On the Existence of Telescopers for Mixed Hypergeometric Terms

Shaoshi Chen[a], Frédéric Chyzak[b]
Ruyong Feng[a], Guofeng Fu[a], Ziming Li[a]

[a] *Key Laboratory of Math.-Mech., AMSS, Chinese Academy of Sciences, 100190 Beijing, China*

[b] *INRIA, 91120 Palaiseau, France*

Abstract

We present a criterion for the existence of telescopers for mixed hypergeometric terms, which is based on additive and multiplicative decompositions. The criterion enables us to determine the termination of Zeilberger's algorithms for mixed hypergeometric inputs, and to verify that certain indefinite sums do not satisfy any polynomial differential equation.

Key words: Creative telescoping, Zeilberger's algorithms, Existence criteria.

1. Introduction

Given a sum $U_m := \sum_{n=0}^{\infty} u_{m,n}$ to be computed, *creative telescoping* is a process that determines a recurrence in m satisfied by the univariate sequence $U = (U_m)$ from a system of recurrences in m and n satisfied by the bivariate summand $u = (u_{m,n})$. A natural counterpart exists for integration. Algorithmic research on this topic has been initiated by Zeilberger in the early 1980s, leading in the 1990s to creative-telescoping algorithms for summands and integrands described by first-order linear equations, i.e., for hypergeometric terms and hyperexponential functions (Zeilberger, 1990a, 1991; Almkvist and Zeilberger, 1990).

Email addresses: `schen@amss.ac.cn`, `frederic.chyzak@inria.fr` (Shaoshi Chen[a], Frédéric Chyzak[b]), `ryfeng@amss.ac.cn`, `fuguofeng@mmrc.iss.ac.cn`, `zmli@mmrc.iss.ac.cn` (Ruyong Feng[a], Guofeng Fu[a], Ziming Li[a]).

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The termination problem of Zeilberger's algorithms has been extensively studied in the last two decades (Wilf and Zeilberger, 1992b; Abramov and Le, 2002; Abramov, 2003; Chen et al., 2005) and can be related to existence problems for other operations, like the computation of diagonals (Lipshitz, 1988). The main output of creative telescoping is the recurrence on the sum U. It is called a *telescoper* for u. Zeilberger's algorithms terminate if and only if telescopers exist, whence the interest to discuss their existence. Zeilberger (1990b) shows that holonomicity, a notion borrowed from the theory of D-modules, implies the existence of telescopers. In particular, the fundamental theorem in (Wilf and Zeilberger, 1992a) states that telescopers always exist for proper hypergeometric terms. However, holonomicity is merely a sufficient condition, i.e., there are cases in which the input functions are not holonomic (proper) but Zeilberger's algorithms still terminate, see Chyzak et al. (2009). Therefore, a challenging problem is to find a necessary and sufficient condition that enables us to determine the existence of telescopers.

In view of the theoretical difficulty, special attention has been focused on the subclass of hypergeometric terms, hyperexponential functions, and mixed hypergeometric terms (see Definition 2.8). In the continuous case, the results by Bernšteĭn (1971), Kashiwara (1978), Lipshitz (1988) and Takayama (1992) show that every hyperexponential function has a telescoper. This implies that Zeilberger's algorithm always succeeds on hyperexponential inputs. However, the situation in other cases turns out to be more involved. In the discrete case, the first complete solution to the termination problem has been given by Le (2001) and Abramov and Le (2002), by deciding whether there exists a telescoper for a given bivariate rational sequence in the (q)-discrete variables m and n. According to their criterion, the rational sequence $\left(1/(m^2+n^2)\right)_{m,n\in\mathbb{Z}^+}$, has no telescoper. The criterion has been extended to the general case of bivariate hypergeometric terms by Abramov (2002, 2003). He proved that a hypergeometric term can be written as a sum of a hypergeometric-summable term and a proper one if it has a telescoper, see (Abramov, 2003, Theorem 10). Similar results have been obtained in the q-discrete case by Chen et al. (2005).

Almkvist and Zeilberger (1990) presented a continuous-discrete analogue of creative telescoping. This analogue is useful in the study of orthogonal polynomials (Koepf, 1998, Chapters 10–13). In analogy with the discrete case, not all mixed hypergeometric terms have telescopers. Therefore, an Abramov-like criterion is also needed in the mixed case.

In order to unify the various cases of mixed rational terms, Chen and Singer (2012) recently presented a criterion that is based on residue analysis for the existence of telescopers for bivariate rational functions. In the present paper, we give a criterion, Theorem 6.3, on the existence of telescopers for mixed hypergeometric terms, including continuous-discrete, continuous-q-discrete and discrete-q-discrete terms. The criterion determines whether Zeilberger's algorithms for mixed hypergeometric terms terminate. Moreover, the non-existence of telescopers makes us able to verify that some indefinite sums do not satisfy any polynomial differential equation. See (Hardouin and Singer, 2008; Schneider, 2010) and Example 6.8 in this paper.

The rest of this paper is organized as follows. An algebraic setting for mixed hypergeometric terms is described in Section 2, and the existence problem of telescopers is stated in Section 3. In Section 4, we define the two notions of exact and proper terms, and we describe two kinds of decompositions: additive and structural. These notions and decompositions are crucial for establishing our criterion. A necessary condition on the existence of telescopers is presented in Section 5. The criterion is given in Section 6, which is based on the fundamental theorem in Wilf and Zeilberger (1992a). The appendix contains a detailed proof of the fundamental theorem in the mixed setting (Theorem 6.1).

2. Preliminaries

The goal of this section is to present an algebraic setting for mixed hypergeometric terms in continuous-discrete, continuous-q-discrete, and discrete-q-discrete cases.

Throughout the paper, we let k be an algebraically closed field of characteristic zero, and q be a nonzero element of k. Assume further that q is not a root of unity. Let $k(x, y)$ be the field of rational functions in x and y over k. For an element $f \in k(x, y)$, the denominator and numerator of f are denoted by $\text{den}(f)$ and $\text{num}(f)$, respectively. They are two coprime polynomials in $k[x, y]$, and numerators are monic with respect to a pre-selected term order.

This section contains three subsections. In §2.1, we describe a field that will serve as ground field in our subsequent algebraic constructions, and we define a (noncommutative) ring of Ore polynomials whose elements will be regarded as linear functional operators. In §2.2, we describe a (commutative) ring extension of the ground field, and recall from (Chen et al., 2011) the notion of compatible rational functions. In §2.3, we define the notion of mixed hypergeometric terms that occur in the study of existence of telescopers. The terms are contained in the ring extension described in the previous subsection.

2.1. *Fields endowed with a pair of operators*

Let $\delta_x = \partial/\partial_x$ be the usual derivation with respect to x, and $\delta_y = \partial/\partial_y$ be that with respect to y. For an element $f \in k(x, y)$, we define the shift operators σ_x and σ_y as

$$\sigma_x(f(x, y)) = f(x + 1, y) \quad \text{and} \quad \sigma_y(f(x, y)) = f(x, y + 1),$$

respectively, and q-shift operators τ_x and τ_y as

$$\tau_x(f(x, y)) = f(qx, y) \quad \text{and} \quad \tau_y(f(x, y)) = f(x, qy),$$

respectively. To describe the mixed cases concisely, we introduce the following notation.

Convention 2.1. Set $\Theta := \{\delta_x, \sigma_x, \tau_x\} \times \{\delta_y, \sigma_y, \tau_y\} \setminus \{(\delta_x, \delta_y), (\sigma_x, \sigma_y), (\tau_x, \tau_y)\}$. A pair (θ_x, θ_y) is always assumed to be in Θ, and is called a *mixed* pair of operators.

Note that $\theta_x \circ \theta_y(f) = \theta_y \circ \theta_x(f)$ for all $f \in k(x, y)$. In the sequel, $k(x, y)$ is usually endowed with a mixed pair (θ_x, θ_y) of operators. The resulting structure is denoted as $(k(x, y), (\theta_x, \theta_y))$. Given a field $(k(x, y), (\theta_x, \theta_y))$, one can define a ring of Ore polynomials (Chyzak and Salvy, 1998), which we denote here by $k(x, y)\langle\partial_x, \partial_y\rangle$. Its commutation rules are $\partial_x\partial_y = \partial_y\partial_x$ and, for every $f \in k(x, y)$ and $z \in \{x, y\}$, $\partial_z f = f\partial_z + \theta_z(f)$ if $\theta_z = \delta_z$; and $\partial_z f = \theta_z(f)\partial_z$ if $\theta_z \in \{\sigma_z, \tau_z\}$.

According to the different choices of operator pairs in Θ, the associated Ore rings correspond to the rings of linear differential-difference, differential-q-difference, and difference-q-difference operators, respectively. Telescopers to be studied in the sequel are regarded as elements of associated Ore rings. Since telescopers need to be studied type by type, we denote ∂_x by D_x, S_x and T_x when θ_x is chosen to be δ_x, σ_x and τ_x, respectively. The same convention applies when x is replaced by y. These fields and associated Ore rings are illustrated in Table 2.3.

Remark 2.2. Table 2.3 contains three choices, although there are six distinct mixed pairs of operators in Θ. This is because the last three pairs can be identified with the first three when the indeterminates x and y are switched.

Choice (θ_x, θ_y)	Ground field $(k(x,y), (\theta_x, \theta_y))$	Associated Ore ring $k(x,y)\langle\partial_x, \partial_y\rangle$
(δ_x, σ_y)	$(k(x,y), (\delta_x, \sigma_y))$	$k(x,y)\langle D_x, S_y\rangle$
(δ_x, τ_y)	$(k(x,y), (\delta_x, \tau_y))$	$k(x,y)\langle D_x, T_y\rangle$
(σ_x, τ_y)	$(k(x,y), (\sigma_x, \tau_y))$	$k(x,y)\langle S_x, T_y\rangle$

Table 2.3. Ground fields and rings of Ore polynomials

2.2. Linear functional operators and compatible rational functions

A first-order mixed linear-functional system is of the form

$$\begin{cases} \theta_x(z) = az, \\ \theta_y(z) = bz, \end{cases} \tag{1}$$

where $(\theta_x, \theta_y) \in \Theta$ and $a, b \in k(x,y)$. For brevity, we call (1) a *first-order mixed system*.

Example 2.4. Let $(\theta_x, \theta_y) = (\delta_x, \sigma_y)$, $a = y/x$ and $b = -x$. The system (1) becomes

$$\begin{cases} \delta_x(z) = \frac{y}{x} z, \\ \sigma_y(z) = -x\, z. \end{cases}$$

It is straightforward to verify that the expression $(-x)^y$ solves this mixed system. Moreover, this system does not have any nonzero rational solution in $k(x,y)$: if it had, we could write such a solution in the form

$$z = \frac{P}{Q} = \frac{p_m y^m + \cdots + p_0}{q_n y^n + \cdots + q_0}, \quad \text{where } p_i \text{ and } q_j \text{ are in } k(x) \text{ with } p_m q_n \neq 0.$$

By the equality $\sigma_y(z) = -xz$, we have $\sigma_y(P)Q = -xP\sigma_y(Q)$. Equating the leading coefficients with respect to y yields $p_m q_n = -x p_m q_n$, which implies that $x = -1$. This is a contradiction with the assumption that x is transcendental over k.

The example given above shows that solving a mixed system generally requires to extend the field $k(x,y)$. This motivates us to consider ring extensions of $k(x,y)$ endowed with a mixed pair of operators.

Definition 2.5. For a pair $(\theta_x, \theta_y) \in \Theta$, we call a tuple $\big(R, (\bar\theta_x, \bar\theta_y)\big)$ a *ring extension* of $(k(x,y), (\theta_x, \theta_y))$ if the following conditions are satisfied.
(i) R is a commutative ring containing $k(x,y)$.
(ii) $\bar\theta_x : R \to R$ is an extension of θ_x, and $\bar\theta_y : R \to R$ is an extension of θ_y.
(iii) $\bar\theta_x$ is a derivation on R if $\theta_x = \delta_x$, and it is a monomorphism if $\theta_x = \sigma_x$ or $\theta_x = \tau_x$.
(iv) $\bar\theta_y$ is a derivation on R if $\theta_y = \delta_y$, and it is a monomorphism if $\theta_y = \sigma_y$ or $\theta_y = \tau_y$.
(v) $\bar\theta_x$ and $\bar\theta_y$ commute.
Moreover, such a ring extension is said to be *simple* if there does not exist any ideal I of R such that $\bar\theta_x(I) \subset I$ and $\bar\theta_y(I) \subset I$ except for $I = R$ and $I = \{0\}$.

