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Effective algorithms for parametrizing linear control systems over Ore algebras

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Abstract In this paper, we study linear control systems over Ore algebras. Within this mathematical framework, we can simultaneously deal with different classes of linear control systems such as time-varying systems of ordinary differential equations (ODEs), differential time-delay systems, underdetermined systems of partial differential equations (PDEs), multidimensional discrete systems, multidimensional convolutional codes, etc. We give effective algorithms which check whether or not a linear control system over some Ore algebra is controllable, parametrizable, flat or π-free.

Keywords Linear systems over Ore algebras · Parametrization · Flatness · Constructive algorithms · Non-commutative Gröbner bases · Module theory

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1 Introduction

Over the last thirty years, for practical and theoretical reasons, different new classes of linear control systems have been introduced such as differential time-delay systems, multidimensional systems, partial differential equations, convolutional codes, hybrid systems ... All these classes of systems are governed by new types of mathematical equations and have needed new techniques in order to analyze their structural properties and to synthesize new control laws. This growth of new types of control systems has led to generalize some previously known results and techniques so that they could be used for more general classes of systems. The main interest is to get similar concepts, techniques and algorithms for studying different classes of systems.

In this paper, we study linear control systems over *Ore algebras*. An Ore algebra is an algebra of non-commutative polynomials. They represent functional operators which satisfy certain commutation rules. For instance, differential/shift/difference/divided difference ... operators can be represented as elements of some Ore algebras. Within this mathematical framework, we can simultaneously deal with different classes of linear control systems such as time-varying ordinary differential systems (ODEs), differential time-delay systems, partial differential equations (PDEs), multidimensional discrete systems, convolutional codes ... Moreover, the recent extension of *Gröbner bases* to some non-commutative polynomial rings allows us to work effectively in Ore algebras [6,7].

The purpose of this paper is to give effective algorithms which check whether or not a linear control system over some Ore algebra is controllable, parametrizable, flat or π-free. These problems have been intensively studied in [12,20,21] for linear differential time-delay systems and, in [22,24–27,29–31,35,39,40], for linear multidimensional systems. The main novelty of this paper is to present algorithms which work for both classes of systems as well as for new ones. In particular, our approach allows us to effectively obtain parametrizations of a controllable plant and flat outputs of a flat system. Let us notice that such algorithms were still missing for linear differential time-delay systems (see [12,20,21] for more details). The results developed in this paper give a new effective machinery for the study of the structural properties of linear systems. We hope that they could play important roles in the study of motion planning or tracking problems [12,20,21].

Ore algebras are rings of non-commutative polynomials that represent linear functional operators in a natural way. A recent extension of the theory of Gröbner bases to this setting [5,7] makes manipulations of (one-sided) modules over Ore algebras effective. We recall the basics of this theory in Section 2. As is the case for linear differential systems in algebraic analysis, a fundamental remark is that a module over an Ore algebra is canonically associated with any linear control system. Before we provide the readers in Section 5 with a dictionary between structural properties of control systems and algebraic properties of modules, we recall in Section 3 the definitions and results of module theory to be used in the rest of the article. Next, the question is to algorithmically decide the algebraic properties of modules. This is done by means of homological algebra in Sections 4 and 6.2. The good point is that the central homological objects as syzygy modules, free resolutions, and extension modules can all be computed by algorithms which we recall and exemplify on cases of application. In passing, the classical notion of involution, which exchanges left and right module structures, is recalled in Section 6.1, where explicit examples of involutions of Ore algebras that appear in control theory are also given.

Beside realizing the applicability of non-commutative Gröbner bases to make the theory algorithmic, the main contribution of the paper lies in the new approach for the proofs of the algorithms in Sections 7 and 8. These sections extend earlier results, previously obtained in special situations only, to the broad setting of linear control systems over Ore algebras. After collecting further results of homological algebra that could prove difficult to find in the literature, Section 7 gives an algorithm to determine whether a control system is parametrizable and, if so, to compute a parametrization, while, if not, to compute a generating set of autonomous elements. Section 8 restricts to control systems over commutative Ore algebras in order to provide algorithms for deciding flatness, studying π-freeness and computing minimal parametrizations. An essential ingredient is an algorithm which computes left and right-inverses of matrices over an Ore algebra, when they exist.

