{lcl@{\qquad\qquad}lcl}
f(0) & = & f_0 & g(0) & = & g_0 \\
f(1) & = & f_0 + f_1 + f_2 & g(1) & = & g_0 + g_1 + g_2 \\
f(-1) & = & f_0 - f_1 + f_2 & g(-1) & = & g_0 - g_1 + g_2 \\
f(2) & = & f_0 + 2f_1 + 4f_2 & g(2) & = & g_0 + 2g_1 + 4g_2 \\
f(\infty) & = & f_2 & g(\infty) & = & g_2
\end{array}$$

Multiplication:

$$h(0) = f(0)g(0), \quad \ldots, \quad h(\infty) = f(\infty)g(\infty)$$

Interpolation: recover h from its values.

$\Longrightarrow$ one can multiply degree-2 polynomials using 5 products instead of 9

Master Theorem: $\mathsf{T}(n) = 5 \cdot \mathsf{T}(n/3) + O(n) \implies \mathsf{T}(n) = O(n^{\log_3(5)}) = O(n^{1.47})$

Generalization of Toom

Let

$$f = f_0 + f_1 x + \cdots + f_{\alpha-1} x^{\alpha-1}, \quad g = g_0 + g_1 x + \cdots + g_{\alpha-1} x^{\alpha-1}$$

and

$$h = fg = h_0 + h_1 x + \cdots + h_{2\alpha-2} x^{2\alpha-2}.$$

Analysis: at each step,

- divide n by α; number of terms in f, g
- and perform $2\alpha - 1$ recursive calls; number of terms in h
- the extra operations count is ℓn, for some ℓ.

Master theorem:

$$\mathsf{T}(n) = O(n^{\log_\alpha(2\alpha-1)}).$$

Examples:

$$\alpha = 100 \implies O(n^{1.15}), \quad \alpha = 1000 \implies O(n^{1.1}), \quad \alpha = 10000 \implies O(n^{1.07})$$

Discrete Fourier Transform (Gentleman-Sande 1966, decimation-in-frequency)

Problem: Given $n = 2^k$, $f \in \mathbb{K}[x]_{<n}$, and $\omega \in \mathbb{K}$ a primitive n-th root of unity, compute $(f(1), f(\omega), \ldots, f(\omega^{n-1}))$

Idea: $\omega = n$-th primitive root of 1 $\Longrightarrow \omega^2 = \frac{n}{2}$-th primitive root of 1, and

$$\begin{aligned} r_0(x) =& f(x) \bmod x^{n/2} - 1 && \Longrightarrow && f(\omega^{2j}) = r_0\left((\omega^2)^j\right) \\ r_1(x) =& f(x) \bmod x^{n/2} + 1 && \Longrightarrow && f(\omega^{2j+1}) = r_1(\omega^{2j+1}) = r_1(\omega x)\big|_{x=(\omega^2)^j} \end{aligned}$$

Moreover, $O(n)$ ops. are enough to get $r_0(x), r_1(x), r_1(\omega x)$ from $f(x)$

Master Theorem: $\mathsf{F}(n) = 2 \cdot \mathsf{F}(n/2) + O(n) \quad \Longrightarrow \quad \mathsf{F}(n) = O(n \log n)$

Discrete Fourier Transform (Cooley-Tukey 1965, decimation-in-frequency)

Problem: Given $n = 2^k$, $f \in \mathbb{K}[x]_{<n}$, and $\omega \in \mathbb{K}$ a primitive n-th root of unity, compute $(f(1), f(\omega), \ldots, f(\omega^{n-1}))$

Idea: Write $f = f_{\text{even}}(x^2) + x f_{\text{odd}}(x^2), \quad$ with $\deg(f_{\text{even}}), \deg(f_{\text{odd}}) < n/2$.

Then $f(\omega^j) = f_{\text{even}}(\omega^{2j}) + \omega^j f_{\text{odd}}(\omega^{2j})$, and $(\omega^{2j})_{0\le j<n} = \frac{n}{2}$-roots of 1.

Master Theorem: $\quad \mathsf{F}(n) = 2 \cdot \mathsf{F}(n/2) + O(n) \quad \Longrightarrow \quad \mathsf{F}(n) = O(n \log n)$

Inverse DFT

Problem: Given $n = 2^k$, $v_0, \ldots, v_{n-1} \in \mathbb{K}$ and $\omega \in \mathbb{K}$ a primitive n-th root of unity, compute $f \in \mathbb{K}[x]_{<n}$ such that $f(1) = v_0, \ldots, f(\omega^{n-1}) = v_{n-1}$

- $V_\omega \cdot V_{\omega^{-1}} = n \cdot I_n \;\rightarrow\;$ performing the inverse DFT in size n amounts to:
  - performing a DFT at
$$\frac{1}{1}, \;\; \frac{1}{\omega}, \;\; \ldots, \;\; \frac{1}{\omega^{n-1}}$$
  - dividing the results by n.
- this new DFT is the same as before:
$$\frac{1}{\omega^i} = \omega^{n-i},$$
  so the outputs are just shuffled.

Consequence: the cost of the inverse DFT is $O(n \log(n))$

FFT polynomial multiplication

Suppose the basefield $\mathbb{K}$ contains enough roots of unity

To multiply two polynomials f, g in $\mathbb{K}[x]$, of degrees $< n$:

- find $N = 2^k$ such that $h = fg$ has degree less than N $\quad N \leq 4n$
- compute DFT(f, N) and DFT(g, N) $\quad O(N \log(N))$
- multiply the values to get DFT(h, N) $\quad O(N)$
- recover h by inverse DFT $\quad O(N \log(N))$

Cost: $O(N \log(N)) = O(n \log(n))$

General case: Create artificial roots of unity $\quad O(n \log(n) \log \log n)$

Strassen's matrix multiplication algorithm

Same idea as for Karatsuba's algorithm: trick in low size + recursion

Additional difficulty: Formulas should be non-commutative

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \times \begin{bmatrix} x & y \\ z & t \end{bmatrix} \iff \begin{bmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & c & d \end{bmatrix} \times \begin{bmatrix} x \\ z \\ y \\ t \end{bmatrix}$$

Crucial remark: If $\varepsilon \in \{0,1\}$ and $\alpha \in \mathbb{K}$, then 1 multiplication suffices for $E \cdot v$, where v is a vector, and E is a matrix of one of the following types:

$$\begin{bmatrix} \alpha & \alpha \\ \varepsilon\alpha & \varepsilon\alpha \end{bmatrix}, \begin{bmatrix} \alpha & -\alpha \\ \varepsilon\alpha & -\varepsilon\alpha \end{bmatrix}, \begin{bmatrix} \alpha & \varepsilon\alpha \\ -\alpha & -\varepsilon\alpha \end{bmatrix}$$

Strassen's matrix multiplication algorithm

Problem: Write

$$M = \begin{bmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & c & d \end{bmatrix}$$

as a sum of less than 8 elementary matrices.

$$M - \underbrace{\begin{bmatrix} a & a & & \\ a & a & & \\ & & & \\ & & & \end{bmatrix}}_{E_1} - \underbrace{\begin{bmatrix} & & & \\ & & & \\ & & d & d \\ & & d & d \end{bmatrix}}_{E_2} = \begin{bmatrix} & b-a & & \\ c-a & d-a & & \\ & & a-d & b-d \\ & & c-d & \end{bmatrix}$$

Strassen's matrix multiplication algorithm

Problem: Write

$$M = \begin{bmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & c & d \end{bmatrix}$$

as a sum of less than 8 elementary matrices.

$$M - E_1 - E_2 = \underbrace{\begin{bmatrix} & & & \\ & d-a & a-d & \\ & d-a & a-d & \\ & & & \end{bmatrix}}_{E_3} - \begin{bmatrix} & b-a & & \\ c-a & & d-a & \\ & a-d & & b-d \\ & & c-d & \end{bmatrix}$$

Strassen's matrix multiplication algorithm

Problem: Write

$$M = \begin{bmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & c & d \end{bmatrix}$$

as a sum of less than 8 elementary matrices.

$$M - E_1 - E_2 - E_3 = \begin{bmatrix} & b-a & & \\ & & & \\ & a-d & & b-d \\ & & & \end{bmatrix} + \begin{bmatrix} & & & \\ c-a & & d-a & \\ & & & \\ & & c-d & \end{bmatrix}$$

Strassen's matrix multiplication algorithm

Problem: Write

$$M = \begin{bmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & c & d \end{bmatrix}$$

as a sum of less than 8 elementary matrices.

$$M - E_1 - E_2 - E_3 = \underbrace{\begin{bmatrix} & b-a & & \\ & & & \\ & (b-d)-(b-a) & & b-d \\ & & & \end{bmatrix}}_{E_4 \quad + \quad E_5} + \underbrace{\begin{bmatrix} & & & \\ c-a & & (c-a)-(c-d) & \\ & & & \\ & & c-d & \end{bmatrix}}_{E_6 \quad + \quad E_7}$$

Strassen's matrix multiplication algorithm

Problem: Write

$$M = \begin{bmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & c & d \end{bmatrix}$$

as a sum of less than 8 elementary matrices.

Conclusion

$$M = E_1 + E_2 + E_3 + E_4 + E_5 + E_6 + E_7$$

$\implies$ one can multiply 2×2 matrices using 7 products instead of 8

Master Theorem:

$\mathsf{MM}(r) = 7 \cdot \mathsf{MM}(r/2) + O(r^2) \quad \implies \quad \mathsf{MM}(r) = O(r^{\log_2(7)}) = O(r^{2.81})$

Inversion of dense matrices

[Strassen, 1969]

To invert a dense matrix $A = \begin{bmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{bmatrix} \in \mathcal{M}_r(\mathbb{K})$:

1. Invert $A_{1,1}$ (recursively)
2. Compute the Schur complement $\Delta := A_{2,2} - A_{2,1} A_{1,1}^{-1} A_{1,2}$
3. Invert Δ (recursively)
4. Recover the inverse of A as

$$A^{-1} = \begin{bmatrix} I & -A_{1,1}^{-1} A_{1,2} \\ & I \end{bmatrix} \times \begin{bmatrix} A_{1,1}^{-1} & \\ & \Delta^{-1} \end{bmatrix} \times \begin{bmatrix} I & \\ -A_{2,1} A_{1,1}^{-1} & I \end{bmatrix}$$

Master Theorem: $\mathsf{C}(r) = 2 \cdot \mathsf{C}\left(\frac{r}{2}\right) + O(\mathsf{MM}(r)) \quad \Longrightarrow \quad \mathsf{C}(r) = O(\mathsf{MM}(r))$

Corollary: inversion A^{-1} and system solving $A^{-1}b$ in time $O(\mathsf{MM}(r))$

Subproduct tree

Problem: Given $a_0, \dots, a_{n-1} \in \mathbb{K}$, compute $A = \prod_{i=0}^{n-1}(x - a_i)$

$$\begin{tikzcd}
 & & & & \boxed{A = \prod_{i=0}^{n-1}(x-a_i)} & & & & \\
 & & \boxed{B_0 = \prod_{i=0}^{n/2-1}(x-a_i)} \arrow[urr] & & & & \boxed{B_1 = \prod_{i=n/2}^{n-1}(x-a_i)} \arrow[ull] & & \\
 & \vdots \arrow[ur] & & \vdots \arrow[ul] & & \vdots \arrow[ur] & & \vdots \arrow[ul] & \\
 & \boxed{(x-a_0)(x-a_1)} & & & & & & \boxed{(x-a_{n-2})(x-a_{n-1})} & \\
\boxed{x-a_0} \arrow[ur] & & \boxed{x-a_1} \arrow[ul] & & \cdots & & \boxed{x-a_{n-2}} \arrow[ur] & & \boxed{x-a_{n-1}} \arrow[ul]
\end{tikzcd}$$

Master Theorem: $\mathsf{C}(n) = 2 \cdot \mathsf{C}(n/2) + O(\mathsf{M}(n)) \quad \Longrightarrow \quad \mathsf{C}(n) = O(\mathsf{M}(n) \log n)$

Fast multipoint evaluation

[Borodin-Moenck, 1974]

Pb: Given $a_0, \dots, a_{n-1} \in \mathbb{K}$ and $P \in \mathbb{K}[x]_{<n}$, compute $P(a_0), \dots, P(a_{n-1})$