Without any possible ambiguity, we denote the operators $\bar\theta_x$ and $\bar\theta_y$ obtained as in the definition given above by θ_x and θ_y, respectively. The reader may find more general ring extensions endowed with derivations, shift and q-shift operators, potentially with respect to the same variable, in (Hardouin and Singer, 2008).

Let $L = \sum_{i,j} a_{i,j}\partial_x^i \partial_y^j$ be an Ore polynomial in $k(x,y)\langle\partial_x, \partial_y\rangle$, where $k(x,y)$ is endowed with a mixed pair (θ_x, θ_y) of operators. Let $(R, (\theta_x, \theta_y))$ be a ring extension of $(k(x,y), (\theta_x, \theta_y))$. The application of L to an element $r \in R$ is defined to be

$$L(r) = \sum_{i,j} a_{i,j}\theta_x^i \circ \theta_y^j(r).$$

So an Ore polynomial can be viewed as a linear functional operator on R. One can verify that multiplication of Ore polynomials and composition of linear functional operators are compatible, that is, $(L_1L_2)(r) = L_1(L_2(r))$ for $L_1, L_2 \in k(x,y)\langle\partial_x, \partial_y\rangle$ and $r \in R$.

We are about to define the constants of a given field $(k(x,y), (\theta_x, \theta_y))$, by describing them in a uniform way as the solutions of specific operators. Define

$$\Delta_x := \partial_x - \partial_x(1) = \begin{cases} D_x & \text{if } \theta_x = \delta_x, \\ S_x - 1 & \text{if } \theta_x = \sigma_x, \\ T_x - 1 & \text{if } \theta_x = \tau_x. \end{cases} \tag{2}$$

We define Δ_y similarly by replacing x with y. An element $c \in R$ is then called a *constant* with respect to the pair (θ_x, θ_y) if

$$\Delta_x(c) = \Delta_y(c) = 0.$$

One can easily verify that $c \in k(x,y)$ is a constant with respect to (θ_x, θ_y) if and only if c is an element of k.

Given a system of the form (1), a basic question is whether there exists a ring extension $(R, (\theta_x, \theta_y))$ containing a nonzero solution of the system. This question is related to compatibility conditions of (1).

Let $k(x,y)$ be endowed with a mixed pair (θ_x, θ_y) of operators. If system (1) has a nonzero solution h in a ring extension $(R, (\theta_x, \theta_y))$, then $\theta_y \circ \theta_x(h) = \theta_x \circ \theta_y(h)$ by the commutativity of θ_x and θ_y. In addition, if θ_x, resp. θ_y, is a monomorphism, then a, resp. b, is nonzero. So (1) satisfies the following compatibility conditions:

$$\begin{cases} \frac{\delta_x(b)}{b} = \sigma_y(a) - a \text{ and } b \neq 0 & \text{if } (\theta_x, \theta_y) = (\delta_x, \sigma_y), \\ \frac{\delta_x(b)}{b} = \tau_y(a) - a \text{ and } b \neq 0 & \text{if } (\theta_x, \theta_y) = (\delta_x, \tau_y), \\ \frac{\sigma_x(b)}{b} = \frac{\tau_y(a)}{a} \text{ and } ab \neq 0 & \text{if } (\theta_x, \theta_y) = (\sigma_x, \tau_y), \end{cases} \tag{3}$$

if it has a nonzero solution in a ring extension of $(k(x,y), (\theta_x, \theta_y))$.

Remark 2.6. The compatibility conditions corresponding to the other three mixed pairs of operators in Θ can be obtained by swapping x with y and a with b in (3).

Definition 2.7. Let $(a, b) \in k(x,y) \times k(x,y)$ and $(\theta_x, \theta_y) \in \Theta$. We say that a and b are *compatible* with respect to (θ_x, θ_y) if the compatibility conditions corresponding to (θ_x, θ_y) in (3) and Remark 2.6 are satisfied.

A first-order mixed system of the form (1) is said to be *compatible* if its coefficients a and b are compatible with respect to (θ_x, θ_y). Conversely, given a compatible mixed system of the form (1), Theorem 2 in (Bronstein et al., 2005) implies that there exists a simple ring extension $(R, (\theta_x, \theta_y))$ of $(k(x,y), (\theta_x, \theta_y))$ containing a nonzero solution of (1). Moreover, under the assumption that k is algebraically closed, R contains no new constant other than the elements of k. Such a simple ring is called a *Picard–Vessiot extension* associated to (1).

2.3. Mixed hypergeometric terms

Hypergeometric terms are a common abstraction of geometric terms, factorials, and binomial coefficients. They play an important role in combinatorics. The continuous analogue of hypergeometric terms is hyperexponential functions: they generalize usual exponential functions and simple radicals. In this paper, we will consider a class of functions in x and y that are solutions of first-order mixed systems, and are therefore intermediate objects between hypergeometric terms and hyperexponential functions.

Definition 2.8. Let $k(x, y)$ be a field endowed with a mixed pair (θ_x, θ_y) of operators. Assume that $(R, (\theta_x, \theta_y))$ is a simple ring extension of $(k(x, y), (\theta_x, \theta_y))$, and that the set of constants in R is equal to k. A *nonzero* element h of R is called a *mixed hypergeometric term* over $(k(x, y), (\theta_x, \theta_y))$ if there exist $a, b \in k(x, y)$ such that

$$\theta_x(h) = ah \quad \text{and} \quad \theta_y(h) = bh.$$

We call a the *certificate of h with respect to θ_x*, and b the *certificate with respect to θ_y*.

The certificates of a mixed term are compatible rational functions, because θ_x and θ_y commute. For brevity, a mixed hypergeometric term will be called a mixed term in the sequel. Viewing mixed terms in an abstract ring allows us to compute their sums, products and inverses legitimately. Moreover, we will never encounter any analytic considerations, such as singularities and the regions of definition. This choice will not do any harm, as the problem we are dealing with is purely algebraic.

We recall some basic facts about mixed terms in this ring setting. These facts are scattered in the literature. We summarize them for the convenience of later references.

The first lemma says that all mixed terms form a multiplicative group, and that two mixed terms with the same certificates differ by a multiplicative constant.

Lemma 2.9. Let the ring extension $(R, (\theta_x, \theta_y))$ be given as in Definition 2.8.

(i) The product of mixed terms is a mixed term, and every mixed term is invertible.

(ii) If two mixed terms have the same certificates, then their ratio belongs to k.

Proof. (i) The closure under product follows from simple calculations with certificates. Now, let h be a mixed term in R, and I be the ideal generated by h in R. Then $\theta_x(h)$ and $\theta_y(h)$ belong to I. It follows that $\theta_x(I) \subset I$ and $\theta_y(I) \subset I$. Since R is simple and h is nonzero, $I = R$, that is, $1 \in I$. Consequently, h is invertible.

(ii) Let h_1 and h_2 be two mixed terms in R. If they have the same certificates, then h_1/h_2 is a constant by a straightforward calculation, that is to say, $h_1 = ch_2$ for some $c \in k$. □

By the second assertion of Lemma 2.9, two mixed terms having the same certificates differ by a multiplicative constant. These constants are irrelevant to the main result of this paper. So we introduce a notation to suppress them.

Let h be a mixed term in R with θ_x-certificate a and θ_y-certificate b. Set

$$\mathcal{H}(a, b) = \{ch \mid c \in k\}.$$

The set consists of zero and mixed terms in R whose respective certificates are a and b. Clearly, $\mathcal{H}(a,b)$ is a one-dimensional linear subspace over k. In the sequel, whenever the notation $\mathcal{H}(a,b)$ is used, a and b are assumed to be compatible rational functions in $k(x,y)$, and $\mathcal{H}(a,b) \subset R$. In particular, for a nonzero rational function $f \in k(x,y)$, the set $f\mathcal{H}(a,b)$ is a subset of R. Indeed, it is the one-dimensional linear subspace spanned by fh over k. By the definition of certificates, we have

$$\theta_x(\mathcal{H}(a,b)) = a\mathcal{H}(a,b) \quad \text{and} \quad \theta_y(\mathcal{H}(a,b)) = b\mathcal{H}(a,b).$$

Let h' be another mixed term in R with θ_x-certificate a' and θ_y-certificate b'. Define

$$\mathcal{H}(a,b)\mathcal{H}(a',b') = \{gg' \mid g \in \mathcal{H}(a,b), g' \in \mathcal{H}(a',b')\},$$

which is equal to the one-dimensional linear subspace spanned by hh' over k.

More rules for manipulating $\mathcal{H}(a,b)$ are given below. They are used for computing the certificates of the product of two mixed terms.

Lemma 2.10. For a field $(k(x,y), (\theta_x, \theta_y))$, we let $\mathcal{H}(a,b)$ and $\mathcal{H}(a',b')$ be given as above, and let f be a nonzero rational function in $k(x,y)$.

(i) If $\theta_x = \delta_x$, then

$$f\mathcal{H}(a,b) = \mathcal{H}\left(a + \frac{\theta_x(f)}{f}, b\frac{\theta_y(f)}{f}\right) \quad \text{and} \quad \mathcal{H}(a,b)\mathcal{H}(a',b') = \mathcal{H}(a+a', bb').$$

(ii) If $\theta_y = \delta_y$, then

$$f\mathcal{H}(a,b) = \mathcal{H}\left(a\frac{\theta_x(f)}{f}, b + \frac{\theta_y(f)}{f}\right) \quad \text{and} \quad \mathcal{H}(a,b)\mathcal{H}(a',b') = \mathcal{H}(aa', b+b').$$

(iii) If either $(\theta_x, \theta_y) = (\sigma_x, \tau_y)$ or $(\theta_x, \theta_y) = (\tau_x, \sigma_y)$, then

$$f\mathcal{H}(a,b) = \mathcal{H}\left(a\frac{\theta_x(f)}{f}, b\frac{\theta_y(f)}{f}\right) \quad \text{and} \quad \mathcal{H}(a,b)\mathcal{H}(a',b') = \mathcal{H}(aa', bb').$$

Proof. Let h be a mixed term in $\mathcal{H}(a,b)$.

(i) Assume that $\theta_x = \delta_x$ and $\theta_y \in \{\sigma_y, \tau_y\}$. It is straightforward to verify that the θ_x-certificate and θ_y-certificate of fh are $a + \theta_x(f)/f$ and $b\theta_y(f)/f$, respectively. Assume further that $h' \in \mathcal{H}(a',b')$ with $hh' \neq 0$. Then the θ_x-certificate of hh' is $a + a'$, and its θ_y-certificate is bb'. It follows that $\mathcal{H}(a,b)\mathcal{H}(a',b') \subset \mathcal{H}(a+a', bb')$. Equality is then a consequence of the two sets being one-dimensional vector spaces. This proves part (i).

Parts (ii) and (iii) can be proved in a similar way. □

Two mixed terms h_1 and h_2 are said to be *similar* if the ratio h_1/h_2 is in $k(x,y) \setminus \{0\}$. Similarity is an equivalence relation. When studying the existence of telescopers, we will encounter at most finitely many mixed terms that are dissimilar to each other. These terms can be regarded as elements in a simple ring extension, because a finite number of Picard–Vessiot extensions associated to compatible first-order mixed systems can be embedded into a simple ring (Li et al., 2006, §2.2). From now on, we assume that R is given as in Definition 2.8. It will be sufficient to consider mixed terms in R.

The next lemma shows that the set consisting of zero and mixed terms similar to each other form a linear space over $k(x,y)$, which is closed under the application of every linear functional operator in $k(x,y)\langle\partial_x, \partial_y\rangle$.