In the appendix in Section 10, we have illustrated the main results and algorithms of the paper with explicit examples. All the computations have been done using the Maple package *OreModules* [8,10] written by the authors based on the library *Mgfun* [6]. These libraries as well as all the example worksheets of the appendix are freely available on the web page given in [8]. Some results are new (e.g., the parametrization of the electric transmission line [20,37]) and we believe that these examples have some educational interest.

Finally, some of the results developed here firstly appeared in [9,10,31].

2 Ore algebras

2.1 Definition and examples

In order to deal with different classes of linear systems (ODEs, PDEs, differential time-delay systems, discrete systems ...) in a unified framework, we represent them by means of matrices with entries in an *Ore algebra* of functional operators. In what follows, we shall denote by k a field.

Definition 1 1. [17] Let A be a domain (i.e., the product of non-zero elements is non-zero) with a unit 1 which is also a k-algebra. The *skew polynomial ring* $A[\partial; \sigma, \delta]$ is the non-commutative ring consisting of all polynomials in ∂ with coefficients in A obeying the commutation rule

$$\forall\, a \in A, \quad \partial\, a = \sigma(a)\, \partial + \delta(a), \tag{1}$$

where σ is a *k-algebra endomorphism of* A, namely, $\sigma : A \to A$ satisfies

$$\forall\, a, b \in A, \quad \begin{cases} \sigma(1) = 1, \\ \sigma(a + b) = \sigma(a) + \sigma(b), \\ \sigma(a\, b) = \sigma(a)\, \sigma(b), \end{cases}$$

and δ is a *σ-derivation of* A, namely, $\delta : A \to A$ satisfies:

$$\forall\, a, b \in A, \quad \begin{cases} \delta(a + b) = \delta(a) + \delta(b), \\ \delta(a\, b) = \sigma(a)\, \delta(b) + \delta(a)\, b. \end{cases}$$

2. [5,7] Let $A = k[x_1, \ldots, x_n]$ be a commutative polynomial ring over a field k (if $n = 0$ then $A = k$). Then, the iterated skew polynomial ring $D = A[\partial_1; \sigma_1, \delta_1] \ldots [\partial_m; \sigma_m, \delta_m]$ is called *Ore algebra* if the σ_i's and δ_j's commute for $1 \leq i, j \leq m$ and satisfy:

$$\forall\, j < i, \quad \sigma_i(\partial_j) = \partial_j, \quad \delta_i(\partial_j) = 0.$$

Remark 1 [5,7,17] Let $D = A[\partial; \sigma, \delta]$ be a skew polynomial ring. Every element P of D has a unique normal form which is given by $P = \sum_{i=0}^{n} a_i\, \partial^i$ for suitable $a_i \in A$ and $n \in \mathbb{Z}_+ = \{0, 1, \ldots\}$. If $a_n \neq 0$, then the degree of P is n. For every Ore algebra, we get a similar normal form of its elements by moving all products of $\partial_1, \ldots, \partial_m$ on the right in each summand.

Example 1 Our first example is the *Weyl algebra* $A_1(k) = k[t][\partial; \sigma, \delta]$, where

$$\sigma = \mathrm{id}_{k[t]}, \quad \delta = \frac{d}{dt}.$$

Rule (1) expresses the commutation of the operator which acts as differentiation with respect to t, namely,

$$\forall\, a \in k[t], \quad \partial\, a = a\, \partial + \frac{da}{dt},$$

which must be compared with $\partial\,(a\, y) = a\, \partial\, y + (da/dt)\, y$.

For instance, the time-varying Kalman system $\dot{x}(t) = A(t)\, x(t) + B(t)\, u(t)$, where $A \in k[t]^{n \times n}$ and $B \in k[t]^{n \times m}$, is defined by the matrix of operators $R = (\partial I_n - A(t) : -B(t)) \in A_1(k)^{n \times (n+m)}$ as we have $R\, (x(t)^T : u(t)^T)^T = 0$.