Naive algorithm: Compute $P(a_i)$ independently $O(n^2)$

Basic idea: Use recursively Bézout's identity $P(a) = P(x) \bmod (x - a)$

Divide and conquer: Same idea as for DFT = evaluation by repeated division

- $P_0 = P \mod (x - a_0) \cdots (x - a_{n/2-1})$
- $P_1 = P \mod (x - a_{n/2}) \cdots (x - a_{n-1})$

$$\Longrightarrow \begin{cases} P_0(a_0) = P(a_0), \quad \dots, \quad P_0(a_{n/2-1}) = P(a_{n/2-1}) \\ P_1(a_{n/2}) = P(a_{n/2}), \quad \dots, \quad P_1(a_{n-1}) = P(a_{n-1}) \end{cases}$$

Fast multipoint evaluation

[Borodin-Moenck, 1974]

Pb: Given $a_0, \ldots, a_{n-1} \in \mathbb{K}$ and $P \in \mathbb{K}[x]_{<n}$, compute $P(a_0), \ldots, P(a_{n-1})$

$$\begin{tikzcd}
 & & & \boxed{P} \arrow[dll, "\mathrm{mod}(\cdot, B_0)"'] \arrow[drr, "\mathrm{mod}(\cdot, B_1)"] & & & \\
 & \boxed{P \bmod B_0} \arrow[dl, "\mathrm{mod}"'] \arrow[dr, "\mathrm{mod}"] & & & & \boxed{P \bmod B_1} \arrow[dl, "\mathrm{mod}"'] \arrow[dr, "\mathrm{mod}"] & \\
\vdots & & \vdots & & \vdots & & \vdots \\
 & \boxed{P \bmod (x-a_0)(x-a_1)} \arrow[dl, "\mathrm{mod}"'] \arrow[dr, "\mathrm{mod}"] & & & & \boxed{P \bmod (x-a_{n-2})(x-a_{n-1})} \arrow[dl, "\mathrm{mod}"'] \arrow[dr, "\mathrm{mod}"] & \\
\boxed{P \bmod (x-a_0)} & & \boxed{P \bmod (x-a_1)} & \cdots & \boxed{P \bmod (x-a_{n-2})} & & \boxed{P \bmod (x-a_{n-1})}
\end{tikzcd}$$

Master Theorem: $\mathsf{C}(n) = 2 \cdot \mathsf{C}(n/2) + O(\mathsf{M}(n)) \implies \mathsf{C}(n) = O(\mathsf{M}(n) \log n)$

Fast interpolation

[Borodin-Moenck, 1974]

Problem: Given $a_0, \dots, a_{n-1} \in \mathbb{K}$ mutually distinct, and $v_0, \dots, v_{n-1} \in \mathbb{K}$, compute $P \in \mathbb{K}[x]_{<n}$ such that $P(a_0) = v_0, \dots, P(a_{n-1}) = v_{n-1}$

Naive algorithm: Linear algebra, Vandermonde system $O(\mathsf{MM}(n))$

Lagrange's algorithm: Use $P(x) = \sum_{i=0}^{n-1} v_i \frac{\prod_{j \neq i}(x - a_j)}{\prod_{j \neq i}(a_i - a_j)}$ $O(n^2)$

Fast algorithm: Modified Lagrange formula

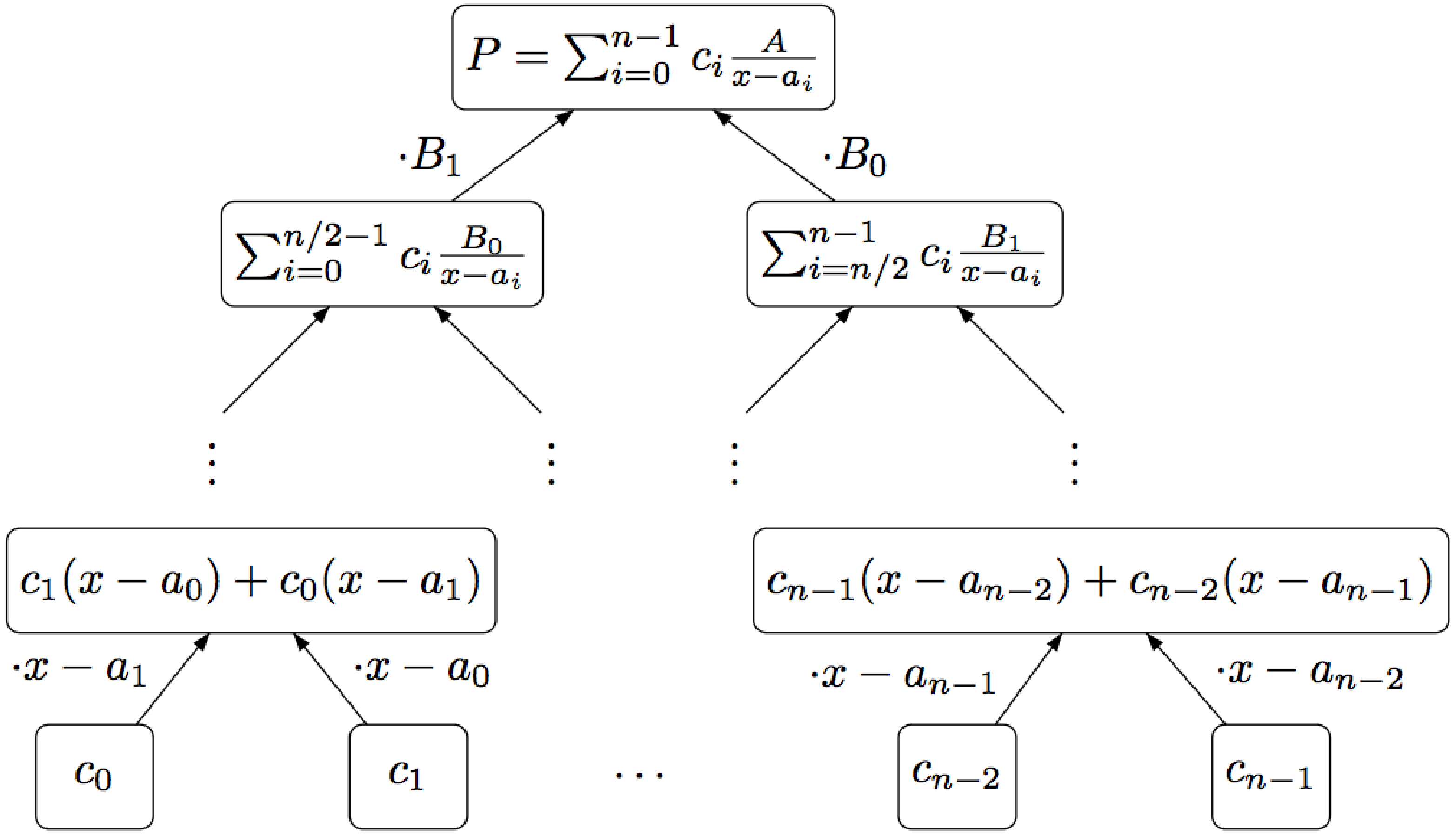
$$P = A(x) \cdot \sum_{i=0}^{n-1} \frac{v_i / A'(a_i)}{x - a_i}$$

- Compute $c_i = v_i / A'(a_i)$ by fast multipoint evaluation $O(\mathsf{M}(n) \log n)$
- Compute $\sum_{i=0}^{n-1} \frac{c_i}{x - a_i}$ by divide and conquer $O(\mathsf{M}(n) \log n)$

Fast interpolation

[Borodin-Moenck, 1974]

Problem: Given $a_0, \dots, a_{n-1} \in \mathbb{K}$ mutually distinct, and $v_0, \dots, v_{n-1} \in \mathbb{K}$, compute $P \in \mathbb{K}[x]_{<n}$ such that $P(a_0) = v_0, \dots, P(a_{n-1}) = v_{n-1}$



$P = \sum_{i=0}^{n-1} c_i \frac{A}{x-a_i}$
$\cdot B_1$
$\cdot B_0$
$\sum_{i=0}^{n/2-1} c_i \frac{B_0}{x-a_i}$
$\sum_{i=n/2}^{n-1} c_i \frac{B_1}{x-a_i}$
$c_1(x-a_0) + c_0(x-a_1)$
$c_{n-1}(x-a_{n-2}) + c_{n-2}(x-a_{n-1})$
$\cdot x - a_1$
$\cdot x - a_0$
$\cdot x - a_{n-1}$
$\cdot x - a_{n-2}$
c_0
c_1
$\cdots$
c_{n-2}
c_{n-1}

Master Theorem: $\mathsf{C}(n) = 2 \cdot \mathsf{C}(n/2) + O(\mathsf{M}(n)) \implies \mathsf{C}(n) = O(\mathsf{M}(n) \log n)$

Decrease and conquer I

Evaluation-interpolation, geometric case

Subproduct tree, geometric case

[B-Schost, 2005]

Problem: Given $q \in \mathbb{K}$, compute $A = \prod_{i=0}^{n-1}(x - q^i)$

Idea: Compute $B_1 = \prod_{i=n/2}^{n-1} (x - q^i)$ from $B_0 = \prod_{i=0}^{n/2-1} (x - q^i)$, by a homothety

$$B_1(x) = B_0\left(\frac{x}{q^{n/2}}\right) \cdot q^{(n/2)^2}$$

Decrease and conquer:

- Compute $B_0(x)$ by a recursive call
- Deduce $B_1(x)$ from $B_0(x)$ $\quad O(n)$
- Return $A(x) = B_0(x)B_1(x)$ $\quad \mathsf{M}(n/2)$

Master Theorem: $\mathsf{C}(n) = \mathsf{C}(n/2) + O(\mathsf{M}(n)) \quad \Longrightarrow \quad \mathsf{C}(n) = O(\mathsf{M}(n))$

Fast multipoint evaluation, geometric case

[Bluestein, 1970]

Problem: Given $q \in \mathbb{K}$ and $P \in \mathbb{K}[x]_{<n}$, compute $P(1), P(q), \ldots, P(q^{n-1})$

The needed values are: $$P(q^i) = \sum_{j=0}^{n-1} c_j q^{ij}, \qquad 0 \le i < n$$

Bluestein's trick: $$ij = \frac{(i+j)^2 - i^2 - j^2}{2} \quad \Longrightarrow \quad q^{ij} = q^{(i+j)^2/2} \cdot q^{-i^2/2} \cdot q^{-j^2/2}$$

$$\Longrightarrow \qquad P(q^i) = q^{-i^2/2} \cdot \underbrace{\sum_{j=0}^{n-1} c_j q^{-j^2/2} \cdot q^{(i+j)^2/2}}_{\text{convolution:}}$$

$$\left[x^{n-1+i}\right] \left(\sum_{k=0}^{n-1} c_k q^{-k^2/2} x^{n-k-1} \right) \left(\sum_{\ell=0}^{2n-2} q^{\ell^2/2} x^{\ell} \right)$$

Conclusion: Fast evaluation on a geometric sequence in $O(\mathsf{M}(n))$

Fast interpolation, geometric case

[B-Schost, 2005]

Problem: Given $q \in \mathbb{K}$, and $v_0, \dots, v_{n-1} \in \mathbb{K}$, compute $P \in \mathbb{K}[x]_{<n}$ such that $P(1) = v_0, \dots, P(q^{n-1}) = v_{n-1}$

Fast algorithm: Modified Lagrange formula

$$P = A(x) \cdot \sum_{i=0}^{n-1} \frac{v_i / A'(q^i)}{x - q^i}, \qquad A = \prod_i (x - q^i)$$

- Compute $\prod_{i=0}^{n-1} (x - q^i)$ by decrease and conquer $O(\mathsf{M}(n))$
- Compute $c_i = v_i / A'(q^i)$ by Bluestein's algorithm $O(\mathsf{M}(n))$
- Compute $\sum_{i=0}^{n-1} \frac{c_i}{x - q^i}$ by decrease and conquer $O(\mathsf{M}(n))$

Fast interpolation, geometric case

[B-Schost, 2005]

Problem: Given $q \in \mathbb{K}$, and $v_0, \ldots, v_{n-1} \in \mathbb{K}$, compute $P \in \mathbb{K}[x]_{<n}$ such that $P(1) = v_0, \ldots, P(q^{n-1}) = v_{n-1}$

Subproblem: Given $c_0, \ldots, c_{n-1} \in \mathbb{K}$, compute $\quad R(x) = \sum_{i=0}^{n-1} \frac{c_i}{x - q^i}$