Lemma 2.11. Let g and h be two mixed terms over $(k(x,y), (\theta_x, \theta_y))$. If g and h are similar, then

(i) $g+h$ is either equal to zero or similar to h;

(ii) for any $L \in k(x,y)\langle \partial_x, \partial_y \rangle$, $L(h)$ is either equal to zero or similar to h.

Proof. (i) Let $r \in k(x,y)$ be equal to g/h. Then $g+h = (r+1)h$.

(ii) Since h is a mixed term, its successive derivatives and (q-)shifts are either equal to zero or similar to h. So $L(h)$ is either equal to zero or similar to h by part (i). $\square$

Remark 2.12. Let h, h_1, h_2 be three mixed terms. If $h = h_1 + h_2$, then the three terms are similar, because the sum of two dissimilar mixed terms is not a mixed term.

The next two examples illustrate how a linear differential or recurrence operator applies to mixed terms similar to a given one. The results will be used for establishing our criterion.

Example 2.13. Consider how to apply D_x^i to rh, where r is a rational function in $k(x,y)$ and h is a mixed term in $\mathcal{H}(u,v)$ for some $u, v \in k(x,y)$.

First, $D_x(rh) = (\delta_x(r) + ru)\, h$. Putting $L_1 = D_x + u$, we rewrite the above relation as $D_x(rh) = L_1(r)h$.

An easy induction shows that

$$D_x^i(rh) = L_i(r)h,$$

where $L_i = (D_x + u)^i \in k(x,y)\langle D_x \rangle$ has coefficients whose common denominators divide some power of $\operatorname{den}(u)$. Moreover, the denominator of $L_i(r)$ divides $(\operatorname{den}(u)\operatorname{den}(r))^{i+1}$.

Example 2.14. Consider how to apply S_x^i to rh, where r and h are the same as those in the above example.

Let $M_i = \left(\prod_{j=0}^{i-1} \sigma_x^j(u)\right) S_x^i$ for $i > 0$. Then an easy induction shows that

$$S_x^i(rh) = M_i(r)h.$$

A similar result holds when the shift operator S_x is replaced by the q-shift operator T_x.

3. Telescopers for mixed hypergeometric terms

The method of creative telescoping was first formulated and popularized in a series of papers by Zeilberger and his collaborators in the early 1990's (Almkvist and Zeilberger, 1990; Zeilberger, 1990a,b, 1991; Wilf and Zeilberger, 1992a). To illustrate the idea of this method, we consider the problem of finding a linear recurrence equation for the integral (if there exists one):

$$H(x) := \int_0^{+\infty} h(x,y)\, dy,$$

where $h(x,y)$ is a mixed term over $(k(x,y), (\sigma_x, \delta_y))$. Suppose that all integrals occurring in the derivation below are well-defined. The key step of creative telescoping tries to find a nonzero linear recurrence operator $L(x, S_x)$ in $k(x)\langle S_x \rangle$ such that

$$L(x, S_x)(h) = D_y(g), \tag{4}$$

for some mixed term g over $k(x, y)$.

Applying the integral sign to both sides of (4) yields

$$L(x, S_x)(H(x)) = g(x, +\infty) - g(x, 0).$$

This implies that $L(x, S_x)$ is indeed the recurrence relation satisfied by $H(x)$ under certain nice boundary condition, say $g(x, +\infty) = g(x, 0)$. For example, consider the integral

$$A(x) = \int_0^{+\infty} y^{x-1} \exp(-y)\, dy.$$

The differential variant of Zeilberger's algorithm in Almkvist and Zeilberger (1990) delivers a pair (L, g) with

$$L = S_x - x \quad \text{and} \quad g = -y^x \exp(-y).$$

Note that, if $x > 0$, then $g(x, +\infty) = g(x, 0) = 0$, which implies that

$$L(A(x)) = A(x + 1) - xA(x) = 0.$$

So we recognize the solution $A(x) = \Gamma(x)$ since the initial value $A(1)$ is equal to 1. For more interesting examples, see the appendix of (Almkvist and Zeilberger, 1990) or the book by Koepf (1998, Chapters 10–13).

Definition 3.1. Let h be a mixed term over $(k(x, y), (\theta_x, \theta_y))$. A nonzero linear operator $L(x, \partial_x) \in k(x)\langle\partial_x\rangle$ is called a *telescoper of type* (∂_x, ∂_y) for h if there exists another mixed term g such that

$$L(x, \partial_x)(h) = \Delta_y(g). \tag{5}$$

Alternatively, the mixed term h is said to be *telescopable of type* (∂_x, ∂_y) if it has a telescoper of type (∂_x, ∂_y).

For a given mixed term, when does a telescoper of certain type exist? And how can one construct telescopers? These are two basic problems related to the method of creative telescoping. In the subsequent sections, we will answer the first one for the mixed cases. More precisely, we solve the following problem, which is equivalent to the termination problem of creative-telescoping algorithms for mixed inputs.

Existence Problem for Telescopers. For a mixed term h over $(k(x, y), (\theta_x, \theta_y))$, find a necessary and sufficient condition on the existence of telescopers of type (∂_x, ∂_y) for h.

Remark 3.2. For a mixed term h, the existence of a telescoper of type (∂_x, ∂_y) does not imply the existence of a telescoper of type (∂_y, ∂_x) (see Example 6.4).

Remark 3.3. By Lemma 2.11, the mixed term g in (5) is similar to h if $\Delta_y(g)$ is nonzero. Otherwise, g can be chosen to be 1.

4. Exact and proper terms

To study the existence problem for telescopers, we need two key notions: exact terms and proper terms. They are related to additive and multiplicative decompositions of mixed terms, respectively.

This section contains three subsections. In §4.1, we define the notion of exact terms and describe an additive decomposition of mixed terms, based on the additive decompositions for univariate (q)-hypergeometric terms and hyperexponential functions in (Abramov and Petkovsek, 2002b; Geddes et al. , 2004; Chen et al., 2005). In §4.2, we recall the notion of split polynomials from (Chen, 2011) and that of spread polynomials from (Abramov, 2003), and describe a relation among the two notions and exact terms. In §4.3, we recall a multiplicative decomposition of mixed terms from (Chen et al., 2011), define the notion of proper terms, and study how to decide whether a mixed term is proper.

4.1. Exact terms amd additive decompostions

For a univariate hypergeometric term $H(y)$, the Gosper algorithm (Gosper, 1978) decides whether it is hypergeometric-summable (also known as Gosper-summable) with respect to y, i.e., whether $H = (S_y - 1)(G)$ for some hypergeometric term G. Based on the Gosper algorithm, Zeilberger (1990a,b) developed his fast version of creative-telescoping algorithms for bivariate hypergeometric terms. Almkvist and Zeilberger (1990) presented a continuous analogue of the Gosper algorithm for deciding the hyperexponential integrability, which leads to a fast algorithm for hyperexponential telescoping. From the viewpoint of creative telescoping, the Gosper algorithm and its continuous analogue decide whether the identity operator, 1, is a telescoper for the inputs.

The following notion of *exact terms* is motivated in the differential case by the existence of an underlying exact form. This differential-form point of view was taken in (Chen et al., 2012) recently.

Definition 4.1. Let h be a mixed term over $(k(x, y), (\theta_x, \theta_y))$. We say that h is *exact with respect to* ∂_x if there exists a mixed term g such that $h = \Delta_x(g)$, where Δ_x is defined in (2). An exact term with respect to ∂_y is defined likewise.

Remark 4.2. In (Abramov and Petkovsek, 2002b) and (Geddes et al. , 2004), an exact term is traditionally called a *(q-)hypergeometric-summable* term in the (q-)discrete case, and a *hyperexponential-integrable* function in the continuous case, respectively. For each choice of ∂_x in $\{D_x, S_x, T_x\}$, it is clear that every exact term with respect to ∂_y has a telescoper of type (∂_x, ∂_y): for instance 1 is such a telescoper.

The next notion to be introduced, related to exact terms, is that of additive decompositions. An algorithm by Abramov and Petkovsek (2002b) decomposes a hypergeometric term $H(y)$ into the sum $\Delta_y(H_1) + H_2$, where H_2 is minimal in some sense. Such a decomposition is called an *additive decomposition* for H with respect to y. Abramov and Petkovšek's algorithm generalizes the capability of the Gosper algorithm in the sense that H is hypergeometric-summable if and only if H_2 is zero. In the continuous case, an algorithm to decompose a hyperexponential function $H(y)$ as $D_y(H_1) + H_2$, where H_1 and H_2 are either zero or hyperexponential, is part of the proof of Lemma 4.2 in Davenport (1986). This remained unknown to Geddes et al. (2004), who later described a similar additive decomposition as a continuous analogue of Abramov and Petkovsek's algorithm, but also prove that H_2 satisfies certain minimality requirements. Based on the continuous analogue, Bostan et al. (2013) presented a reduction algorithm that decomposes a hyperexponential function into the sum of an integrable one and a non-integrable one in a unique way. On the other hand, a q-discrete analogue is presented in (Chen et al., 2005). When H is a rational function, additive decompositions are more classical; they were presented by Ostrogradskiĭ (1845) and Hermite (1872) for the continuous case, and by Abramov (1975, 1995) for the discrete and q-discrete cases.

For a mixed term h over $(k(x, y), (\theta_x, \theta_y))$, we can perform three kinds of additive decompositions with respect to y according to the choice of θ_y. We recall now the notions related to additive decompositions.

Definition 4.3. Let K be a field of characteristic zero, and a be a nonzero polynomial in $K[z]$. Denote by δ_z, σ_z, and τ_z the usual derivation, shift and q-shift operators with respect to z on $K(z)$, respectively.

(i) a is said to be *δ_z-free*, or *squarefree*, if $\gcd(a, \delta_z(a)) = 1$.
(ii) a is said to be *σ_z-free*, or *shift-free*, if $\gcd(a, \sigma_z^i(a)) = 1$ for every nonzero integer i.
(iii) Let $a = z^m\tilde{a}$ with $\tilde{a} \in K[z]$ and $z \nmid \tilde{a}$. Then a is said to be *τ_z-free*, or *q-shift-free*, if

$$\gcd\left(\tilde{a}, \tau_z^i(\tilde{a})\right) = 1 \quad \text{for every nonzero integer } i.$$

Moreover, let f be a nonzero rational function in $K(z)$. Set $a = \operatorname{num}(f)$ and $b = \operatorname{den}(f)$.

(iv) f is said to be *δ_z-reduced* if $\gcd(b, a - i\delta_z(b)) = 1$ for all $i \in \mathbb{Z}$;
(v) f is said to be *σ_z-reduced* if $\gcd(b, \sigma_z^i(a)) = 1$ for all $i \in \mathbb{Z}$; and
(vi) f is said to be *τ_z-reduced* if $\gcd(b, \tau_z^i(a)) = 1$ for all $i \in \mathbb{Z}$.

A univariate polynomial is δ_z-free if and only if it has no multiple root. It is σ_z-free if and only if its distinct roots do not differ by an integer additively; and it is τ_z-free if and only if its distinct nonzero roots do not differ by a power of q multiplicatively.

A univariate rational function is δ_z-reduced if it has no integral residue at any simple pole by Lemma 2 in (Geddes et al. , 2004). It is σ_z-reduced (resp. τ_z-reduced) if any root of the numerator and any root of the denominator do not differ by an integer additively (resp. a power of q multiplicatively).

For a polynomial p in $k[x, y]$, we say that it is θ_y-free when p is θ_y-free as a polynomial in y over $k(x)$. The same convention applies to θ_y-reduced functions.

Finally, we define an additive decomposition in the setting of mixed terms.

Definition 4.4. Let h be a mixed term over $(k(x, y), (\theta_x, \theta_y))$. Assume that

$$h = \Delta_y(h_1) + h_2 \tag{6}$$

where h_1 is a mixed term, and h_2 is equal to either zero or a mixed term. We call (6) an *additive decomposition* of h with respect to ∂_y if there exist $r \in k(x, y)$ with a θ_y-free denominator, and compatible rational functions $u, v \in k(x, y)$ with v being θ_y-reduced such that

$$h_2 \in r\mathcal{H}(u, v). \tag{7}$$

The additive decomposition with respect to ∂_x is defined likewise.