Similar to polynomial rings in $2\,n$ indeterminates, we can define the so-called Weyl algebra $A_n(k) = k[x_1, \ldots, x_n][\partial_1; \sigma_1, \delta_1] \ldots [\partial_n; \sigma_n, \delta_n]$, where σ_i and δ_i on $k[x_1, \ldots, x_n]$ are the maps

$$\sigma_i = \mathrm{id}_{k[x_1, \ldots, x_n]}, \quad \delta_i = \frac{\partial}{\partial\, x_i}, \quad i = 1, \ldots, n,$$

and every other commutation rule is prescribed by Definition 1. In particular, we have the relations

$$\partial_i\, x_j = x_j\, \partial_i + \delta_{ij}, \quad 1 \leq i, j \leq n,$$

where the Kronecker symbol δ_{ij} is defined by $\delta_{ij} = 1$ if $i = j$ and 0 else.

Example 2 The algebra of shift operators with polynomial coefficients is another case of an Ore algebra. For $h \in \mathbb{R}$, we define $S_h = k[t][\delta_h; \sigma_h, \delta]$ by:

$$\forall\, a \in k[t], \quad \sigma_h(a)(t) = a(t - h), \quad \delta(a) = 0.$$

Hence, the commutation rule $\delta_h\, t = (t - h)\, \delta_h$ actually represents the action of the shift operator on polynomials. We note that δ_h is a *time-delay operator* if $h > 0$ and δ_h is an *advance operator* if $h < 0$.

For instance, the time-delay system $x(t) = x(t - h) + u(t - 2\,h)$ is defined by the matrix $R = (1 - \delta_h : -\delta_h^2) \in S_h^{1 \times 2}$, i.e., we have $R\, (x(t) : u(t))^T = 0$.

Example 3 In order to treat differential time-delay systems, we mix the constructions of the two preceding examples. We define the Ore algebra $D_h = k[t][\partial; \sigma_1, \delta_1]\,[\delta_h; \sigma_2, \delta_2]$, where:

$$\sigma_1 = \mathrm{id}_{k[t]}, \quad \delta_1 = \frac{d}{dt}, \quad \forall\, a \in k[t], \quad \sigma_2(a)(t) = a(t-h), \quad \delta_2 = 0, \quad h \in \mathbb{R}.$$

For instance, the matrix of operators $R = (\partial I - A(t) \;:\; -B(t)\,\delta_h) \in D_h^{n\times(n+m)}$, where A and B are two polynomial matrices in t, defines the following linear system $R\,(x(t)^T \;:\; u(t)^T)^T = \dot{x}(t) - A(t)\,x(t) - B(t)\,u(t-h) = 0$.

If the considered system also involves an advance operator σ_3 of amplitude $l \in \mathbb{R}$ such that h and l are non-commensurate, i.e., the $\mathbb{Q}$-vector space generated by h and l is supposed to be 2-dimensional, then we can work with the Ore algebra defined by $H_{(h,l)} = k[t][\partial; \sigma_1, \delta_1][\delta_h; \sigma_2, \delta_2][\tau_l; \sigma_3, \delta_3]$, where $\sigma_i, \delta_i, i = 1, 2$, are as above and:

$$\forall\, a \in k[t], \quad \sigma_3(a)(t) = a(t+l), \quad \delta_3 = 0, \quad l > 0.$$

Now, if we want to consider the case where $h = l$, we can then define the k-algebra H_h formed by $H_{(h,l)}$ modulo the relations $\delta_h\,\tau_l - 1$ and $\tau_l\,\delta_h - 1$. We note that H_h is not an Ore algebra but a quotient of the Ore algebra $H_{(h,l)}$ by the two-sided ideal generated by $\tau_l\,\delta_h - 1$.

Example 4 In order to study multidimensional discrete linear systems, we define the Ore algebra $D = k[x_1, \dots, x_n][\partial_1; \sigma_1, \delta_1]\dots[\partial_n; \sigma_n, \delta_n]$, where σ_i and δ_i on $k[x_1, \dots, x_n]$ are the maps:

$$\sigma_i(a)(x_1, \dots, x_n) = a(x_1, \dots, x_{i-1}, x_i + 1, x_{i+1}, \dots, x_n),\ \delta_i = 0,\ 1 \le i \le n.$$

Similarly, we can define an Ore algebra formed by the backward shift operators τ_i defined by:

$$\tau_i(a)(x_1, \dots, x_n) = a(x_1, \dots, x_{i-1}, x_i - 1, x_{i+1}, \dots, x_n),\ \delta_i = 0,\ 1 \le i \le n.$$

Ore algebras based on other functional operators can also be defined (e.g., Eulerian operators, divided differences). We refer to [7,17] for more details.