Idea: change of representation – enough to compute $R \mod x^n$

Second idea: $R \bmod x^n$ = multipoint evaluation at $\{1, q^{-1}, \ldots, q^{-(n-1)}\}$:

$$\sum_{i=0}^{n-1} \frac{c_i}{x - q^i} \mod x^n = -\sum_{i=0}^{n-1} \left(\sum_{j=0}^{n-1} c_i q^{-i(j+1)} x^j \right) = -\sum_{j=0}^{n-1} C(q^{-j-1}) x^j$$

Conclusion: Algorithm for interpolation at a geometric sequence in $O(\mathsf{M}(n))$ (generalization of the IDFT)

Product of polynomial matrices

[B-Schost, 2005]

Problem: Given $A, B \in \mathcal{M}_r(\mathbb{K}[x]_{<n})$, compute $C = AB$

Idea: change of representation – evaluation-interpolation at a geometric sequence $\mathcal{G} = \{1, q, q^2, \ldots, q^{2n-2}\}$

- Evaluate A and B at $\mathcal{G}$ $\quad O(r^2\,\mathsf{M}(n))$
- Multiply values $C(v) = A(v)B(v)$ for $v \in \mathcal{G}$ $\quad O(n\,\mathsf{MM}(r))$
- Interpolate C from values $\quad O(r^2\,\mathsf{M}(n))$

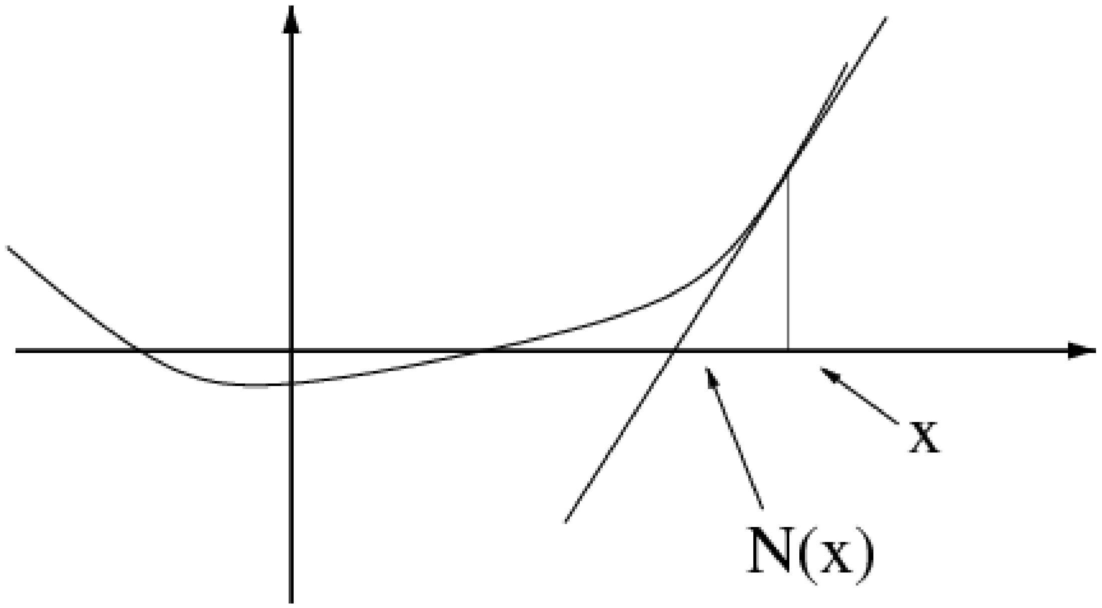
Total complexity $\quad O(r^2\,\mathsf{M}(n) + n\,\mathsf{MM}(r))$

Decrease and conquer II

Newton iteration

Newton's tangent method: real case

[Newton, 1671]



$$\mathbf{x_{\kappa+1} = N(x_\kappa) = x_\kappa - (x_\kappa^2 - 2)/(2x_\kappa), \quad x_0 = 1.5}$$

$$\mathbf{x_1 = 1.41666666666666666666666666666667}$$

$$\mathbf{x_2 = 1.4142156862745098039215686274510}$$

$$\mathbf{x_3 = 1.41421356237468991062629557889013}$$

$$\mathbf{x_4 = 1.41421356237309504880168962350253}$$

Newton's tangent method: power series case

Let $\varphi : \mathbb{K}[[x]] \to \mathbb{K}[[x]]$. To solve $\varphi(g) = 0$ in $\mathbb{K}[[x]]$, iterate

$$g_{\kappa+1} = g_\kappa - \frac{\varphi(g_\kappa)}{\varphi'(g_\kappa)} \mod x^{2^{\kappa+1}}$$

- The number of correct coefficients doubles after each iteration
- Total cost $\quad = \quad \mathbf{2} \times \Big($ the cost of the last iteration $\Big)$

Theorem [Cook (1966), Sieveking (1972) & Kung (1974), Brent (1975)]
Division, logarithm and exponential of power series in $\mathbb{K}[[x]]$ *can be computed at precision* $\mathbf{N}$ *using* $\mathbf{O}(\mathsf{M}(\mathbf{N}))$ operations in $\mathbb{K}$

Division and logarithm of power series

[Sieveking-Kung, 1972]

To compute the reciprocal of $f \in \mathbb{K}[[x]]$, choose $\varphi(g) = 1/g - f$:

$$g_0 = \frac{1}{f_0} \quad \text{and} \quad g_{\kappa+1} = g_\kappa + g_\kappa(1 - f g_\kappa) \quad \text{mod } x^{2^{\kappa+1}} \qquad \text{for } \kappa \geq 0$$

Master Theorem: $\mathsf{C}(N) = \mathsf{C}(N/2) + O(\mathsf{M}(N)) \qquad \Longrightarrow \qquad \mathsf{C}(N) = O(\mathsf{M}(N))$

Corollary: division of power series at precision N in $O(\mathsf{M}(N))$

Corollary: Logarithm $\log(f) = -\sum_{i\geq 1} \frac{(1-f)^i}{i}$ of $f \in 1 + x\mathbb{K}[[x]]$ in $O(\mathsf{M}(N))$:

- compute the Taylor expansion of $h = f'/f$ modulo x^{N-1} $\qquad O(\mathsf{M}(N))$
- take the antiderivative of h $\qquad O(N)$

Application: polynomial division

[Strassen, 1973]

Pb: Given $F, G \in \mathbb{K}[x]_{\leq N}$, compute (Q, R) in Euclidean division $F = QG + R$

Naive algorithm: $O(N^2)$

Idea: look at $F = QG + R$ from the infinity: $Q \sim_{+\infty} F/G$

Let $N = \deg(F)$ and $n = \deg(G)$. Then $\deg(Q) = N - n$, $\deg(R) < n$ and

$$\underbrace{F(1/x)x^N}_{\mathrm{rev}(F)} = \underbrace{G(1/x)x^n}_{\mathrm{rev}(G)} \cdot \underbrace{Q(1/x)x^{N-n}}_{\mathrm{rev}(Q)} + \underbrace{R(1/x)x^{\deg(R)}}_{\mathrm{rev}(R)} \cdot x^{N-\deg(R)}$$

Algorithm:

- Compute $\mathrm{rev}(Q) = \mathrm{rev}(F)/\mathrm{rev}(G) \mod x^{N-n+1}$ $O(\mathsf{M}(N))$
- Recover Q $O(N)$
- Deduce $R = F - QG$ $O(\mathsf{M}(N))$

Application: extension of recurrences

[Shoup, 1991]

Problem: Given $N \in \mathbb{N}$ and the first n terms $u_0, \ldots, u_{n-1}$ of a recurrent sequence with constant coefficients of order n, compute $u_n, \ldots, u_N$

Naive algorithm: unroll the recurrence $O(N^2)$

Idea: $\sum_{i \geq 0} u_i x^i$ is rational $A(x)/B(x)$, with B given by the input recurrence, and $\deg(A) < \deg(B)$

Example (Fibonacci): $F_{i+2} = F_{i+1} + F_i \quad \Longleftrightarrow \quad \sum_i F_i x^i = \frac{F_0 + (F_1 - F_0)x}{1 - x - x^2}$

Algorithm:

- Compute A from B and $u_0, \ldots, u_{n-1}$ $O(\mathsf{M}(n))$
- Expand A/B modulo x^{N+1} $O(\mathsf{M}(N))$

Exponentials of power series and 1st order LDE

[Brent, 1975]

To compute the exponential $\exp(f) = \sum_{i\geq 0} \frac{f^i}{i!}$, choose $\varphi(g) = \log(g) - f$:

$$g_0 = 1 \quad \text{and} \quad g_{\kappa+1} = g_\kappa - g_\kappa \left(\log(g_\kappa) - f\right) \mod x^{2^{\kappa+1}} \quad \text{for } \kappa \geq 0.$$

Master Theorem: $\mathsf{C}(N) = \mathsf{C}(N/2) + O(\mathsf{M}(N)) \implies \mathsf{C}(N) = O(\mathsf{M}(N))$

Corollary: Solve first order linear differential equations $af' + bf = c$ in $O(\mathsf{M}(N))$

- if $c = 0$ then the solution is $f_0 = \exp\left(-\int b/a\right)$ $\quad O(\mathsf{M}(N))$
- else, variation of constants: $f = f_0 g$, where $g' = c/(af_0)$ $\quad O(\mathsf{M}(N))$

▶ main difficulty for higher orders: for non-commutativity reasons, the matrix exponential $Y(x) = \exp(\int A(x))$ is not a solution of $Y' = A(x)Y$.

Application: conversion coefficients $\leftrightarrow$ power sums

[Schönhage, 1982]

Any polynomial $F = x^n + a_1 x^{n-1} + \cdots + a_n$ in $\mathbb{K}[x]$ can be represented by its first n power sums $S_i = \sum_{F(\alpha)=0} \alpha^i$

Conversions coefficients $\leftrightarrow$ power sums can be performed

- either in $O(n^2)$ using Newton identities (naive way):

$$i a_i + S_1 a_{i-1} + \cdots + S_i = 0, \quad 1 \leq i \leq n$$

- or in $O(\mathsf{M}(n))$ using generating series

$$\frac{\mathsf{rev}(F)'}{\mathsf{rev}(F)} = -\sum_{i\geq 0} S_{i+1} x^i \quad \Longleftrightarrow \quad \mathsf{rev}(F) = \exp\left(-\sum_{i\geq 1} \frac{S_i}{i} x^i\right)$$

Application: special bivariate resultants

[B-Flajolet-Salvy-Schost, 2006]

Composed products and sums: manipulation of algebraic numbers

$$F \otimes G = \prod_{F(\alpha)=0, G(\beta)=0} (x - \alpha\beta), \quad F \oplus G = \prod_{F(\alpha)=0, G(\beta)=0} (x - (\alpha+\beta))$$

Output size: $N = \deg(F)\deg(G)$

Linear algebra: χ_{xy}, χ_{x+y} in $\mathbb{K}[x,y]/(F(x), G(y))$ $O(\mathsf{MM}(N))$

Resultants: $\mathrm{Res}_y\left(F(y), y^{\deg(G)} G(x/y)\right), \ \mathrm{Res}_y\left(F(y), G(x-y)\right)$ $O(N^{1.5})$

Better: $\otimes$ and $\oplus$ are easy in Newton representation $O(\mathsf{M}(N))$

$$\sum \alpha^s \sum \beta^s = \sum (\alpha\beta)^s \qquad \text{and}$$

$$\sum \frac{\sum(\alpha+\beta)^s}{s!} x^s = \left(\sum \frac{\sum \alpha^s}{s!} x^s\right)\left(\sum \frac{\sum \beta^s}{s!} x^s\right)$$

Corollary: Fast polynomial shift $P(x+a) = P(x) \oplus (x+a)$ $O(\mathsf{M}(\deg(P)))$

Newton for differential systems – a glimpse

The previous iteration for $\exp(f)$ rewrites

$$g_0 = 1 \quad \text{and} \quad g_{\kappa+1} = g_\kappa - g_\kappa \int g_\kappa^{-1} \left(g_\kappa' - f' g_\kappa\right) \mod x^{2^{\kappa+1}}.$$

➥ It computes simultaneously $\exp(f)$ and $\exp(-f)$.