We remark that the additive decompositions given in Definition 4.4 are more weakly constrained than those in (Abramov and Petkovšek, 2001) and (Geddes et al. , 2004). For example, h_2 is not necessarily equal to zero when h is an exact term.

4.2. Split and spread polynomials

Split polynomials defined in (Chen, 2011) play the same role as integer-linear polynomials in the difference case. A polynomial $p \in k[x, y]$ is said to be *split* if it is of the form $p_1(x)p_2(y)$ with $p_1 \in k[x]$ and $p_2 \in k[y]$. A rational function $r \in k(x, y)$ is said to be *split* if it is of the form $r_1(x)r_2(y)$ with $r_1 \in k(x)$ and $r_2 \in k(y)$.

A rational function $f \in k(x, y)$ can always be decomposed as $f_1(x)f_2(y)f_3(x, y)$, where $f_1 \in k(x)$, $f_2 \in k(y)$ and neither $\operatorname{num}(f_3)$ nor $\operatorname{den}(f_3)$ have split factors except constants. We call f_1f_2 and f_3 the *split* and *non-split parts* of f, respectively. Both are defined up to a nonzero multiplicative constant.

Remark 4.5. For $p \in k[x, y]$, one may decide whether it is split by comparing all monic normalized coefficients of p with respect to y. More precisely, p is split if and only if all those are equal. In an implementation, one would abort as soon as a mismatch is found.

Remark 4.6. A nontrivial and nonsplit polynomial $p \in k[x, y]$ has at least two terms. Since q is not a root of unity, $\tau_y^i(p)$ and $\tau_y^j(p)$ are coprime for all $i, j \in \mathbb{Z}$ with $i \neq j$ if p is irreducible.

The notion of spread polynomials is introduced by Abramov (2003) for establishing his criterion on the existence of telescopers for hypergeometric terms. We extend this notion to the mixed setting so as to connect split rational functions with exact terms.

Definition 4.7. Let K be a field of characteristic zero, and δ_z, σ_z, and τ_z be the usual derivation, shift and q-shift operators on $K[z]$, respectively. For a polynomial $a \in K[z]$ of positive degree, we say that:

(i) a is *δ_z-spread* if every nontrivial irreducible factor of a has multiplicity > 1;
(ii) a is *σ_z-spread* if, for every nontrivial irreducible factor b of a, $\sigma_z^i(b) \mid a$ for some nonzero integer i;
(iii) a is *τ_z-spread* if, for every nontrivial irreducible factor b of a with $z \nmid b$, $\tau_z^i(b) \mid a$ for some nonzero integer i.

Clearly, δ_z-spread polynomials are not δ_z-free. And there are polynomials that are neither δ_z-spread nor δ_z-free. The same observations hold for σ_z-spread polynomials and for τ_z-spread polynomials that have at least two terms.

For a polynomial p in $k[x, y]$ of positive degree in y, we say that it is θ_y-spread when p is θ_y-spread as a polynomial in y over $k(x)$. Case (iii) of Definition 4.7 restricts the factors b in a way that makes no constraint on the multiplicity of z in a. This is because we shall only consider τ_y-spread *non-split* polynomials in what follows.

Next, we present a mixed analogue of Theorem 8 in (Abramov, 2003), which is the basis of a key argument in the proof of the main conclusion in Section 5.

Proposition 4.8. Let h be a mixed term over $(k(x, y), (\theta_x, \theta_y))$, and assume $h \in f\mathcal{H}(u, v)$, where $f, u, v \in k(x, y)$. Let w be the non-split part of $\operatorname{den}(f)$. Assume that h is exact with respect to ∂_y, and that w is not in k.

(i) If $\theta_y = \delta_y$ and $\operatorname{den}(v)$ is split, then w is δ_y-spread.
(ii) If $\theta_y \in \{\sigma_y, \tau_y\}$ and v is split, then w is θ_y-spread.

Proof. Since h is exact, there exists a mixed term g such that $h = \Delta_y(g)$. By Remark 3.3, g is similar to h, that is, $g \in r\mathcal{H}(u,v)$ for some $r \in k(x,y)$. Accordingly,

$$f\mathcal{H}(u,v) = \Delta_y(r\mathcal{H}(u,v)). \tag{8}$$

Let p be a non-split irreducible factor of w with $\deg_x p > 0$ and $\deg_y p > 0$.

(i) Assume that $\theta_y = \delta_y$ and $\mathrm{den}(v)$ is split. By (8), there exists $c \in k \setminus \{0\}$ such that

$$cf = \delta_y(r) + rv.$$

Since $\mathrm{den}(v)$ is split and p divides $\mathrm{den}(f)$, p is an irreducible factor of $\mathrm{den}(r)$. So there exists an integer $i > 1$ such that $p^i \mid \mathrm{den}(\delta_y(r))$ and $p^i \nmid \mathrm{den}(r)$. Therefore, $p^i \nmid \mathrm{den}(rv)$. It follows that $p^i \mid \mathrm{den}(f)$, and, thus, $p^i \mid w$. This proves part (i).

(ii) Assume that $\theta_y = \sigma_y$ and v is split. By (8), there exists $c \in k \setminus \{0\}$ such that

$$cf = v\sigma_y(r) - r. \tag{9}$$

Since v is split, $p \mid \mathrm{den}(r)$ or $p \mid \mathrm{den}(\sigma_y(r))$. So, the set

$$L := \{\ell \in \mathbb{Z} \text{ such that } \sigma_y^\ell(p) \mid \mathrm{den}(r)\}$$

is nonempty (consider $0 \in L$ and $-1 \in L$, respectively) and finite. Therefore, there exist $i, j \in \mathbb{Z}$ with $i > j$ such that $i-1 \in L$, $i \notin L$, $j-1 \notin L$, $j \in L$, that is

$$\sigma_y^i(p) \mid \mathrm{den}(\sigma_y(r)),\ \ \sigma_y^i(p) \nmid \mathrm{den}(r),\ \ \sigma_y^j(p) \nmid \mathrm{den}(\sigma_y(r)),\ \text{ and } \sigma_y^j(p) \mid \mathrm{den}(r).$$

It follows from (9) that both $\sigma_y^i(p)$ and $\sigma_y^j(p)$ divide $\mathrm{den}(f)$. Since $i \neq j$, we can find $\ell \neq 0$ in $\{i, j\}$. Then, both p and $\sigma_y^\ell(p)$ divide w. So w is σ_y-spread, and part (ii) holds.

The case $\theta_y = \tau_y$ can be handled in the same vein by Remark 4.6 □

4.3. Structural decompositions and proper terms

By the Ore-Sato theorem in (Ore, 1930; Sato, 1990), every bivariate hypergeometric term is the product of a rational function and a factorial term. Using this multiplicative decomposition, one can define the notion of proper hypergeometric terms. Wilf and Zeilberger (1992a) show that every proper hypergeometric term has a telescoper. In this section, we apply a mixed analogue of the Ore-Sato theorem in (Chen et al., 2011) to define the notion of proper mixed terms.

Let (θ_x, θ_y) be a mixed pair of operators in Θ. For a pair (a, b) of compatible rational functions in $k(x,y) \times k(x,y)$, we obtain the conclusions listed in Table 4.9 by Proposition 6.1 in (Chen et al., 2011). In fact, the conclusions can also be derived from Lemmas 3.1, 3.2 and 3.3 in (Chen et al., 2011), because we are only concerned with bivariate mixed terms over $(k(x,y), (\theta_x, \theta_y))$.

Example 4.10. Let $h \in \mathcal{H}(a,b)$ be a mixed term over $(k(x,y), (\theta_x, \theta_y))$.

If $(\theta_x, \theta_y) = (\delta_x, \sigma_y)$, then there exist $f \in k(x,y)$, $\beta \in k(x)$, a univariate hyperexponential function $E(x)$, and a univariate hypergeometric term $G(y)$ such that

$$h = f(x,y)\beta(x)^y E(x)G(y)$$

by Proposition 6.1 in (Chen et al., 2011). Setting $\alpha(x)$ to be $\delta_x(E)/E$ and $\gamma(y)$ to be $\sigma_y(G)/G$, we see that part 1 in Table 4.9 holds by Lemma 2.10 (i).

	Choice	$(a,b) \in k(x,y) \times k(x,y)$ compatible w.r.t. (θ_x, θ_y)
1.	$(\theta_x, \theta_y) = (\delta_x, \sigma_y)$	$\exists\ f \in k(x,y)$, $\alpha, \beta \in k(x)$, and $\gamma \in k(y)$ such that $\mathcal{H}(a,\, b) = f\mathcal{H}\left(y\frac{\delta_x(\beta)}{\beta} + \alpha,\, \beta \cdot \gamma\right)$.
2.	$(\theta_x, \theta_y) = (\sigma_x, \delta_y)$	$\exists\ f \in k(x,y)$, $\alpha, \beta \in k(y)$, and $\gamma \in k(x)$ such that $\mathcal{H}(a,\, b) = f\mathcal{H}\left(\beta \cdot \gamma,\, x\frac{\delta_y(\beta)}{\beta} + \alpha\right)$.
3.	3.1. $(\theta_x, \theta_y) = (\delta_x, \tau_y)$ 3.2. $(\theta_x, \theta_y) = (\tau_x, \delta_y)$ 3.3. $(\theta_x, \theta_y) = (\sigma_x, \tau_y)$ 3.4. $(\theta_x, \theta_y) = (\tau_x, \sigma_y)$	$\exists\ f \in k(x,y)$, $\alpha \in k(x)$, and $\beta \in k(y)$ such that $\mathcal{H}(a,\, b) = f\mathcal{H}(\alpha,\, \beta)$.

Table 4.9. Multiplicative decompositions of mixed terms

If $(\theta_x, \theta_y) = (\sigma_x, \tau_y)$, then there exist $f \in k(x,y)$, $\alpha \in k(x)$ and $\beta \in k(y)$ such that

$$a = \frac{\sigma_x(f)}{f}\alpha \quad \text{and} \quad b = \frac{\tau_y(f)}{f}\beta$$

by Lemma 3.3 in (Chen et al., 2011). Part 3.3 in Table 4.9 holds by Lemma 2.10 (iii).

The decomposition of sets in Table 4.9 induces a decomposition of mixed terms.

Definition 4.11. Given a mixed term $h \in \mathcal{H}(a,b)$, the product fh' of a rational function $f \in k(x,y)$ and a mixed term h' is called a *structural decomposition* of h if $h = fh'$ and if h' has certificates given by the rational functions α, β, γ in Table 4.9 for the same f.

A mixed term may have more than one structural decompositions, as illustrated below.

Example 4.12. Let $h = x(x+y)x^y \exp(x)\Gamma(y)$ be a mixed term over $(k(x,y), (\delta_x, \sigma_y))$. Then the two certificates of h are, respectively,

$$a := \frac{\delta_x(h)}{h} = \frac{x^2 + 2xy + y^2 + 2x + y}{x(x+y)} \quad \text{and} \quad b := \frac{\sigma_y(h)}{h} = \frac{xy(x+y+1)}{x+y}.$$

The mixed term h has at least two structural decompositions $h = f_i h_i$ with $i \in \{1,2\}$,

$$f_i \in k(x,y) \quad \text{and} \quad h_i \in \mathcal{H}\left(y\frac{\delta_x(\beta_i)}{\beta_i} + \alpha_i,\, \beta_i \cdot \gamma_i\right),$$

where $f_1 = x + y$, $\beta_1 = x$, $\alpha_1 = (x+1)/x$, and $\gamma_1 = y$; and $f_2 = x(x+y)$, $\beta_2 = x/2$, $\alpha_2 = 1$, and $\gamma_2 = 2y$.

The next definition is fundamental for our existence criterion.

Definition 4.13. Let h be a mixed term over $(k(x,y),\, (\theta_x, \theta_y))$. We say that h is a *proper term* if there exists a structural decomposition $h = fh'$ for which $\mathrm{den}(f)$ is split.

Two questions concern this definition of proper terms. First, defining properness by the existence of *some* structural decomposition with a prefactor f whose denominator is split raises the question whether *any* structural decomposition of a given proper h will have the property witnessing properness. In order to derive a properness test, it is indeed of importance to understand if any structural decomposition will do, or if one needs to consider a restricted type of structural decompositions. Second, a decomposition of the form (7) in Definition 4.4 can be computed easily, but it is not necessarily a *structural* decomposition. This raises a question of algorithmic importance, namely, to what extend one could decide properness without appealing to any structural decomposition.

In the rest of this subsection, we answer these two questions.

Lemma 4.14. Let f and r be two nonzero rational functions in $k(x, y)$.

(i) If the denominator of $\theta(f)/f$ is split for some $\theta \in \{\delta_x, \delta_y\}$, so is f.
(ii) If $\theta(f)/f$ is split for some $\theta \in \{\sigma_x, \sigma_y, \tau_x, \tau_y\}$, so is f.
(iii) If both $\mathrm{den}(r)$ and f/r are split, so is $\mathrm{den}(f)$.

Proof. (i) It is immediate from the logarithmic derivative identity.

(ii) Suppose that f is not split. Then there exists a nontrivial non-split irreducible polynomial p dividing $\mathrm{den}(f)\cdot\mathrm{num}(f)$. Assume further that $\sigma_x^i(p)$ does not divide $\mathrm{den}(f)\cdot\mathrm{num}(f)$ for all $i > 0$. Then $\sigma_x(p)$ is a factor of either the numerator or the denominator of the rational function $\sigma_x(f)/f$. Thus, $\sigma_x(f)/f$ is not split, a contradiction. The same argument applies to $\tau_x(f)/f$ by Remark 4.6. Likewise, part (ii) holds for $\theta \in \{\sigma_y, \tau_y\}$.

(iii) Note that

$$\frac{f}{r} = \frac{\mathrm{num}(f)\cdot \mathrm{den}(r)}{\mathrm{den}(f)\cdot \mathrm{num}(r)}.$$

Suppose for a contradiction that $\mathrm{den}(f)$ has a non-split irreducible factor p. Since f/r is split, p divides $\mathrm{num}(f)\cdot\mathrm{den}(r)$. Since $\mathrm{den}(f)$ and $\mathrm{num}(f)$ are coprime, p divides $\mathrm{den}(r)$, a contradiction to the assumption that $\mathrm{den}(r)$ is split. $\square$

Proposition 4.15. Let h be a mixed term over $(k(x, y), (\theta_x, \theta_y))$. Assume $h \in r\mathcal{H}(u, v)$ for some $r, u, v \in k(x, y)$ and that $\mathrm{den}(r)$ is split. Then the following statements hold:

(i) Assume that $\theta_x = \delta_x$. If $\mathrm{den}(u)$ or v is split, then h is proper.
(ii) Assume that $\theta_y = \delta_y$. If u or $\mathrm{den}(v)$ is split, then h is proper.
(iii) Assume that $(\theta_x, \theta_y) = (\sigma_x, \tau_y)$ or $(\theta_x, \theta_y) = (\tau_x, \sigma_y)$. If u or v is split, then h is proper.

Proof. (i) We assume that $(\theta_x, \theta_y) = (\delta_x, \sigma_y)$. For any structural decomposition $h = fh'$ given by $f \in k(x, y)$, $\alpha, \beta \in k(x)$ and $\gamma \in k(y)$ as in part 1 in Table 4.9, Lemma 2.10 (i) implies that

$$\frac{\delta_x(h)}{h} = \frac{\delta_x(f)}{f} + y\frac{\delta_x(\beta)}{\beta} + \alpha \quad \text{and} \quad \frac{\sigma_y(h)}{h} = \frac{\sigma_y(f)}{f}\beta\gamma.$$

which, together with the assumption $h \in r\mathcal{H}(u, v)$, further implies that

$$\frac{\delta_x(f)}{f} + y\frac{\delta_x(\beta)}{\beta} + \alpha = \frac{\delta_x(r)}{r} + u \quad \text{and} \quad \frac{\sigma_y(f)}{f}\beta\gamma = \frac{\sigma_y(r)}{r}v. \tag{10}$$

In view of Definition 4.13 and the structural decomposition $h = fh'$, it remains to show that $\mathrm{den}(f)$ is split. It suffices to show that f/r is split by Lemma 4.14 (iii) and by the hypothesis that $\mathrm{den}(r)$ is split.

Case 1. Assume that $\operatorname{den}(u)$ is split. By the first equality in (10),

$$\frac{\delta_x(f/r)}{f/r} = u - y\frac{\delta_x(\beta)}{\beta} - \alpha,$$

which, together with the splitness of α, β and $\operatorname{den}(u)$, implies that the denominator of $\delta_x(f/r)/(f/r)$ is split. Thus, f/r is split by Lemma 4.14 (i).

Case 2. Assume that v is split. By the second equality in (10),

$$\frac{\sigma_y(f/r)}{f/r} = \frac{v}{\beta\gamma},$$

which, together with the splitness of β, γ and v, implies that $\sigma_y(f/r)/(f/r)$ is split. Thus, f/r is split by Lemma 4.14 (ii).

The case $(\theta_x, \theta_y) = (\delta_x, \tau_y)$ can be handled in the same vein by part 3.1 of Table 4.9 and Remark 4.6.

(ii) It can be proved by switching x with y in (i).

(iii) It can be proved by a similar argument used for Case 2 in part (i), because, by parts 3.1 and 3.2 of Table 4.9, (10) becomes

$$\frac{\theta_x(f)}{f}\alpha = \frac{\theta_x(r)}{r}u \quad \text{and} \quad \frac{\theta_y(f)}{f}\beta = \frac{\theta_y(r)}{r}v.$$

for some $\alpha \in k(x)$ and $\beta \in k(y)$. $\square$

Remark that Proposition 4.15 assumes $h = r\bar{h}$ for $\bar{h} \in \mathcal{H}(u, v)$ with conditions in each of the cases (i), (ii), and (iii) that are satisfied if $h = r\bar{h}$ is a structural decomposition. The proof establishes that for any structural decomposition $h = fh'$ and f, α, β, γ defined as in the right-hand column of Table 4.9, $\operatorname{den}(f)$ is split whenever $\operatorname{den}(r)$ is split. As a consequence, given a mixed term h attested to be proper by a structural decomposition $h = f_1h_1$ in which $\operatorname{den}(f_1)$ is split, then $\operatorname{den}(f_2)$ is split in any other structural decomposition $h = f_2h_2$. Thus, the notion of proper terms is independent of the structual decomposition under consideration. So Proposition 4.15 answers the first question.

To answer the second question, we connect split polynomials and reduced rational functions via Table 4.9, as done in the next lemma. Observe before the proof that cases (i) and (ii) are not symmetric, owing to the hypothesis on v, and similarly with the two subcases of (iii).

Lemma 4.16. Let $u, v \in k(x, y)$ be compatible with respect to a mixed pair (θ_x, θ_y) of operators. Assume that v is θ_y-reduced.

(i) If $\theta_x = \delta_x$, then both $\operatorname{den}(u)$ and v are split.
(ii) If $\theta_y = \delta_y$, then both u and $\operatorname{den}(v)$ are split.
(iii) If (θ_x, θ_y) is equal to either (σ_x, τ_y) or (τ_x, σ_y), then both u and v are split.

Proof. There exist $f, u', v' \in k(x, y)$ such that

$$\mathcal{H}(u, v) = f\mathcal{H}(u', v'), \tag{11}$$

where $f \in k(x, y)$, and u', v' are given by univariate rational functions α, β, γ in Table 4.9.

(i) First, we set $(\theta_x, \theta_y) = (\delta_x, \sigma_y)$. By part 1 of Table 4.9 and Lemma 2.10 (i), (11) can be rewritten as

$$u = \frac{\delta_x(f)}{f} + y\frac{\delta_x(\beta)}{\beta} + \alpha \quad \text{and} \quad v = \frac{\sigma_y(f)}{f}\beta\gamma. \tag{12}$$

We claim that f is split. Suppose on the contrary that f is non-split. Then there exists a non-split irreducible polynomial $p \in k[x, y]$ dividing $\mathrm{den}(f) \cdot \mathrm{num}(f)$. We only treat the case of p dividing $\mathrm{num}(f)$; the case of p dividing $\mathrm{den}(f)$ would be dealt with similarly. Then there exist two integers i and j with $i \leq j$ such that both $\sigma_y^i(p)$ and $\sigma_y^j(p)$ divide $\mathrm{num}(f)$, but $\sigma_y^\ell(p)$ does not divide $\mathrm{num}(f)$ for all ℓ with $\ell < i$ or $\ell > j$. It follows from the second equality in (12) and the splitness of $\beta\gamma$ that $\sigma_y^i(p)$ divides $\mathrm{den}(v)$ and $\sigma_y^{j+1}(p)$ divides $\mathrm{num}(v)$. Hence, v is not σ_y-reduced, a contradiction. This proves our claim that f is split, which, together with (12), implies that v and $\mathrm{den}(u)$ are also split.

Second, we set $(\theta_x, \theta_y) = (\delta_x, \tau_y)$. Then (11) becomes

$$u = \frac{\delta_x(f)}{f} + \alpha \quad \text{and} \quad v = \frac{\sigma_y(f)}{f}\beta \quad \text{for some } f \in k(x, y),\ \alpha \in k(x) \text{ and } \beta \in k(y).$$

by part 3.1 in Table 4.9 and Lemma 2.10 (i). It follows from the same argument as given above that both $\mathrm{den}(u)$ and v are split.

(ii) Set $(\theta_x, \theta_y) = (\sigma_x, \delta_y)$. Then (11) becomes

$$u = \frac{\sigma_x(f)}{f}\beta\gamma \quad \text{and} \quad v = \frac{\delta_y(f)}{f} + x\frac{\delta_y(\beta)}{\beta} + \alpha \tag{13}$$

by part 2 of Table 4.9 and Lemma 2.10 (ii). Since v is δ_y-reduced, every irreducible factor of $\mathrm{den}(f) \cdot \mathrm{num}(f)$ is either in $k[x]$ or in $k[y]$ by Lemma 2 in (Geddes et al. , 2004) and the second equality in (13). So f is again split. Both u and $\mathrm{den}(v)$ are split by (13).

Consider the case $(\theta_x, \theta_y) = (\tau_x, \delta_y)$. Then (11) becomes

$$u = \frac{\sigma_x(f)}{f}\alpha \quad \text{and} \quad v = \frac{\delta_y(f)}{f} + \beta \quad \text{for some } f \in k(x, y),\ \alpha \in k(x) \text{ and } \beta \in k(y).$$

by part 3.2 of Table 4.9 and Lemma 2.10 (ii). The same argument used in the above case implies that both u and $\mathrm{den}(v)$ are split.

(iii) We only treat the case $(\theta_x, \theta_y) = (\sigma_x, \tau_y)$; the case $(\theta_x, \theta_y) = (\tau_x, \sigma_y)$ would be dealt with similarly. By part 3.3 in Table 4.9 and Lemma 2.10 (iii), (11) becomes

$$u = \frac{\sigma_x(f)}{f}\alpha \quad \text{and} \quad v = \frac{\tau_y(f)}{f}\beta. \tag{14}$$

The lemma follows from a similar argument used for the first case in part (i). □

The last proposition answers the second question by providing a properness criterion that requires considering no structural decomposition.

Proposition 4.17. Let h be a mixed term over $(k(x, y), (\theta_x, \theta_y))$. Assume $h \in r\mathcal{H}(u, v)$ for some $r, u, v \in k(x, y)$. If $\mathrm{den}(r)$ is split and v is θ_y-reduced, then h is proper.

Proof. If $\theta_y = \delta_y$, then $\mathrm{den}(v)$ is split by Lemma 4.16 (ii). It follows from Proposition 4.15 (ii) that h is proper. If $\theta_y \in \{\sigma_y, \tau_y\}$, then v is split by Lemma 4.16 (i) and (iii). Thus, h is proper by Proposition 4.15 (i) and (iii). □

By the above proposition, the mixed term h_2 in the additive decomposition (6) is proper if r in (7) has a split denominator.