Idea [BCOSSS, 2007]: the similar iteration on polynomial matrices

$$Y_0 = I_r \quad \text{and} \quad Y_{\kappa+1} = Y_\kappa - Y_\kappa \int Y_\kappa^{-1} \left(Y_\kappa' - A Y_\kappa\right) \mod x^{2^{\kappa+1}}.$$

computes (truncated) solutions of $Y' = A(x)Y$ and of $Z' = -ZA(x)$.

Newton iteration on power series matrices

To solve an equation $\phi(Y) = 0$, with $\phi : \mathcal{M}_r(\mathbb{K}[[x]]) \to \mathcal{M}_r(\mathbb{K}[[x]])$:

Define the sequence $Y_{\kappa+1} = Y_\kappa - U_{\kappa+1}$, where

- $U_{\kappa+1}$ is a solution of valuation $\geq 2^{\kappa+1}$ of the linearized equation
$$D\phi|_{Y_\kappa} \cdot U = \phi(Y_\kappa),$$
- $D\phi|_{Y_\kappa}$ is the differential of ϕ at Y_κ.

Then, the sequence Y_κ converges quadratically to the solution Y.

Examples:

- inversion of power series matrices $A(x)^{-1}$
- quasi-exponential of power series matrices $Y' = A(x)Y$

Application: inversion of power series matrices

[Schulz, 1933]

To compute the inverse Z of a matrix of power series $Y \in \mathcal{M}_r(\mathbb{K}[[x]])$:

- choose the map $\phi : Z \mapsto I - YZ$ with differential $Z \mapsto -YZ$
- the equation for U becomes $\quad -YU = I - YZ_\kappa \mod x^{2^{\kappa+1}}$
- solution $\quad U = -Y^{-1}\,(I - YZ_\kappa) = -Z_\kappa\,(I - YZ_\kappa) \bmod x^{2^{\kappa+1}}$

This yields the following Newton-type iteration for Y^{-1}

$$Z_{\kappa+1} = Z_\kappa + Z_\kappa(I_r - YZ_\kappa) \mod x^{2^{\kappa+1}}$$

Master Theorem:

$$\mathsf{C}_{\mathrm{inv}}(N) = \mathsf{C}_{\mathrm{inv}}(N/2) + O(\mathsf{M}(r,N)) \qquad \Longrightarrow \qquad \mathsf{C}_{\mathrm{inv}}(N) = O(\mathsf{M}(r,N))$$

Application: quasi-exponential of power series matrices

[B-Chyzak-Ollivier-Salvy-Schost-Sedoglavic 2007]

To compute the solution $Y \in \mathcal{M}_r(\mathbb{K}[[x]])$ of the system $Y' = AY$

- choose the map $\phi : Y \mapsto Y' - AY$, with differential ϕ.
- the equation for U is $\quad U' - AU = Y'_\kappa - AY_\kappa \mod x^{2^{\kappa+1}}$
- the method of variation of constants yields the solution
  $U = Y_\kappa V_\kappa \bmod x^{2^{\kappa+1}}, \; Y'_\kappa - AY_\kappa = Y_\kappa V'_\kappa \bmod x^{2^{\kappa+1}}$

This yields the following Newton-type iteration for Y:

$$Y_{\kappa+1} = Y_\kappa - Y_\kappa \int Y_\kappa^{-1} \left(Y'_\kappa - AY_\kappa\right) \mod x^{2^{\kappa+1}}$$

Master Theorem:

$\mathsf{C}_{\text{solve}}(N) = \mathsf{C}_{\text{solve}}(N/2) + O(\mathsf{M}(r, N)) \quad \Longrightarrow \quad \mathsf{C}_{\text{solve}}(N) = O(\mathsf{M}(r, N))$

Decrease and conquer III

Recurrences with constant coefficients

Binary powering

Problem: Given a ring $\mathbb{A}$, $a \in \mathbb{A}$ and $N \geq 1$, compute a^N

Naive algorithm: $O(N)$

Better algorithm (Pingala, 200 BC): $O\big(\log N\big)$

Compute a^N recursively, using square-and-multiply

$$a^N = \begin{cases} (a^{N/2})^2, & \text{if } N \text{ is even,} \\ a \cdot (a^{\frac{N-1}{2}})^2, & \text{else.} \end{cases}$$

Modular exponentiation:

- $\mathbb{A} = \mathbb{Z}/A\mathbb{Z}$ N-th decimal of $1/A$ via $(10^N \bmod A)$ in $O\big(\log N\big)$
- $\mathbb{A} = \mathbb{K}[x]/(P)$ if $P, Q \in \mathbb{K}[x]_{<d}$, then $\big(Q^N \bmod P\big)$ in $O\big(\mathsf{M}(d) \log N\big)$

Fibonacci sequence: N-th term

$$F_{n+2} = F_{n+1} + F_n, \quad n \geq 0, \qquad F_0 = 1,\ F_1 = 1.$$

Naive algorithm: N ops. to compute F_N.

Folckore trick:

$$\begin{bmatrix} F_N \\ F_{N+1} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}}_{C} \begin{bmatrix} F_{N-1} \\ F_N \end{bmatrix} = C^N \begin{bmatrix} F_0 \\ F_1 \end{bmatrix}, \qquad N \geq 1.$$

Binary powering: compute C^N recursively, using

$$C^N = \begin{cases} (C^{N/2})^2, & \text{if } N \text{ is even,} \\ C \cdot (C^{\frac{N-1}{2}})^2, & \text{else.} \end{cases}$$

Cost: $O(\log N)$ products of 2×2 matrices $\longrightarrow$ $O(\log N)$ ops. to compute F_N.

Fibonacci sequence: N-th term

$$\begin{bmatrix} F_{n-2} & F_{n-1} \\ F_{n-1} & F_n \end{bmatrix} = C^n \quad\Longrightarrow\quad \begin{bmatrix} F_{2n-2} & F_{2n-1} \\ F_{2n-1} & F_{2n} \end{bmatrix} = \begin{bmatrix} F_{n-2} & F_{n-1} \\ F_{n-1} & F_n \end{bmatrix}^2$$

The previous algorithm computes (by squarings of 2×2 symetric matrices)

$$(F_0, F_1, F_2) \to (F_2, F_3, F_4) \to (F_6, F_7, F_8) \to (F_{14}, F_{15}, F_{16}) \to \dots$$

Cost: $5 \times$ and $3 +$ per arrow

Shortt's variant (1978): uses $\begin{cases} F_{2n-2} & = F_{n-2}^2 + F_{n-1}^2 \\ F_{2n-1} & = F_{n-1}^2 + 2F_{n-1}F_{n-2} \end{cases}$ and computes

$$(F_0, F_1) \to (F_2, F_3) \to (F_6, F_7) \to (F_{14}, F_{15}) \to \dots$$

Cost: $3 \times$ and $3 +$ per arrow

Fibonacci sequence: N-th term

Fiduccia's algorithm (1985): binary powering in the ring $\mathbb{K}[x]/(x^2-x-1)$:

$$\begin{aligned} C^n = &\quad \text{matrix of} \quad (x^n \bmod x^2-x-1) \\ \Longrightarrow \quad F_{n-2}+xF_{n-1} = &\quad x^n \bmod x^2-x-1 \end{aligned}$$

Cost: $O(\log N)$ products in $\mathbb{K}[x]/(x^2-x-1) \longrightarrow O(\log N)$ ops. for F_N

Explains Shortt's algorithm:

$$F_{2n-2}+xF_{2n-1} = (F_{n-2}+xF_{n-1})^2 \bmod x^2-x-1$$

N-th term, general case

$$a_{n+d} = c_{d-1}a_{n+d-1} + \cdots + c_0 a_n, \qquad n \geq 0,$$

rewrites

$$\underbrace{\begin{bmatrix} a_N \\ a_{N+1} \\ \vdots \\ a_{N+d-1} \end{bmatrix}}_{v_N} = \underbrace{\begin{bmatrix} & 1 & & \\ & & \ddots & \\ & & & 1 \\ c_0 & c_1 & \cdots & c_{d-1} \end{bmatrix}}_{C^T} \underbrace{\begin{bmatrix} a_{N-1} \\ a_N \\ \vdots \\ a_{N+d-2} \end{bmatrix}}_{v_{N-1}} = (C^T)^N \underbrace{\begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{d-1} \end{bmatrix}}_{v_0}, \qquad N \geq 1.$$

Folklore trick: compute $(C^T)^N$ by binary powering $O(\mathsf{MM}(d)\log(N))$

Fiduccia's algorithm: binary powering in $\mathbb{K}[x]/(P)$, with $P = x^d - \sum_{i=0}^{d-1} c_i x^i$

$$a_N = e \cdot v_N = \left(C^N \cdot e^T\right)^T \cdot v_0 = \langle\, x^N \bmod P,\ v_0 \,\rangle$$

where $e = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}$.

Cost: $O(\log N)$ products in $\mathbb{K}[x]/(P)$ $O(\mathsf{M}(d)\log N)$

High order lifting: statement

Problem: Given an invertible polynomial matrix A of degree d, compute the high order components $(C_0, C_1), (C_2, C_3), (C_6, C_7), (C_{14}, C_{15}), \dots$ in the Taylor expansion of its inverse

$$A^{-1} = \sum_{i \geq 0} C_i P^i, \qquad \text{with} \quad P = x^d, \quad C_i \in \mathcal{M}_n\left(\mathbb{K}[x]_{<d}\right)$$

Particular cases:

- If $d = 1$ and $A = I_n - xM$, then $C_i = M^i$, and the high order components can be computed fast by binary powering $O(\mathsf{MM}(n)\,\log(N))$
- If $n = 1$, then high order component $=$ N-th term of a recurrent sequence $\longrightarrow$ can be computed fast by Fiduccia's algorithm $O(\mathsf{M}(d)\,\log(N))$

Upcoming: Storjohann's algorithm $O(\mathsf{MM}(n, d)\,\log(N))$

Generalized Newton identity

Theorem (generalized Newton identity) The following holds modulo P^{s+t+2} :

$$C_{s+1}P^{s+1}+\cdots+C_{s+t+1}P^{s+t+1} = \left(C_0+\cdots+C_tP^t\right)\cdot(I_n - A\cdot(C_0+\cdots+C_sP^s))$$

Particular case: If $s = t = 2^i$, we recover Schulz's Newton-type iteration

Proof:

$$I - A(C_0 + C_1P + \cdots + C_sP^s) = AP^{s+1}(C_{s+1} + C_{s+2}P + \cdots)$$

$$\implies \quad \text{RHS} = \underbrace{(C_0 + C_1P + \cdots + C_tP^t)A}_{I-P^{t+1}(C_{t+1}+\cdots)A}\cdot P^{s+1}(C_{s+1} + C_{s+2}P + \cdots)$$

$$= \text{LHS} \bmod P^{s+t+2}$$

High order lifting: algorithm

[Storjohann, 2002]

Corollary (Storjohann 2002): For all $s, t \geq 0$:

$$C_{s+t+1} = -\{\ (C_{t-1} + C_t P) \cdot \{A \cdot C_s\}\ \}$$

Here, $\{\ B\ \}$ denotes the coefficient of P^1 in B.