5. A necessary condition on the existence of telescopers

In the rest of this paper, we provide a necessary and sufficient condition on the existence of telescopers for mixed terms. Recall that a mixed term is telescopable if it has a telescoper. The goal of this section is to prove that a telescopable term can be written as the sum of an exact term and a proper one. More precisely, we are going to prove

Theorem 5.1. Let h be a mixed term over $(k(x,y), (\theta_x, \theta_y))$. Let

$$h = \Delta_y(h_1) + h_2$$

be an additive decomposition with respect to ∂_y. If h is telescopable of type (∂_x, ∂_y), then h_2 is either zero or proper.

As pointed out in Remark 3.2, a telescopable term of type (∂_x, ∂_y) is not necessarily telescopable of type (∂_y, ∂_x). So there are six mixed pairs of operators to be considered. We prove Theorem 5.1 for the continuous-discrete and continuous-q-discrete cases in §5.1, and for the discrete-q-discrete case in §5.2.

Owing to the length and case distinction of our proof, we make a notational convention.

Convention 5.2. Assume that h_2 given by the additive decomposition in Theorem 5.1 is nonzero. By Definition 4.4, there exist $r, u, v \in k(x,y)$ such that

(i) $\operatorname{den}(r)$ is θ_y-free;
(ii) u and v are compatible with respect to (θ_x, θ_y), and v is θ_y-reduced;
(iii) h_2 belongs to $r\mathcal{H}(u,v)$.
(iv) Assume further that L is a telescoper of type (∂_x, ∂_y) for h.
(v) Set

$$L = \sum_{i=0}^{\rho} e_i \partial_x^i \in k(x)\langle\partial_x\rangle, \quad \text{where } e_i \in k(x) \text{ and } e_\rho \neq 0, \tag{15}$$

$$f = \sum_{i=0}^{\rho} u_i \theta_x^i(r), \tag{16}$$

where

$$u_i = \begin{cases} \text{the coefficient of } D_x^i \text{ in } \sum_{i=0}^{\rho} e_i (D_x + u)^i \text{ if } \theta_x = \delta_x; \\ e_i \prod_{j=0}^{i-1} \theta_x^j(u) \text{ if } \theta_x = \sigma_x \text{ or } \theta_x = \tau_x. \end{cases}$$

By Examples 2.13 and 2.14 there exists a mixed term $g \in \mathcal{H}(u,v)$ such that

$$L(h_2) = fg. \tag{17}$$

(vi) Set w to be the non-split part of $\operatorname{den}(f)$.

The strategy for our proof is as follows. By the above convention and Proposition 4.17, it suffices to prove that $\operatorname{den}(r)$ is split. Suppose on the contrary that $\operatorname{den}(r)$ is not split. We show that w is neither in k nor θ_y-spread by (16) and the θ_y-freeness of $\operatorname{den}(r)$. On the other hand, the exactness of fg with respect to ∂_y enables us to prove that w is either in k or θ_y-spread by the next lemma. This contradiction asserts that $\mathrm{den}(r)$ has to be split.

Lemma 5.3. With Convention 5.2, we assume that w is not in k. Then w is θ_y-spread.

Proof. Note that f is nonzero, because w is not in k. Hence, fg is nonzero, which, together with Convention 5.2 (iv) and equation (17), implies that fg is an exact term.

By Convention 5.2 (ii), v is θ_y-reduced. If $\theta_y = \delta_y$, then $\mathrm{den}(v)$ is split by Lemma 4.16 (ii). It follows from Proposition 4.8 (i) applied to the exact term fg that w is δ_y-spread. If $\theta_y \in \{\sigma_y, \tau_y\}$, then v is split by Lemma 4.16 (i) and (iii). Thus, w is θ_y-spread by Proposition 4.8 (ii). $\square$

5.1. Continuous-discrete and continuous-q-discrete cases

In this section, we prove that Theorem 5.1 holds for a mixed pair of operators, in which either $\theta_x = \delta_x$ or $\theta_y = \delta_y$. By Convention 5.2 (ii), (iii) and Proposition 4.17, it suffices to show that $\mathrm{den}(r)$ is split.

Case $(\theta_x, \theta_y) = (\delta_x, \sigma_y)$. The telescoper L for h is of type (D_x, S_y). So $\partial_x = D_x$ in (15). The u_i in (16) have split denominators, because $\mathrm{den}(u_i)$ divides a power of $\mathrm{den}(u)$, which is split by Lemma 4.16 (i). Moreover, Leibniz's rule for δ_x implies that $u_\rho = e_\rho$, which is a univariate rational function of x by (15).

Suppose for a contradiction that $\mathrm{den}(r)$ is not split. Then there exists a non-split irreducible polynomial $p \in k[x, y]$ such that $p \mid \mathrm{den}(r)$. Assume that m is the multiplicity of p in $\mathrm{den}(r)$. Then the multiplicity of p in $\mathrm{den}(u_i \delta_x^i(r))$ is less than or equal to $m + i$, because $\mathrm{den}(u_i)$ is split. In particular, the multiplicity of p in $\mathrm{den}\left(u_\rho \delta_x^\rho(r)\right)$ is equal to $m + \rho$, because $u_\rho \in k(x)$. By (16), p is an irreducible factor of $\mathrm{den}(f)$. Accordingly, the non-split part w of $\mathrm{den}(f)$ is nontrivial. Since all the $\mathrm{den}(u_i)$ are split, w divides a power of $\mathrm{den}(r)$ by (16). Since $\mathrm{den}(r)$ is σ_y-free, so is w. Hence, w is neither in k nor σ_y-spread, a contradiction to Lemma 5.3. Therefore, $\mathrm{den}(r)$ is split.

Case $(\theta_x, \theta_y) = (\sigma_x, \delta_y)$. The telescoper L for h is of type (S_x, D_y). So $\partial_x = S_x$ in (15). The u_i in (16) are all split, because u is split by Lemma 4.16 (ii). In particular, u_ρ is nonzero because e_ρ is nonzero.

Suppose for a contradiction that $\mathrm{den}(r)$ is not split. Then there exists a non-split irreducible polynomial $p \in k[x, y]$ such that $p \mid \mathrm{den}(r)$ and $\sigma_x^i(p) \nmid \mathrm{den}(r)$ for all $i > 0$. Since $\mathrm{den}(r)$ is δ_y-free, p is a nontrivial simple factor of $\mathrm{den}(r)$ in $k(x)[y]$. Since u_ρ is split, $\sigma_x^\rho(p)$ is a simple and irreducible factor of $\mathrm{den}\left(u_\rho \sigma_x^\rho(r)\right)$. Since $\sigma_x^i(p) \nmid \mathrm{den}(r)$ for any $i > 0$, $\sigma_x^\rho(p)$ is not an irreducible factor of $\mathrm{den}\left(u_j \sigma_x^j(r)\right)$ for any j with $0 \le j \le \rho - 1$. It follows from (16) that $\sigma_x^\rho(p)$ is a simple factor of w. So w is neither in k nor δ_y-spread, a contradiction to Lemma 5.3. Therefore, $\mathrm{den}(r)$ is split.

By Remark 4.6, we can replace σ_x and σ_y by τ_x and τ_y in the above proof, respectively. This completes the proof of Theorem 5.1 in both continuous-discrete and continuous-q-discrete cases.

5.2. Discrete-q-discrete case

For the case in which either $(\theta_x, \theta_y) = (\sigma_x, \tau_y)$ or $(\theta_x, \theta_y) = (\tau_x, \sigma_y)$, we need not only similar arguments used in §5.1, but also two mixed analogues of the results given by Abramov and Petkovsek (2002a) and by Abramov (2003), respectively. Such analogues are not necessary for the proof in the continuous-discrete case, because a power of a σ_y-free polynomial is again σ_y-free, and the least common multiple of finitely many δ_y-free polynomials is again δ_y-free. Unfortunately, similar situations do not occur in the discrete-q-discrete case.

The first lemma is a mixed analogue of Theorem 7 in (Abramov and Petkovsek, 2002a).

Lemma 5.4. Let p be an irreducible polynomial in $k[x,y]$. Assume that $\sigma_x^i \tau_y^j(p) = cp$ for some $i, j \in \mathbb{Z}$ and $c \in k$.

(i) If $i \neq 0$, then $p \in k[y]$.
(ii) If $j \neq 0$, then either $p \in k[x]$ or $p = \lambda y$ for some $\lambda \in k$.

Proof. (i) Assume that there exist $i, j \in \mathbb{Z}$ with $i \neq 0$ and $c \in k$ such that $\sigma_x^i \tau_y^j(p) = cp$. We consider two cases. First, if $\tau_y^j(p) = p$, then $\sigma_x^i(p) = cp$, so that $c = 1$ by comparing the leading coefficients with respect to x. Thus, p is free of x since $i \neq 0$.

Second, assume that $\tau_y^j(p) \neq p$. Then $d = \deg_y(p) > 0$ and $j \neq 0$. Write

$$p = p_d(x)y^d + \cdots + p_0(x),$$

where $p_0, \ldots, p_d \in k[x]$ and $p_d \neq 0$. Then

$$\sigma_x^i \tau_y^j(p) = p_d(x+i)q^{jd}y^d + \cdots + p_0(x+i).$$

By the equality $\sigma_x^i \tau_y^j(p) = cp$, we have that

$$p_\ell(x+i)q^{j\ell} = cp_\ell(x) \quad \text{for all } \ell \text{ with } 0 \leq \ell \leq d.$$

Assume that ℓ is an integer in $\{0, \ldots, d\}$ such that $p_\ell(x) \neq 0$. Then $c = q^{j\ell}$ by considering leading coefficients with respect to x in the above equation. This implies that $p_\ell(x+i) = p_\ell(x)$. It follows from $i \neq 0$ that p_ℓ belongs to k. Thus, p is again free of x.

(ii) First, we show that $\sigma_x^i(p) = p$ and $\tau_y^j(p) = cp$. If $i = 0$, then there is nothing to prove. Otherwise, p belongs to $k[y]$ by (i). So $\sigma_x^i(p) = p$. The result follows from $\sigma_x \circ \tau_y = \tau_y \circ \sigma_x$.

Second, we show that p is a monomial in y over $k[x]$. Suppose that there are d_1 and d_2 in $\mathbb{N}$ with $d_1 > d_2$ such that

$$p = p_{d_1}y^{d_1} + p_{d_2}y^{d_2} + \text{ terms of lower degree in } y,$$

where $p_{d_1}, p_{d_2} \in k[x]$ and $p_{d_1}p_{d_2} \neq 0$. Applying τ_y^j to p yields

$$\tau_y^j(p) = p_{d_1}q^{jd_1}y^{d_1} + p_{d_2}q^{jd_2}y^{d_2} + \text{ lower terms in } y.$$

The equality $\tau_y^j(p) = cp$ implies that $c = q^{jd_1}$ and $c = q^{jd_2}$. Hence, $q^{j(d_1-d_2)} = 1$, a contradiction to the assumption that q is not a root of unity. So p is a monomial in y.

Set $p = \lambda y^s$ for some $\lambda \in k[x]$ and $s \in \mathbb{N}$. Since p is irreducible, s is equal to 0 or 1. If $s = 0$, then p is an irreducible polynomial in $k[x]$. If $s = 1$, then λ belongs to k, again because p is irreducible. Part (ii) of the lemma holds. □

The next lemma is a mixed analogue of Theorem 9 in (Abramov, 2003). Its proof is also similar to that in (Abramov, 2003).

Lemma 5.5. Let $r \in k(x,y)$ be a nonzero rational function whose denominator is not split. Let L be a nonzero element in $k(x,y)\langle\partial_x\rangle$ whose coefficients are all split. Then the following statements hold.

(i) If $\partial_x = T_x$, and $\mathrm{den}(r)$ is σ_y-free, then the non-split part of the denominator of $L(r)$ is neither in k nor σ_y-spread.

(ii) If $\partial_x = S_x$, and $\mathrm{den}(r)$ is τ_y-free, then the non-split part of the denominator of $L(r)$ is neither in k nor τ_y-spread.

Proof. Set $a = \mathrm{num}(r)$, $b = \mathrm{den}(r)$ and $L = \sum_{i=0}^{\rho} u_i \partial_x^i$, where $u_0, u_1, \ldots, u_\rho \in k(x, y)$ are split and u_ρ is nonzero.