Corollary (Storjohann 2002): For all $i \geq 2$, the following equalities hold

$$\begin{cases} C_{2^i-2} & = \ -\ \{\ (C_{2^{i-1}-2} + C_{2^{i-1}-1} P) \cdot \{A \cdot C_{2^{i-1}-2}\}\ \}, \\ C_{2^i-1} & = \ -\ \{\ (C_{2^{i-1}-2} + C_{2^{i-1}-1} P) \cdot \{A \cdot C_{2^{i-1}-1}\}\ \}, \end{cases}$$

and allow to compute the *high order components*

$$(C_0, C_1) \to (C_2, C_3) \to (C_6, C_7) \to (C_{14}, C_{15}) \to \dots$$

Cost: $O(\mathsf{MM}(n,d))$ ops. per arrow $O(\mathsf{MM}(n,d)\ \log(N))$

Generalizes simultaneously binary powering ($d = 1$) and Fiduccia ($n = 1$)

Example (Fibonacci revisited)

$$\frac{1}{1-x-x^2} = C_0 + C_1 P + C_2 P^2 + \cdots, \qquad P = x^2, \quad C_n = F_{2n} + F_{2n+1} x$$

The Storjohann identities become

$$\begin{cases} C_{2^i-2} & = & -\left\{ \left(C_{2^{i-1}-2} + C_{2^{i-1}-1} x^2\right) \cdot \left\{ \left(1-x-x^2\right) \cdot C_{2^{i-1}-2} \right\} \right\}, \\ C_{2^i-1} & = & -\left\{ \left(C_{2^{i-1}-2} + C_{2^{i-1}-1} x^2\right) \cdot \left\{ \left(1-x-x^2\right) \cdot C_{2^{i-1}-1} \right\} \right\}, \end{cases}$$

and allow to compute the *high order components*

$$(F_0, F_1, F_2, F_3) \to (F_4, F_5, F_6, F_7) \to (F_{12}, F_{13}, F_{14}, F_{15}) \to \dots \quad \text{by}$$

$$\begin{cases} F_{2^{i+1}-2} = F_{2^i-2} \cdot F_{2^i} + F_{2^i-1} \cdot F_{2^i-3}, \\ F_{2^{i+1}-4} = F_{2^i-2}^2 + F_{2^i-3}^2, \\ F_{2^{i+1}-1} = F_{2^i-1} \cdot F_{2^i} + F_{2^i-2} \cdot F_{2^i-1}, \\ F_{2^{i+1}-3} = F_{2^i-1} \cdot F_{2^i-2} + F_{2^i-2} \cdot F_{2^i-3}, \end{cases} \iff \underbrace{\begin{cases} F_{2n-2} = F_{n-2}^2 + F_{n-1}^2 \\ F_{2n-1} = F_{n-1}^2 + 2F_{n-1}F_{n-2} \end{cases}}_{\text{Shortt's algorithm}}$$

Keller-Gehrig algorithm

Problem: Given $M \in \mathcal{M}_n(\mathbb{K})$ and $v \in \mathbb{K}^n$, compute the Krylov sequence

$$\mathcal{K} = \Big(v, \quad Mv, \quad M^2 v, \quad \dots, \quad M^{n-1}v\Big)$$

Interest: If M is *generic*, $\mathcal{K}$ forms a basis of $\mathbb{K}^n$, and the matrix C of $v \mapsto Mv$ w.r.t. $\mathcal{K}$ is companion $\Longrightarrow$ the characteristic polynomial $\det(xI_n - M)$ reads off $C = P^{-1}MP$, where $P = \big[\, v \mid Mv \mid \cdots \mid M^{n-1}v \,\big]$.

Naive algorithm: Compute iteratively $v_{i+1} = Mv_i$, $v_0 = v$.

Cost: $O(n)$ matrix-vector products $\longrightarrow O(n^3)$ ops. in $\mathbb{K}$.

Keller-Gehrig algorithm (1985) Compute

(1) $M_0 = M,\ M_1 = M^2,\ M_2 = M^4,\ M_3 = M^8, \dots$ (binary powering)

(2) $\Big[\, M^{2^k}v \mid \cdots \mid M^{2^{k+1}-1}v \,\Big] := M_k \times \Big[\, v \mid Mv \mid \cdots \mid M^{2^k-1}v \,\Big], \quad k \geq 0$

Cost: $O(\log(n))$ matrix products for both (1) and (2) $\longrightarrow O(\mathsf{MM}(n)\log n)$

Solving linear systems with polynomial coefficients

Problem: Given $A \in \mathcal{M}_n(\mathbb{K}[x]_{\leq d})$ invertible, and $b \in \mathbb{K}[x]^n_{<d}$, compute $A^{-1}b$

[Moenck-Carter, 1979]: One can recover the exact solution $y = A^{-1}b \in \mathbb{K}(x)^n$ of $Ay = b$ from its approximation

$$y_{2nd} = A^{-1}b \bmod x^{2nd} = c_0 + c_1 P + c_2 P^2 + \cdots + c_{2n} P^{2n}, \qquad \text{where } \; c_i \in \mathbb{K}[x]^n_{<d}$$

Conversion $y_{2nd} \to y$ by Padé approximation $\qquad O(n\,\mathsf{M}(nd)\log(nd))$

Particular case: If $d = 1$ and $A = I_n - xM$, with $M \in \mathcal{M}_n(\mathbb{K})$ and $b \in \mathbb{K}^n$, then $c_i = M^i b$, and $c_0, \ldots, c_{2n}$ can be computed fast by the Keller-Gehrig algorithm.

Storjohann's algorithm

Problem: Given $A \in \mathcal{M}_n(\mathbb{K}[x]_{\leq d})$ invertible, and $b \in \mathbb{K}[x]^n_{<d}$, compute $y_N = A^{-1} b \bmod x^N$

Theorem (Storjohann 2002): For all $s, t \geq 0$:

$$c_{s+t+1} = -\left\{ \, (C_{t-1} + C_t P) \cdot \{ A \cdot c_s \} \, \right\}$$

Here, $\{ \, v \, \}$ denotes the coefficient of P^1 in $v \in \mathbb{K}[x]^n$.

Corollary (Storjohann 2002): For all $i \geq 2$, the following equality holds

$$\left[\, c_{2^k} \mid \cdots \mid c_{2^{k+1}-1} \, \right] = -\left\{ \left(C_{2^k-2} + C_{2^k-1} P \right) \cdot \left\{ A \cdot \left[\, c_0 \mid \cdots \mid c_{2^k-1} \, \right] \right\} \right\},$$

which allows to compute

$$(c_0, c_1) \to (c_2, c_3) \to (c_4, c_5, c_6, c_7) \to (c_8, c_9, c_{10}, c_{11}, c_{12}, c_{13}, c_{14}, c_{15}) \to \dots$$

Generalizes Keller-Gehrig's algorithm

Cost: $O(\mathsf{MM}(n,d))$ ops. per arrow $\qquad O(\mathsf{MM}(n,d)\log(n))$

Baby steps / giant steps

Baby steps / giant steps for polynomial evaluation

Problem: Given a $\mathbb{K}$-algebra $\mathbb{A}$, $a \in \mathbb{A}$ and $P \in \mathbb{K}[x]_{<N}$, compute $P(a)$

Horner rule: $O(N)$ products in $\mathbb{A}$

Better algorithm (Paterson-Stockmeyer, 1973): $O(\sqrt{N})$ products in $\mathbb{A}$

Write $P(x) = P_0(x) + \cdots + P_{\ell-1}(x) \cdot (x^\ell)^{\ell-1}$, with $\ell = \sqrt{N}$ and $\deg(P_i) < \ell$

(BS) Compute $a^2, \quad \dots, \quad a^\ell =: b$ $O(\sqrt{N})$ products in $\mathbb{A}$

(GS) Compute $b_0 = 1,\ b_1 = b, \quad \dots, \quad b_{\ell-1} = b^{\ell-1}$ $O(\sqrt{N})$ products in $\mathbb{A}$

Evaluate $c_0 = P_0(a), \quad \dots, \quad c_{\ell-1} = P_{\ell-1}(a)$ no product in $\mathbb{A}$

Return $P(a) = b_0 c_0 + \cdots + b_{\ell-1} c_{\ell-1}$ $O(\sqrt{N})$ products in $\mathbb{A}$

Application: evaluation of $P \in \mathbb{K}[x]_{<N}$ at a matrix in $\mathcal{M}_r(\mathbb{K})$ $O(\sqrt{N}\,\mathsf{MM}(r))$

Baby steps / giant steps, application to factorials

Problem: Compute $N! = 1 \times 2 \times \cdots \times N$

Naive algorithm: unroll the recurrence $O(N)$

Better algorithm (Strassen, 1976): BS-GS strategy $O\big(\mathsf{M}(\sqrt{N}\,)\log N\big)$

(BS) Compute $P = (x+1)(x+2)\cdots(x+\sqrt{N})$ $O\big(\mathsf{M}(\sqrt{N}\,)\log N\big)$

(GS) Evaluate P at $0, \sqrt{N}, 2\sqrt{N}, \ldots, (\sqrt{N}-1)\sqrt{N}$ $O\big(\mathsf{M}(\sqrt{N}\,)\log N\big)$

Return $u_N = P((\sqrt{N}-1)\sqrt{N})\cdots P(\sqrt{N})\cdot P(0)$ $O(\sqrt{N})$

Baby steps / giant steps, application to recurrences

Problem: Compute the N-th term u_N of a P-recursive sequence

$$p_r(n)u_{n+r} + \cdots + p_0(n)u_n = 0, \qquad (n \in \mathbb{N})$$

Naive algorithm: unroll the recurrence $O(N)$

Better algorithm: $U_n = (u_n, \ldots, u_{n+r-1})^T$ satisfies the 1st order recurrence

$$U_{n+1} = \frac{1}{p_r(n)} A(n) U_n \quad \text{with} \quad A(n) = \begin{bmatrix} & p_r(n) & & \\ & & \ddots & \\ & & & p_r(n) \\ -p_0(n) & -p_1(n) & \ldots & -p_{r-1}(n) \end{bmatrix}.$$

$\Longrightarrow u_N$ reads off the matrix factorial $A(N-1)\cdots A(0)$

Chudnovsky-Chudnovsky, 1987: (BS)-(GS) strategy $O\big(\mathsf{M}(\sqrt{N}\,)\log N\big)$

Baby steps / giant steps, application to point counting

Problem: count the number n of solutions of the equation $y^2 = f(x)$ over $\mathbb{F}_p$

Basic idea (Deuring, 1941): if $\deg(f) = 3$, then $(n \bmod p)$ is $-[x^{p-1}]f(x)^{(p-1)/2}$

Explanation: z is a non-zero square in $\mathbb{F}_p$ exactly when $z^{(p-1)/2} = 1$

Generalization (Cartier-Manin, 1956): for a genus-g curve $y^2 = f(x)$, $(n \bmod p)$ reads off the Hasse-Witt matrix $(h_{i,j})_{i,j=1}^{g}$, with $h_{i,j} = [x^{ip-j}]f(x)^{(p-1)/2}$

Corollary (B-Gaudry-Schost, 2007): hyperelliptic point counting / $\mathbb{F}_p$ in $\tilde{O}(\sqrt{p})$

Based on (Flajolet-Salvy, 1997): $h = f^N$ satisfies the differential equation $fh' - Nf'h = 0$, thus its coefficient sequence is P-recursive.

Tellegen’s transposition principle

Tellegen's transposition principle

Let $\mathbf{M}$ *be a* $\mathsf{m} \times \mathsf{n}$ *matrix, with no zero rows and no zero columns. Any linear algorithm of complexity* L *that computes the matrix-vector product* $\mathbf{M}{\cdot}\mathbf{v}$ *can be transformed into a linear algorithm of complexity*

$$\mathsf{L} - \mathsf{n} + \mathsf{m}$$

that computes the transposed matrix-vector product $\mathbf{M}^{\mathbf{T}}{\cdot}\mathbf{w}$.

- A precise formulation depends on the *model of computation.*

A particular case

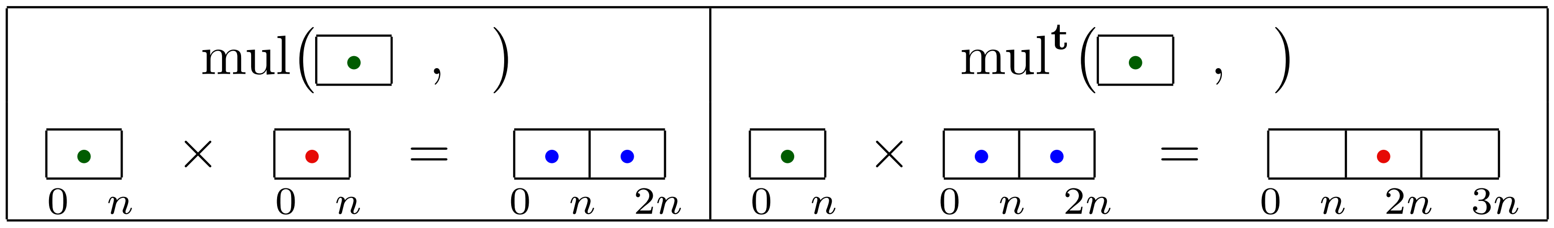
Suppose that the naive algorithm $\mathcal{N}$ is used to compute $M \cdot v$. Define its dual $\mathcal{N}^T$ as the naive algorithm for computing $M^T \cdot w$. Then:

$$\begin{array}{llllllll} \mathcal{N} & \text{uses} & m(n-1) & \text{ops.} & \pm & \text{and} & mn & \text{ops.} \quad \times \\ \mathcal{N}^t & \text{uses} & n(m-1) & \text{ops.} & \pm & \text{and} & mn & \text{ops.} \quad \times. \end{array}$$

$\longrightarrow$ Tellegen's theorem is trivially true for generic matrices

$\longrightarrow$ it becomes interesting when M is structured or sparse

Transposed polynomial multiplication = middle product (MP)



Example:

$$\mathrm{mul}\,(2+3x, a+bx) \qquad \mathrm{mul}^{\mathbf{t}}\,(3+2x, a+bx+cx^2)$$

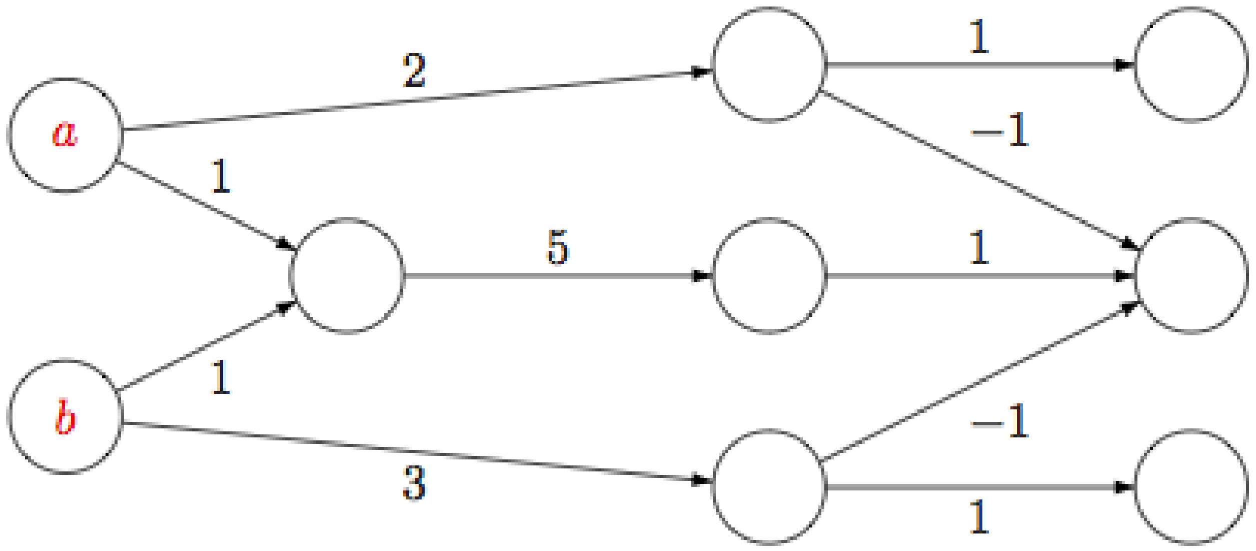
$$\begin{bmatrix} 2 & 0 \\ 3 & 2 \\ 0 & 3 \end{bmatrix} \times \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 2a \\ 3a+2b \\ 3b \end{bmatrix} \quad \begin{bmatrix} 2 & 3 & 0 \\ 0 & 2 & 3 \end{bmatrix} \times \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 2a+3b \\ 2b+3c \end{bmatrix}$$

Theorem [Hanrot-Quercia-Zimmerman, 2002]

From any linear algorithm for multiplication in degree n of cost $\mathsf{M}(\mathbf{n})$, one can derive an algorithm for the $(n, 2n)$ MP, of cost $\mathsf{M}(\mathbf{n}) + \mathbf{O}(\mathbf{n})$.

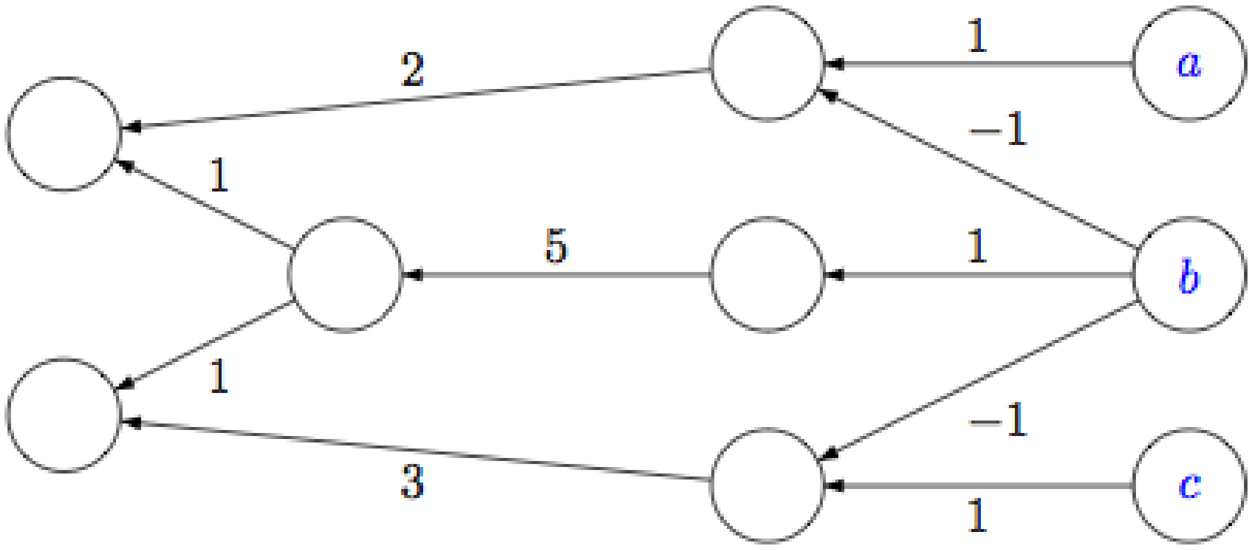
▶ Particular instance of Tellegen's theorem, for Toeplitz-band matrices

A DAG computing $(2+3x)(a+bx)$ *à la Karatsuba*



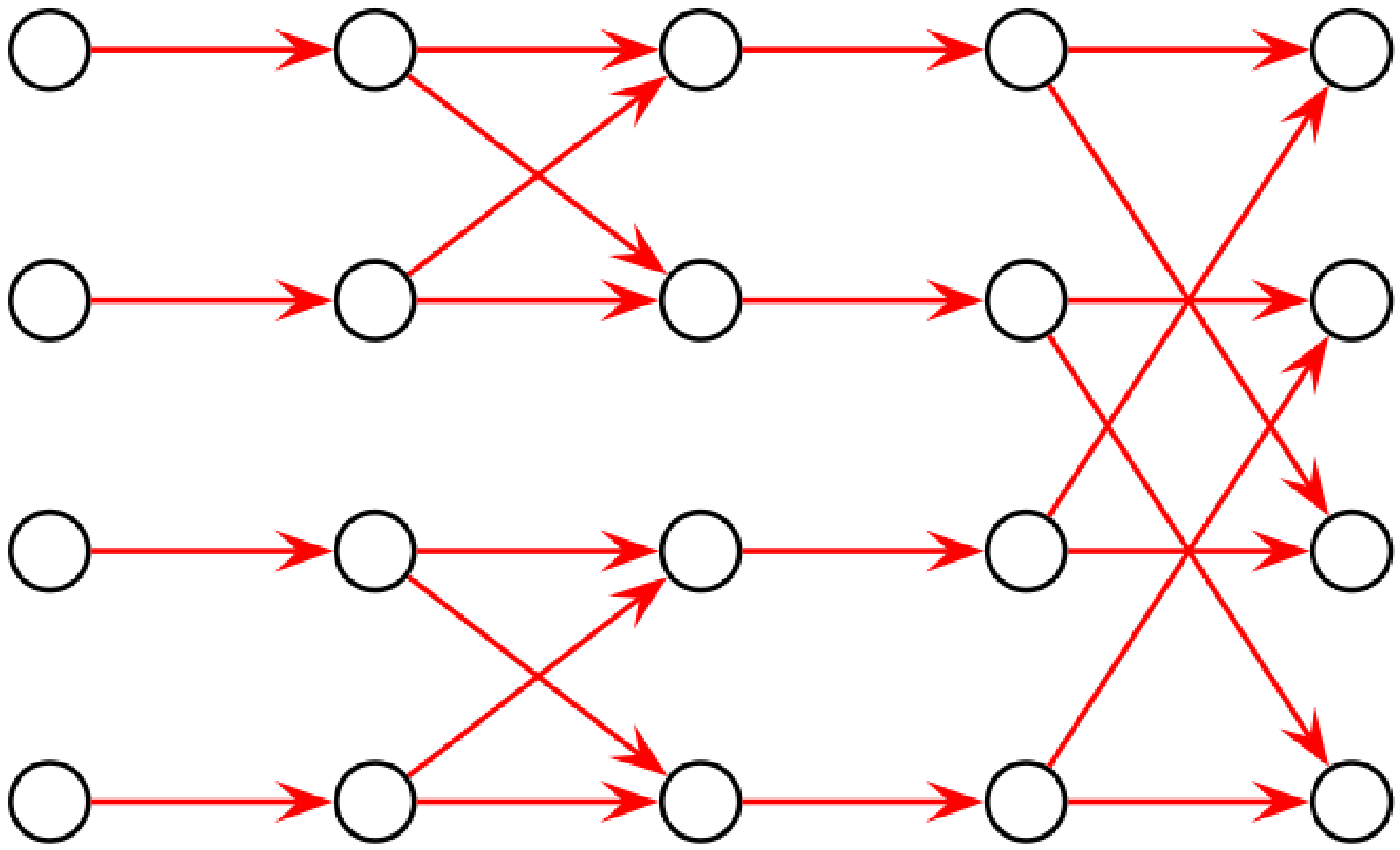
$$\begin{bmatrix} 2 & 0 \\ 3 & 2 \\ 0 & 3 \end{bmatrix} \times \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 2a \\ 3a+2b \\ 3b \end{bmatrix} = \begin{bmatrix} 2a \\ 5(a+b)-2a-3b \\ 3b \end{bmatrix}$$

The transposed DAG computes the middle product of $3 + 2x$ and $a + bx + cx^2$ *à la Karatsuba*



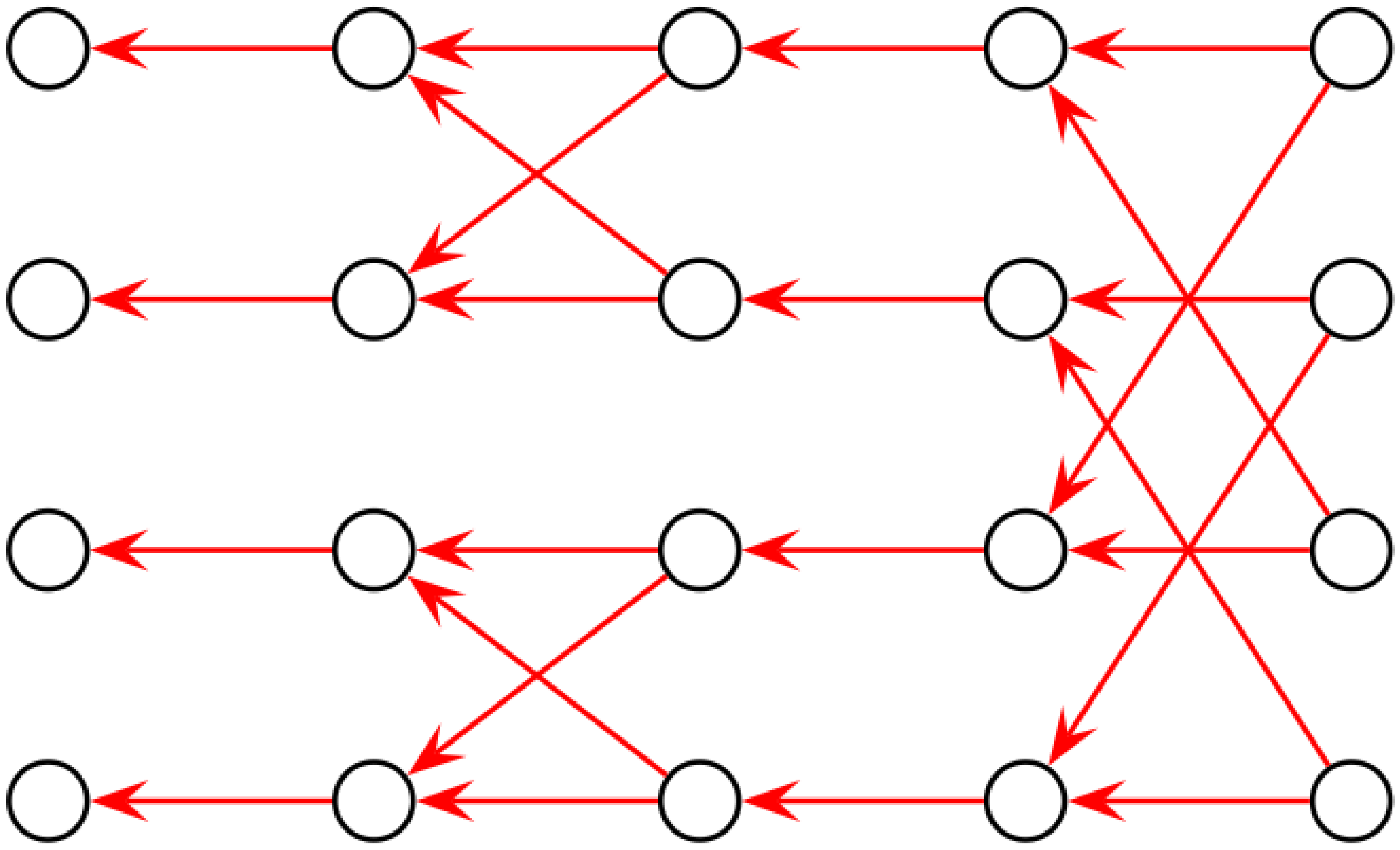
$$\begin{bmatrix} 2 & 3 & 0 \\ 0 & 2 & 3 \end{bmatrix} \times \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 2a + 3b \\ 2b + 3c \end{bmatrix} = \begin{bmatrix} 2(a-b) + 5b \\ 3(b-c) + 5b \end{bmatrix}$$