(i) Applying L to r, we have that

$$L(r) = \sum_{i=0}^{\rho} u_i \tau_x^i \left(\frac{a}{b}\right). \tag{18}$$

There exists a non-split and irreducible polynomial p such that p divides b but $\tau_x^\mu(p)$ does not divide b for any positive integer $\mu > 0$, because b is not split. It follows that the irreducible polynomial $\tau_x^\rho(p)$ divides the denominator of $\tau_x^\rho\,(a/b)$ but does not divide that of $\tau_x^i\,(a/b)$ for any i with $0 \le i \le \rho - 1$. Thus, $\tau_x^\rho(p)$ divides the non-split part B of $\mathrm{den}(L(r))$. In particular, B does not belong to k.

It remains to prove that B is not σ_y-spread. Suppose on the contrary that B is σ_y-spread. Then there exists $j_0 \in \mathbb{Z}$ with $j_0 \neq 0$ such that $\sigma_y^{j_0} \tau_x^\rho(p) \mid B$. Furthermore, we have that $B \mid \prod_{i=0}^{\rho} \tau_x^i(b)$ by (18) and the splitness of the u_i. The two divisibilities and irreducibility of p imply that there exists ℓ_0 in $\{0, \ldots, \rho\}$ such that $\sigma_y^{j_0} \tau_x^\rho(p) \mid \tau_x^{\ell_0}(b)$. Note that $\ell_0 \neq \rho$, for otherwise, $\sigma_y^{j_0}(p)$ would divide b, which, together with $p \mid b$, would imply that b is not shift-free, a contradiction. Therefore, $\ell_0 < \rho$. Since $\sigma_y^{j_0} \tau_x^{\rho - \ell_0}(p) \mid b$, there exists a non-negative integer i_0 such that

$$\sigma_y^{j_0} \tau_x^{\rho - \ell_0 + i_0}(p) \mid b \quad \text{but} \quad \sigma_y^{j_0} \tau_x^{\rho - \ell_0 + i_0 + \mu}(p) \nmid b$$

for any positive integer μ. It follows from $\rho - \ell_0 > 0$ that $\rho - \ell_0 + i_0 > 0$.

Repeating the above process for the irreducible factor $\sigma_y^{j_0} \tau_x^{\rho - \ell_0 + i_0}(p)$ instead of the factor p, we can find $j_1 \in \mathbb{Z}$ with $j_1 \neq 0$, $i_1 \in \mathbb{N}$ and $\ell_1 \in \{0, \ldots, \rho - 1\}$ such that $\rho - \ell_1 + i_1 > 0$ and

$$\sigma_y^{j_0 + j_1} \tau_x^{(\rho - \ell_0 + i_0) + (\rho - \ell_1 + i_1)}(p) \mid b \quad \text{but} \quad \sigma_y^{j_0 + j_1} \tau_x^{(\rho - \ell_0 + i_0) + (\rho - \ell_1 + i_1) + \mu}(p) \nmid b$$

for any positive integer μ. Continuing this process yields a sequence of irreducible factors of b. Since b has only finitely many irreducible factors, there exist $m, n \in \mathbb{N}$ with $n < m$ such that

$$\sigma_y^{j_0 + \cdots + j_n} \tau_x^{(\rho - \ell_0 + i_0) + \cdots + (\rho - \ell_n + i_n)}(p) = c\sigma_y^{j_0 + \cdots + j_m} \tau_x^{(\rho - \ell_0 + i_0) + \cdots + (\rho - \ell_m + i_m)}(p)$$

for some $c \in k$. This implies that

$$\sigma_y^{-j_{n+1} - \cdots - j_m} \tau_x^{-(\rho - \ell_{n+1} + i_{n+1}) - \cdots - (\rho - \ell_m + i_m)}(p) = cp.$$

Note that $(\rho - \ell_{n+1} + i_{n+1}) + \cdots + (\rho - \ell_m + i_m) \neq 0$. By Lemma 5.4 (ii), p is split, a contradiction to the assumption that p is non-split.

(ii) It follows from the same argument given as above, in which τ_x and σ_y are replaced by σ_x and τ_y, respectively. □

We are ready to prove that Theorem 5.1 holds for the discrete-q-discrete cases.

Case $(\theta_x, \theta_y) = (\tau_x, \sigma_y)$. With Convention 5.2, suppose for a contradiction that $\mathrm{den}(r)$ is not split. Note that the u_i in (16) are split by Lemma 4.16 (iii). Thus, Lemma 5.5 (i) implies that the non-split part w of $\mathrm{den}(f)$ is neither in k nor σ_y-spread, a contradiction to Lemma 5.3. We conclude that $\mathrm{den}(r)$ is split.

Case $(\theta_x, \theta_y) = (\sigma_x, \tau_y)$. The assertion can be proved by a similar argument, in which Lemma 5.5 (ii) is applied instead of Lemma 5.5 (i).

We have completed a proof of Theorem 5.1 for all the six mixed pairs of operators.

6. Determining the existence of telescopers

In this section, we first establish a criterion on the existence of telescopers for mixed terms in §6.1, then present an algorithm with a few examples in §6.2.

6.1. A criterion on the existence of telescopers

In this section, we prove the converse of Theorem 5.1, leading to the criterion of Theorem 6.3. Our proof is based on the following theorem.

Theorem 6.1. A proper term over $(k(x, y), (\theta_x, \theta_y))$ is telescopable of type (∂_x, ∂_y).

In fact, the above theorem is part of the fundamental theorem in (Wilf and Zeilberger, 1992a). In their paper, Wilf and Zeilberger present an elementary proof of the existence of telescopers for proper hypergeometric terms and indicate that their argument should be applicable to the mixed setting. For the sake of completeness, we elaborate a proof of Theorem 6.1 in the appendix of this paper.

Next, we show the converse of Theorem 5.1.

Lemma 6.2. Let h be a mixed term over $(k(x, y), (\theta_x, \theta_y))$. Let

$$h = \Delta_y(h_1) + h_2$$

be an additive decomposition with respect to ∂_y. If h_2 is either zero or proper, then h is telescopable of type (∂_x, ∂_y).

Proof. If h_2 is zero, then 1 is a telescoper of h. Therefore, we assume that h_2 is a proper mixed term. By Theorem 6.1 below, h_2 is telescopable of type (∂_x, ∂_y). We claim that that so is h, which ends the proof of the lemma.

Because h_2 has a telescoper L of type (∂_x, ∂_y), there exists indeed a mixed term g such that $L(h_2) = \Delta_y(g)$. Note that $L\Delta_y = \Delta_y L$, because $L \in k[x]\langle\partial_x\rangle$ and $\Delta_y \in k\langle\partial_y\rangle$. It follows that

$$L(h) = L\Delta_y(h_1) + L(h_2) = \Delta_y(L(h_1)) + \Delta_y(g). \tag{19}$$

If $L(h) = 0$, then $L(h) = \Delta_y(1)$ and L is a telescoper for h; if either $\Delta_y(L(h_1)) = 0$ or $\Delta_y(g) = 0$, then L is a telescoper for h by (19). Otherwise, the three mixed terms $L(h)$, $\Delta_y(L(h_1))$ and $\Delta_y(g)$ are similar to each other by Remark 2.12. Therefore, $L(h_1) + g$ is either a mixed term or zero. Set g' to be the sum if it is nonzero. Otherwise, set $g' = 1$. Then $L(h) = \Delta_y(g')$, that is, h is telescopable of type (∂_x, ∂_y). □

As both Theorem 5.1 and its converse Lemma 6.2 hold, we obtain a criterion on the existence of telescopers for mixed terms, which is the main result of this article.

Theorem 6.3. Let h be a mixed term over $(k(x, y), (\theta_x, \theta_y))$. Assume that

$$h = \Delta_y(h_1) + h_2$$

is an additive decomposition of h. Then h has a telescoper of type (∂_x, ∂_y) if and only if h_2 is either zero or a proper mixed term.

6.2. Algorithm and Examples

For a given mixed term, we can decide the existence of telescopers by Theorem 6.3. First, we use the algorithms in (Abramov and Petkovsek, 2002b), (Geddes et al. , 2004), and (Chen et al., 2005) to perform the respective additive decompositions to obtain (6). Second, we test whether the denominator of r in (7) is split by Remark 4.5. The decision procedure is given in Table 6.6.

Example 6.4. It is possible that a mixed term has a telescoper of type (S_x, D_y) but no telescoper of type (D_x, S_y) or (D_x, T_y). Consider the rational function

$$h = \frac{1}{(x+y)^2}.$$

Applying Hermite reduction to h with respect to δ_y yields $h = D_y\left(-1/(x+y)\right)$, which implies that 1 is a telescoper of type (S_x, D_y) for h. Note that $h = \Delta_y(1) + h$ is an additive decomposition when $\theta_y = \sigma_y$ or $\theta_y = \tau_y$, because $\mathrm{den}(h) = (x+y)^2$ is both shift-free and q-shift-free with respect to y. But h is not proper, because $x + y$ is not split. Hence, h has no telescoper of type (D_x, S_y) and (D_x, T_y).

Similarly, consider the rational function

$$h = \frac{1}{(x+y)(x+y+1)}.$$

Since $h = (S_y - 1)(-1/(x+y))$, 1 is a telescoper of type (D_x, S_y) for h. However, h has no telescoper of type (S_x, D_y) or (T_x, D_y), because $(x+y)(x+y+1)$ is squarefree with respect to δ_y and it is not split.

Example 6.5. [2] Consider the rational function

$$h = \frac{x^5 + 2yx^3 + xy^2 - y}{(y+x^2)^2 y}.$$

An additive decomposition of h with respect to D_y is

$$h = D_y\left(\frac{1}{y+x^2}\right) + \frac{x}{y}.$$

Since y is split, x/y is proper. So h is telescopable of the type (S_x, Dy) by Theorem 6.3. An additive decomposition of h with respect to Δ_y is

$$h = \Delta_y(1) + h.$$

Since the denominator $(y+x^2)^2 y$ of h is not split, h is not proper. So h is not telescopable of type (D_x, S_y) by Theorem 6.3.

[2] This example was proposed by an anonymous referee.

Algorithm IsTelescopable
Input: a mixed term $h \in \mathcal{H}(a, b)$ over $(k(x, y), (\theta_x, \theta_y))$.
Output: true, if h has a telescoper of type (θ_x, θ_y); false, otherwise.

(1) Compute an additive decomposition of h with respect to θ_y and get
$$h = \Delta_y(h_1) + rg,$$
where $r \in k(x, y)$ and $g \in \mathcal{H}(u, v)$ are as given by (6) and (7) in Definition 4.4.

(2) Compute the primitive part p of $\mathrm{den}(r)$ with respect to y.

(3) If p is in $k[y]$, then return true, otherwise, return false.

Table 6.6. Algorithms for deciding the existence of telescopers

As we mentioned before, properness is only a sufficient condition for the existence of telescopers. The following example illustrates this fact.

Example 6.7. Consider the mixed term over $(k(x, y), (\sigma_x, \delta_y))$
$$h = \frac{-y + 2xy + 2x^2}{(x + y)^2 x} \cdot y^x \cdot e^{-y}.$$
A structural decomposition of h is given by $\alpha = -1$, $\beta = y$, $\gamma = 1$, and $f = h/(y^x e^{-y})$. Note that the denominator of f is equal to $x(x + y)^2$, which is not split. So h is not proper. But h has a telescoper of type (S_x, D_y) since it can be decomposed into
$$h = D_y\left(\frac{1}{y(x + y)} \cdot y^x \cdot e^{-y}\right) + y^{x-1} \cdot e^{-y},$$
where $y^{x-1} \cdot e^{-y}$ is proper, because the rational function in the corresponding structural decomposition is 1, and therefore its denominator is split.
Similarly, h also has a telescoper of type (T_x, D_y).

The last example presents another application of Theorem 6.3.