Duality between two classes of FFT algorithms



The Cooley-Tukey decimation-in-time DFT, on 4 points

Duality between two classes of FFT algorithms



The Gentleman-Sande decimation-in-frequency DFT, on 4 points

Automatic discovery of Horner's rule

$$[p_0, \ldots, p_n] \mapsto \sum_{i=0}^{n} p_i a^i$$

Automatic discovery of Horner's rule

$$\mathbf{M} = [1, a, \ldots, a^n]$$

$$[p_0, \ldots, p_n] \mapsto \sum_{i=0}^{n} p_i a^i$$

Automatic discovery of Horner's rule

$$\mathbf{M} = [1, a, \ldots, a^n]$$

$$x_0 \mapsto [x_0, ax_0, \ldots, a^n x_0] \qquad [p_0, \ldots, p_n] \mapsto \sum_{i=0}^{n} p_i a^i$$

Automatic discovery of Horner's rule

$$\mathbf{M} = [1, a, \ldots, a^n]$$

$x_0 \mapsto [x_0, ax_0, \ldots, a^n x_0]$ $\qquad$ $[p_0, \ldots, p_n] \mapsto \sum_{i=0}^n p_i a^i$

```
Input $x_0$.
$p_0 \leftarrow x_0$;
  for $j$ from $1$ to $n$ do
    $p_j \leftarrow p_{j-1}$;
    $p_j \leftarrow ap_j$;
Output $p = [p_0, \ldots, p_n]$.
```

Automatic discovery of Horner's rule

$$\mathbf{M} = [1, a, \ldots, a^n]$$

$$x_0 \mapsto [x_0, ax_0, \ldots, a^n x_0] \qquad [p_0, \ldots, p_n] \mapsto \textstyle\sum_{i=0}^{n} p_i a^i$$

```
**Input** $x_0$.
$p_0 \leftarrow x_0$;
  **for** $j$ **from** $1$ **to** $n$ **do**
    $p_j \leftarrow p_{j-1}$;
    $p_j \leftarrow a p_j$;
**Output** $p = [p_0, \ldots, p_n]$.
```

```
**Input** $p = [p_0, \ldots p_n]$.




**Output** $x_0$.
```

Automatic discovery of Horner's rule

$$\mathbf{M} = [1, a, \ldots, a^n]$$

$x_0 \mapsto [x_0, ax_0, \ldots, a^n x_0]$

$[p_0, \ldots, p_n] \mapsto \sum_{i=0}^{n} p_i a^i$

```
**Input** $x_0$.
$p_0 \leftarrow x_0$;
  **for** $j$ **from** $1$ **to** $n$ **do**
    $p_j \leftarrow p_{j-1}$;
    $p_j \leftarrow a p_j$;
**Output** $p = [p_0, \ldots, p_n]$.
```

```
**Input** $p = [p_0, \ldots p_n]$.
  **for** $j$ **from** $n$ **downto** $1$ **do**




**Output** $x_0$.
```

Automatic discovery of Horner's rule

$$\mathbf{M} = [1, a, \ldots, a^n]$$

$$x_0 \mapsto [x_0, ax_0, \ldots, a^n x_0] \qquad [p_0, \ldots, p_n] \mapsto \sum_{i=0}^n p_i a^i$$

```
Input $x_0$.
$p_0 \leftarrow x_0$;
  for $j$ from 1 to $n$ do
    $p_j \leftarrow p_{j-1}$;
    $p_j \leftarrow ap_j$;
Output $p = [p_0, \ldots, p_n]$.
```

```
Input $p = [p_0, \ldots p_n]$.
  for $j$ from $n$ downto 1 do
    $p_j \leftarrow ap_j$;
Output $x_0$.
```

Automatic discovery of Horner's rule

$$\mathbf{M} = [1, a, \ldots, a^n]$$

$$x_0 \mapsto [x_0, ax_0, \ldots, a^n x_0] \qquad [p_0, \ldots, p_n] \mapsto \sum_{i=0}^{n} p_i a^i$$

```
Input $x_0$.
$p_0 \leftarrow x_0$;
  for $j$ from $1$ to $n$ do
    $p_j \leftarrow p_{j-1}$;
    $p_j \leftarrow a p_j$;
Output $p = [p_0, \ldots, p_n]$.
```

```
Input $p = [p_0, \ldots p_n]$.
  for $j$ from $n$ downto $1$ do
    $p_j \leftarrow a p_j$;
    $p_{j-1} \leftarrow p_j + p_{j-1}$;

Output $x_0$.
```

Automatic discovery of Horner's rule

$$\mathbf{M} = [1, a, \ldots, a^n]$$

$$x_0 \mapsto [x_0, ax_0, \ldots, a^n x_0] \qquad [p_0, \ldots, p_n] \mapsto \sum_{i=0}^{n} p_i a^i$$

```
**Input** $x_0$.
$p_0 \leftarrow x_0$;
  **for** $j$ **from** $1$ **to** $n$ **do**
    $p_j \leftarrow p_{j-1}$;
    $p_j \leftarrow a p_j$;
**Output** $p = [p_0, \ldots, p_n]$.
```

```
**Input** $p = [p_0, \ldots p_n]$.
  **for** $j$ **from** $n$ **downto** $1$ **do**
    $p_j \leftarrow a p_j$;
    $p_{j-1} \leftarrow p_j + p_{j-1}$;
$x_0 \leftarrow p_0$;
**Output** $x_0$.
```

Karatsuba's algorithm and its transpose

$$\boxed{a \mid b} \times \boxed{c \mid d} \mapsto \underset{|--W--|}{\boxed{U \mid V}} \qquad \boxed{a \mid b} \times^{\mathbf{t}} \underset{|--W--|}{\boxed{U \mid V}} \mapsto \boxed{c \mid d}$$

mul

```
Input $(c, d)$.

$e \leftarrow c + d$ ;
  $U \leftarrow \mathrm{mul}(a, c)$ ;
  $V \leftarrow \mathrm{mul}(b, d)$ ;
  $W \leftarrow \mathrm{mul}(a + b, e)$ ;
$W \leftarrow W - U - V$ ;

Output $(U, V, W)$.
```

Karatsuba's algorithm and its transpose

$$\boxed{\;a\;|\;b\;} \times \boxed{\;c\;|\;d\;} \mapsto \begin{array}{c}\boxed{\quad U \quad|\quad V \quad} \\ |--W--|\end{array} \qquad\qquad \boxed{\;a\;|\;b\;} \times^{\mathbf{t}} \begin{array}{c}\boxed{\quad U \quad|\quad V \quad} \\ |--W--|\end{array} \mapsto \boxed{\;c\;|\;d\;}$$

mul

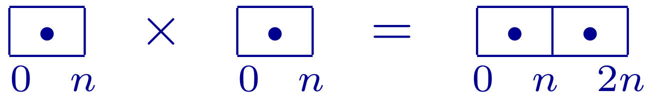
```
**Input** $(c, d)$.

$e \leftarrow c + d$ ;
  $U \leftarrow \mathrm{mul}(a, c)$ ;
  $V \leftarrow \mathrm{mul}(b, d)$ ;
  $W \leftarrow \mathrm{mul}(a + b, e)$ ;
$W \leftarrow W - U - V$ ;

**Output** $(U, V, W)$.
```

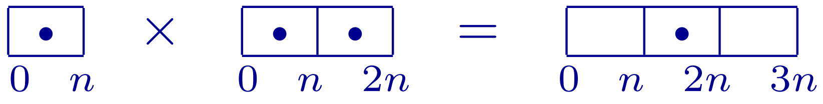
$\mathrm{mul}^{\mathbf{t}}$

```
**Input** $(U, V, W)$.
$V \leftarrow V - W$ ;
$U \leftarrow U - W$ ;
  $e \leftarrow \mathrm{mul}^{\mathbf{t}}(a + b, W)$ ;
  $d \leftarrow \mathrm{mul}^{\mathbf{t}}(b, V)$ ;
  $c \leftarrow \mathrm{mul}^{\mathbf{t}}(a, U)$ ;
$c \leftarrow c + e$ ;
$d \leftarrow d + e$ ;
**Output** $(c, d)$.
```

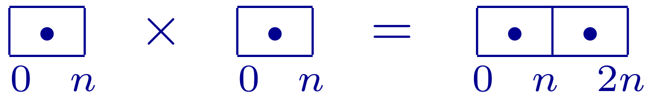
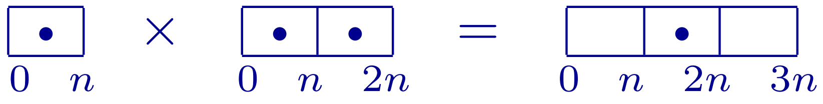
Tellegen's Polynomial Dictionary

direct problem	transposed problem
multiplication	middle product

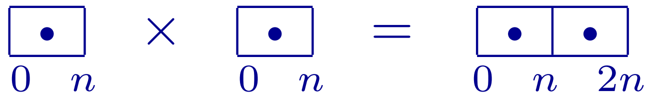
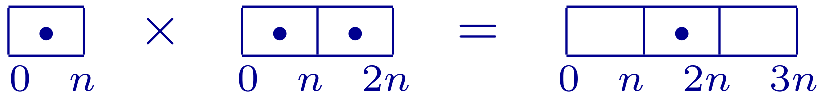




Tellegen's Polynomial Dictionary

direct problem	transposed problem
multiplication	middle product
	
division with remainder	extension of recurrences
$A \mapsto A \mod P$	$(a_0, \ldots, a_{n-1}) \mapsto (a_0, \ldots, a_{2n-1})$

Tellegen's Polynomial Dictionary

direct problem	transposed problem
multiplication	middle product
	
division with remainder	extension of recurrences
$A \mapsto A \mod P$	$(a_0, \ldots, a_{n-1}) \mapsto (a_0, \ldots, a_{2n-1})$
multipoint evaluation	generalized power sums
$P \mapsto (P(a_0), \ldots, P(a_{n-1}))$	$(p_0, \ldots, p_{n-1}) \mapsto (\sum p_i, \ldots, \sum p_i a_i^{n-1})$

Tellegen's Polynomial Dictionary

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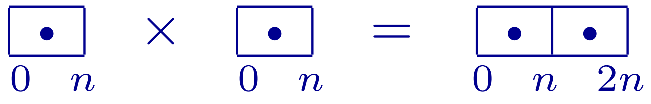	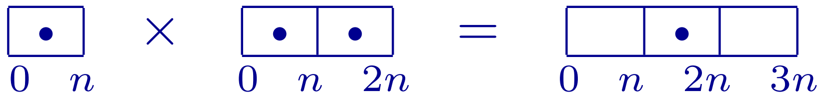
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shift of polynomials	evaluation in falling factorial basis
$P(x) \mapsto P(x+1)$	$P = \sum a_i x^{\underline{i}} \mapsto (P(0), \ldots, P(n-1))$

Tellegen's Polynomial Dictionary

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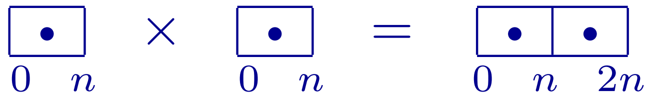	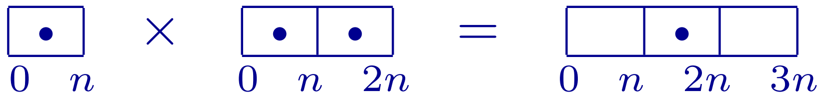
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shift of polynomials	evaluation in falling factorial basis
$P(x) \mapsto P(x+1)$	$P = \sum a_i x^{\underline{i}} \mapsto (P(0), \dots, P(n-1))$
polynomial base change	series composition
$\dots$	$\dots$

Chebyshev polynomials

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x),$$

$$\begin{aligned}
T_0(x) &= 1, \\
T_1(x) &= x, \\
T_2(x) &= 2x^2 - 1, \\
T_3(x) &= 4x^3 - 3x, \\
T_4(x) &= 8x^4 - 8x^2 + 1, \\
T_5(x) &= 16x^5 - 20x^3 + 5x, \\
T_6(x) &= 32x^6 - 48x^4 + 18x^2 - 1, \\
T_7(x) &= 64x^7 - 112x^5 + 56x^3 - 7x, \\
T_8(x) &= 128x^8 - 256x^6 + 160x^4 - 32x^2 + 1
\end{aligned}$$

Conversion Chebyshev basis $\longrightarrow$ monomial basis

Conversion matrix from the Chebyshev basis to the monomial basis

$$\mathcal{T} = \begin{bmatrix}
1/2 & 0 & -1 & 0 & 1 & 0 & -1 & 0 & 1 & 0 & -1 \\
 & 1 & 0 & -3 & 0 & 5 & 0 & -7 & 0 & 9 & 0 \\
 & & 2 & 0 & -8 & 0 & 18 & 0 & -32 & 0 & 50 \\
 & & & 4 & 0 & -20 & 0 & 56 & 0 & -120 & 0 \\
 & & & & 8 & 0 & -48 & 0 & 160 & 0 & -400 \\
 & & & & & 16 & 0 & -112 & 0 & 432 & 0 \\
 & & & & & & 32 & 0 & -256 & 0 & 1120 \\
 & & & & & & & 64 & 0 & -576 & 0 \\
 & & & & & & & & 128 & 0- & 1280 \\
 & & & & & & & & & 256 & 0 \\
 & & & & & & & & & & 512
\end{bmatrix}$$

Question: Perform efficiently the matrix-vector product by $\mathcal{T}$

Transposed Chebyshev conversion

Write:

$$\sum_{n\geq 0} T_n(x)t^n \; = \; \sum_{i\geq 0} C_i(t)x^i.$$

Then, the transposed Chebyshev conversion (in degree $< N$) is the linear map

$$(f_0,\dots,f_{N-1}) \in \mathbb{K}^N \quad \longmapsto \quad f_0C_0(t) + \cdots + f_{N-1}C_{N-1}(t) \bmod t^N$$

General key-fact: (transposed basis conversion amounts to series composition)

$$\sum_{i\geq 0} C_i(t)x^i = \frac{v(t)}{1 - x\cdot h(t)} \quad \Longrightarrow \quad \sum_i C_i(t)f_i \bmod t^N = v(t)\cdot F(h(t)) \bmod t^N$$

Specific key-fact (orthogonality): $\displaystyle \sum_{n\geq 0} T_n(x)t^n = \frac{1}{2}\cdot\frac{1-t^2}{1+t^2}\cdot\frac{1}{1 - x\cdot\frac{2t}{1+t^2}}$

Conclusion: Transposed Chebyshev conversion amounts to composition by $\frac{2t}{1+t^2}$

Fast Chebyshev conversion

Transposed Chebyshev conversion in degree $< N$ means $F\left(\frac{2t}{1+t^2}\right) \bmod t^N$.

Fact: The identity

$$\frac{2t}{1+t^2} = \sqrt{1-\left(\frac{2}{1+t^2}-1\right)^2}$$

implies that the composition $F\left(\frac{2t}{1+t^2}\right) \bmod t^N$ can be performed using a constant number of shifts in degree N.

Consequence: Algorithm of complexity $O(\mathsf{M}(N))$ for transposed Chebyshev conversion in degree $< N$

By Tellegen's principle: Algorithm of complexity $O(\mathsf{M}(N))$ for Chebyshev conversion in degree $< N$

(B-Salvy-Schost 2008): same works for the conversion to a basis $(P_i(x))$ of Sheffer polynomials $\sum_{i\geq 0} P_i(x)\, t^i/i! = v(t)\cdot e^{x\cdot h(t)}$ with "nice" $v(t), h(t)$

Conclusion

Open questions

- Over a general field $\mathbb{K}$, is it possible to compute
  - polynomial products in $\mathbb{K}[x]_{\leq n}$ in $O(n \log n)$? $\quad O(n \log n \log \log n)$
  - matrix products in $\mathcal{M}_r(\mathbb{K})$ in $\tilde{O}(r^2)$? $\quad O(r^{2.38})$
  - compositions in $\mathbb{K}[[x]]/(x^n)$ in $\tilde{O}(n)$? $\quad \tilde{O}(n^{3/2})$
  - resultants in $\mathbb{K}[x,y]_{\leq (n,n)}$ in $\tilde{O}(n^2)$? $\quad \tilde{O}(n^3)$
  - char. polynomials in $\mathcal{M}_r(\mathbb{K}[x]_{\leq d})$ in $\tilde{O}(\mathsf{MM}(r,d))$? $\quad \tilde{O}(r^{2.6972}\, n)$
  - conversions in $\mathbb{K}[x]_{\leq n}$ between bases of special polynomials in $\tilde{O}(n)$?
  - the N-th term of a P-recursive sequence in less than $\tilde{O}(\sqrt{N})$?
  - factorizations in $\mathbb{F}_p[x]_{\leq n}$ in deterministic polynomial time in $(n, \log(p))$?

- Are there $O(\mathsf{M}(n))$ algorithms for
  - evaluation-interpolation at $(1, 2, \ldots, n)$
  - for special gcds, e.g. $\gcd(f, f')$, or $\gcd(f, \prod_{i=1}^{n}(x-k))$
  - rational interpolation at n points in geometric progression

Open questions

- Is DFT optimal? Is polynomial multiplication by FFT optimal?

- Can multiplication be done faster for power series than for polynomials? (Is there an FFT version of Mulder's short product?)

- Structured (algebraic / differential) Hermite-Padé approximants

- Evaluation of $P \in \mathbb{K}[x, y]$, or of $L \in \mathbb{K}[x]\langle D_x \rangle$, at $y = y(x) \in \mathbb{K}[[x]]$

- Deterministic $O(\mathsf{MM}(r))$ algorithm for characteristic polynomials in $\mathcal{M}_r(\mathbb{K})$

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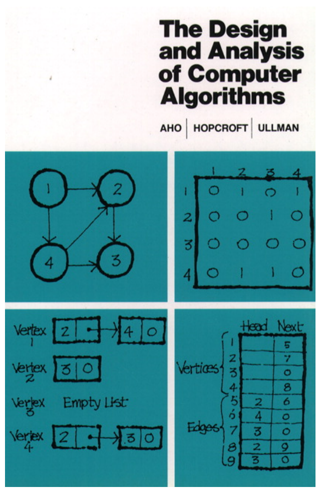
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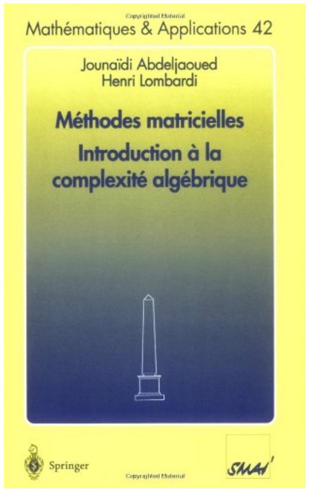
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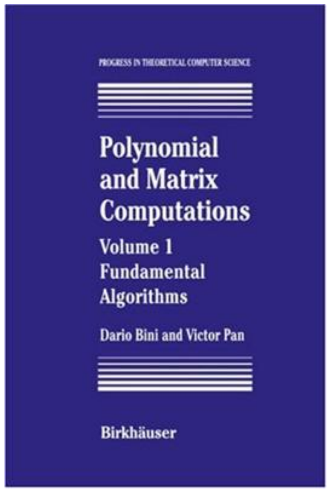
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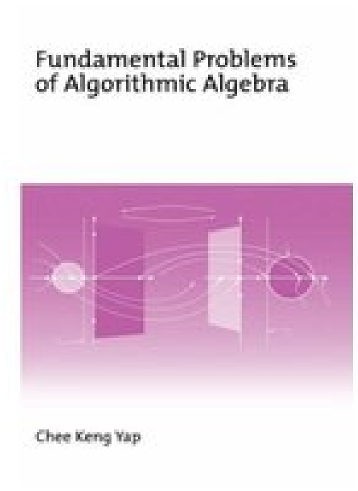
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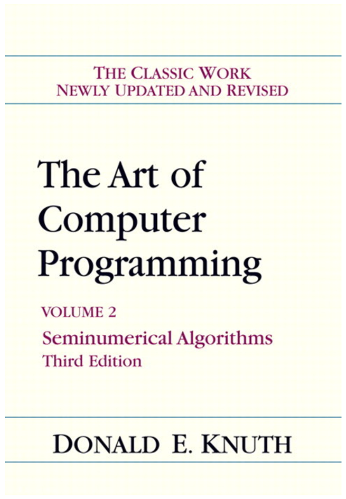
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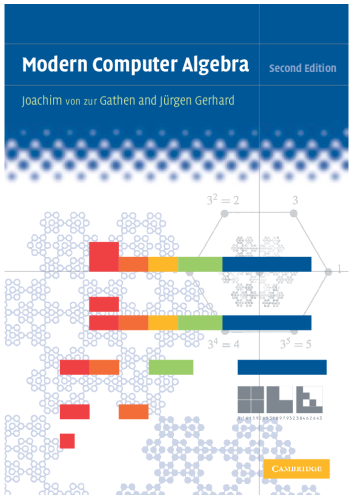
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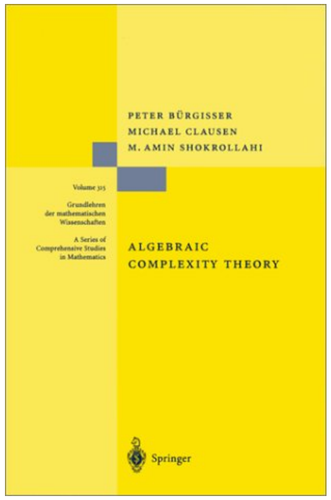
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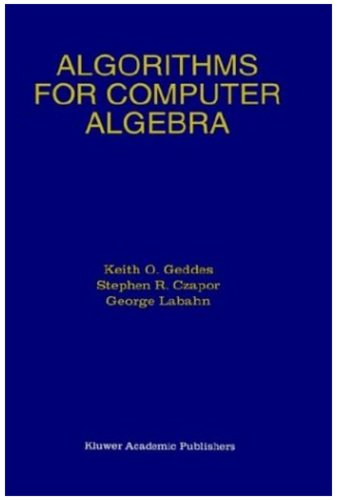
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What can be computed in 1 minute with a CA system

polynomial product[a] in degree 14,000,000 (>1 year with schoolbook)

gcd of two polynomials of degree 600,000

resultant of two polynomials of degree 40,000

factorization of a univariate polynomial of degree 4,000

factorization of a bivariate polynomial of total degree 500

resultant of two bivariate polynomials of total degree 100 (output 10,000)

product/sum of two algebraic numbers of degree 450 (output 200,000)

determinant (char. polynomial) of a matrix with 4,500 (2,000) rows

determinant of an integer matrix with 32-bit entries and 700 rows

product of two integers with 500,000,000 binary digits

factorial of $N = 20,000,000$ (output of 140,000,000 digits);

[a] in $\mathbb{K}[x]$, for $\mathbb{K} = \mathbb{F}_{67108879}$