Example 6.8. Let $f = 1/(x + y)^s$, where s is any fixed positive integer. Note that the denominator of f is non-split and shift-free with respect to σ_y. By Theorem 6.3, there is no linear differential operator $L(x, D_x) \in k(x)\langle D_x\rangle$ and $g \in k(x, y)$ such that $L(x, D_x)(f) = \Delta_y(g)$, which, together with Proposition 3.1 in (Hardouin and Singer, 2008) and the descent argument similar to that given in the proof of Corollary 3.2 in (Hardouin and Singer, 2008) (or Section 1.2.1 of (Di Vizio and Hardouin, 2012)), implies that the incomplete Hurwitz zeta function
$$\zeta(x, y, s) := \sum_{i=1}^{y-1} \frac{1}{(i + x)^s} \qquad \text{(satisfying } S_y(F) - F = f)$$
satisfies no polynomial differential equation $P(x, y, s, F, D_x(F), D_x^2(F), \ldots) = 0$.

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7. Appendix: A Proof of Theorem 6.1

Our proof of Theorem 6.1 consists of two steps. First, we show that a mixed term in the continuous-discrete case is telescopable if its certificate with respect to the shift operator has a split denominator. This is done by a well-known linear algebra argument borrowed from Lipshitz (1988). Second, we prove the theorem by expressing a proper term as a linear combination of the mixed terms treated in the first step. The coefficients in the linear combination are mixed terms free of y. Note that the first step is not necessary for the continuous-q-discrete and discrete-q-discrete cases, because the structural decompositions in part 3 of Table 4.9 are simpler than those in parts 1 and 2. Four technical lemmas prepare the ground: Lemma 7.1 completes the first step, and Lemmas 7.3, 7.4, and 7.5 prepare the second.

As said, the next lemma enables us to use structural decompositions to show that Theorem 6.1 holds in the continuous-discrete case.

Lemma 7.1. Let h be a mixed term over $(k(x, y), (\theta_x, \theta_y))$, where $(\theta_x, \theta_y) = (\delta_x, \sigma_y)$ or $(\theta_x, \theta_y) = (\sigma_x, \delta_y)$. If its certificate with respect to the shift operator has a split denominator, then h is telescopable of type (D_x, S_y) and (S_x, D_y).

Proof. The proof is reminiscent of the linear algebra argument given by Lipshitz (1988); however, it is based on linear algebra and filtrations over $k(x)$ instead of k. Let μ be the maximum of the degrees in y of the numerators and denominators of the two certificates of h. Moreover, let a and b be the denominators of the θ_x- and θ_y-certificates, respectively.

Assume that $(\theta_x, \theta_y) = (\delta_x, \sigma_y)$. Since θ_y is a shift operator, the θ_y-certificate of h has a split denominator, that is, b is split. So there are $b_1 \in k[x]$ and $b_2 \in k[y]$ such that $b = b_1 b_2$. It follows that

$$S_y^j(h) = \frac{p}{\prod_{\ell=0}^{j-1} \sigma_y^\ell(b_2)} h$$

for some $p \in k(x)[y]$ with $\deg_y(p) \leq j\mu$. Since b_2 is free of x,

$$D_x^i S_y^j(h) = \frac{D_x^i\,(ph)}{\prod_{\ell=0}^{j-1} \sigma_y^\ell(b_2)} = \frac{wh}{a^i \prod_{\ell=0}^{j-1} \sigma_y^\ell(b_2)}$$

for some $w \in k(x)[y]$ with $\deg_y(w) \leq (i + j)\mu$.

If $i+j \leq N$, then $D_x^i S_y^j(h)$ is an element of the $k(x)$-linear space

$$\mathcal{W}_N = \operatorname{span}_{k(x)} \left\{ \frac{y^m h}{a^N \prod_{\ell=0}^{N-1} \sigma_y^\ell(b_2)} \mid m \leq 2\mu N \right\}. \tag{20}$$

Accordingly, there is a well-defined $k(x)$-linear map ϕ_N from

$$\mathcal{F}_N = \operatorname{span}_{k(x)}\{D_x^i S_y^j \mid i+j \leq N\}$$

to $\mathcal{W}_N$ that sends L to $L(h)$ for all $L \in \mathcal{F}_N$. Since the dimension of $\mathcal{F}_N$ over $k(x)$ is $\binom{N+2}{2}$ while that of $\mathcal{W}_N$ is $2\mu N + 1$, the kernel of ϕ_N is nontrivial when N is sufficiently large.

Let A be a nonzero element of $\ker(\phi_N)$ whose degree in S_y is minimal. Since A is free of y, $A = L + (S_y - 1)N$ for some $L \in k(x)\langle D_x\rangle$ and $N \in k(x,y)\langle D_x, S_y\rangle$. Accordingly, L is nonzero by the same argument used by Graham et al. (1994, §5.8, p. 241)[3] (see also (Petkovšek et al., 1996, §6.2, p. 108)). Since $A(h) = 0$, $L(h) = (S_y - 1)(N(h))$. So L is a telescoper of type (D_x, S_y) for h.

For the case $(\theta_x, \theta_y) = (\sigma_x, \delta_y)$, we can replace the linear space $\mathcal{W}_N$ in (20) by

$$\operatorname{span}_{k(x)} \left\{ \frac{y^m h}{b_2^N \prod_{\ell=0}^{N-1} \sigma_x^\ell(a)} \mid m \leq 2\mu N \right\},$$

and prove that h is telescopable of type (S_x, D_y) by a similar argument. $\square$

Remark 7.2. With the map from $\mathcal{F}_N$ to $\mathcal{W}_F$, the proof above has obtained, in the terminology of Chyzak et al. (2009), a polynomial growth of 1 for the ideal of annihilating operators of h, from which the existence of a telescoper follows by Theorem 3 in there.

Next, we prepare for the second step of the proof. The following lemma reflects the intuition that a mixed term H is "free of" y if $\Delta_y(H) = 0$, where Δ_y is defined in (2).

Lemma 7.3. Let H be a mixed term over $(k(x,y), (\theta_x, \theta_y))$. If $\Delta_y(H) = 0$, then its θ_x-certificate belongs to $k(x)$ and its θ_y-certificate is equal to $\theta_y(1)$.

Proof. Let the θ_x-certificate and θ_y-certificate of H be a and b, respectively.

First, assume that $\theta_y = \delta_y$. Then $\Delta_y = D_y$. It follows from $D_y(H) = bH = 0$ that $b = 0$. Consequently, $b = \delta_y(1)$. It remains to show that $a \in k(x)$. If $\theta_x = \sigma_x$, then $\delta_y(a)/a = \sigma_x(b) - b$ by (3) and Remark 2.6. Since $b = 0$, $\delta_y(a) = 0$, that is, a is free of y. Similarly, a is free of y if $\theta_x = \tau_x$.

Second, assume that $\theta_y = \tau_y$. Then $\Delta_y(H) = 0$ implies that $\tau_y(H) = bH = H$. It follows that $b = 1$, that is, $b = \tau_y(1)$. To show that $a \in k(x)$, we make a case distinction.

Case 1. If $\theta_x = \delta_x$, then $\delta_x(b)/b = \tau_y(a) - a$ by (3). It follows from $b = 1$ that $\tau_y(a) = a$. Thus, $a \in k(x)$ because q is not a root of unity.

Case 2. If $\theta_x = \sigma_x$, then $\sigma_x(b)/b = \tau_y(a)/a$, which, together with $b = 1$, implies that $\tau_y(a) = a$, and so that $a \in k(x)$.

Third, assume that $\theta_y = \sigma_y$. Then $b = 1$ and $a \in k(x)$ by a similar argument as in the second case. $\square$

[3] This section is not part of the first edition.

Lemma 7.4. Let h and H be two mixed terms over $(k(x,y), (\theta_x, \theta_y))$ with $\Delta_y(H) = 0$. If h is telescopable of type (∂_x, ∂_y), so is Hh.

Proof. Assume that $H \in \mathcal{H}(a,b)$. A straightforward induction shows that, for all $i \in \mathbb{N}$,

$$H\partial_x^i(h) = L_i(Hh) \text{ for some } L_i \in k(x)\langle\partial_x\rangle, \tag{21}$$

where $L_i = (\partial_x - a)^i$ if $\partial_x = D_x$ and $L_i = \left(\prod_{j=0}^{i-1} \theta_x^j(a)\right)^{-1} \partial_x^i$ if $\partial_x \in \{S_x, T_x\}$. Furthermore, $L_i \in k(x)\langle\partial_x\rangle$, because $a \in k(x)$ by Lemma 7.3.

Let $L \in k(x)\langle\partial_x\rangle$ be a telescoper for h of the form $L = \sum_{i=0}^{\rho} e_i \partial_x^i$, where e_ρ, $e_{\rho-1}$, $\ldots$, e_0 are in $k(x)$ with $e_\rho \neq 0$. Since $L(h) = \Delta_y(g)$ for some mixed term g,

$$HL(h) = \sum_{i=0}^{\rho} e_i L_i(Hh) = \Delta_y(Hg)$$

by (21). Set $M = \sum_{i=0}^{\rho} e_i L_i$. Then $M(Hh) = \Delta_y(Hg)$. Moreover, $M \neq 0$ because L_i is of order i in ∂_x. Therefore, M is a telescoper for Hh. $\square$

Lemma 7.5. Similar and telescopable terms of type (∂_x, ∂_y), together with zero, form a linear space over $k(x)$, which is closed under Δ_y.

Proof. Let h be a telescopable term of type (∂_x, ∂_y). Set

$$\mathcal{V}_h = \{g \mid g \text{ is similar to } h \text{ and telescopable of the same type}\} \cup \{0\}.$$

For two telescopable terms in $\mathcal{V}_h$, a common left multiple of their telescopers is a telescoper of the sum whenever the sum is nonzero. Then $\mathcal{V}_h$ is a linear space over $k(x)$ by Lemma 7.4. The space is closed under Δ_y, because Δ_y commutes with every element of $k(x)\langle\partial_x\rangle$. $\square$

We are ready to show Theorem 6.1, which states that every proper term is telescopable.

Proof of Theorem 6.1. Assume that h is a proper term over $(k(x,y), (\theta_x, \theta_y))$. By Definition 4.13, a structural decomposition of h is of the form $h = (u/v)g$ and $g \in \mathcal{H}(a,b)$, where $u \in k(x)[y]$, $v \in k[y]$, and $\mathcal{H}(a,b)$ are given by α, β and γ in Table 4.9. In particular, the denominators of a and b are split by their expressions in the table. By Lemma 2.10, we can move the univariate polynomial $v(y)$ into $\mathcal{H}(a,b)$, yielding $h = uG$ and $G \in \mathcal{H}(a,B)$, where B has a split denominator.

Write $u = \sum_{i=0}^{m} u_i y^i$, where $u_0, u_1, \ldots, u_m \in k(x)$, and set $G_i = y^i G$. Again by Lemma 2.10, $G_i \in \mathcal{H}(a, B_i)$ for some $B_i \in k(x,y)$ whose denominator is split. Since h is equal to $\sum_{i=0}^{m} u_i G_i$ and the G_i are similar to each other, by Lemma 7.5 it suffices to prove that G_i is telescopable of type (∂_x, ∂_y) for all i with $0 \leq i \leq m$.

In the continuous-discrete case, G_i is telescopable by Lemma 7.1, because both a and B_i have split denominators.

In the continuous-q-discrete and discrete-q-discrete cases, $a \in k(x)$ and $b \in k(y)$ by part 3 of Table 4.9. Therefore, $B_i \in k(y)$ by Lemma 2.10. It follows again from Lemma 2.10 that $\mathcal{H}(a, B_i) = \mathcal{H}(a, c')\, \mathcal{H}(c'', B_i)$, where $c' = 0$ and $c'' = 1$ if $\theta_y = \delta_y$, $c' = 1$ and $c'' = 0$ if $\theta_x = \delta_x$, and, otherwise, $c' = c'' = 1$. Accordingly, $G_i = G_i' G_i''$ for some G_i' in $\mathcal{H}(a, c')$ and G_i'' in $\mathcal{H}(c'', B_i)$. It follows that

$$\Delta_y(G_i') = 0 \quad \text{and} \quad \Delta_x(G_i'') = 0.$$

The second equality implies that G_i'' is telescopable of type (∂_x, ∂_y), which, together with the first equality and Lemma 7.4, implies that G_i is telescopable of type (∂_x, ∂_y). The proof of Theorem 6.1 is completed.