2.2 Properties of Ore algebras & Gröbner bases

We summarize the most important properties of Ore algebras that will enable us to computationally deal with modules over Ore algebras. In order to do that, we first recall some definitions. See [17,36] for more details.

Definition 2 1. A ring A is called a *left* (resp., *right*) *noetherian ring* if every left (resp., right) ideal of A is finitely generated as a left (resp., right) A-module.
2. A domain A is said to have the *left* (resp., *right*) *Ore property* if, for each pair $(a_1, a_2) \in A \times A$, there is a non-trivial pair $(b_1, b_2) \in A \times A$ such that $b_1\,a_1 = b_2\,a_2$ (resp., $a_1\,b_1 = a_2\,b_2$). In that case, A is called a *left* (resp., *right*) *Ore ring*.

Proposition 1 *[17] If D is a left (resp., right) noetherian ring, then D has the left (resp., right) Ore property.*

Proposition 2 *[17] If A is a domain and σ is injective, then the skew polynomial ring $A[\partial; \sigma, \delta]$ is a domain.*

Proposition 3 *[17] If A is a left (resp., right) noetherian ring and σ is an automorphism (e.g., A_n, S_h, D_h, $H_{(h,l)}$), then the skew polynomial ring $A[\partial; \sigma, \delta]$ is a left (resp., right) noetherian ring.*

Proposition 4 *[7] If A has the left (resp., right) Ore property, then $A[\partial; \sigma, \delta]$ also has the left (resp., right) Ore property.*

In order to study systems over (non-commutative) polynomial rings effectively, we need to introduce some algorithmic methods based on Gröbner bases. We first need monomial orders to compare (non-commutative) polynomials.

Definition 3 1. Let D be an Ore algebra. A *monomial order* $<$ on Mon (D) is defined as a total order on the set of monomials $\mathrm{Mon}(D)$ satisfying $1 < m$ for all monomials $1 \neq m \in \mathrm{Mon}(D)$ and, if $m_1 < m_2$ holds for two monomials $m_1, m_2 \in \mathrm{Mon}(D)$, then $n \cdot m_1 < n \cdot m_2$ for all $n \in \mathrm{Mon}(D)$.

2. Given a polynomial $0 \neq P \in D$ and a monomial order $<$ on Mon (D), we can compare the monomials with a non-zero coefficient in P with respect to $<$. The greatest of these monomials is the *leading monomial* $\mathrm{lm}(P)$ of P.

Definition 4 [1] Let A be a polynomial ring and I an ideal of A. A set of non-zero polynomials $G = \{g_1, \ldots, g_t\} \subset I$ is called a *Gröbner basis* for I if for all $0 \neq f \in I$, there exists $1 \leq i \leq t$ such that $\mathrm{lm}(g_i)$ divides $\mathrm{lm}(f)$.

Gröbner bases for left ideals in Ore algebras are defined analogously for monomial orders on $\partial_1, \ldots, \partial_m$ and $x_1, \ldots, x_n$ [7].

Remark 2 A consequence of the condition that defines Gröbner bases is that every polynomial f in I is *reduced* to 0 modulo G, i.e., by subtracting suitable left multiples of the $g_i \in G$ from f, we obtain the zero polynomial.

For the case of commutative polynomial rings, Buchberger's algorithm ([1,2]) computes Gröbner bases of polynomial ideals. The next theorem states that this algorithm can be extended to certain Ore algebras. Every Ore algebra within our scope is of this kind.

Theorem 1 *[5,7,13] Let k be a computable field [2] (e.g., $k = \mathbb{Q}$, $\mathbb{F}_p$), $A = k[x_1, \ldots, x_n]$ the polynomial ring with n indeterminates over k and $A[\partial_1; \sigma_1, \delta_1] \ldots [\partial_m; \sigma_m, \delta_m]$ an Ore algebra with*

$$\sigma_i(x_j) = a_{ij}\, x_j + b_{ij}, \quad \delta_i(x_j) = c_{ij}, \quad 1 \leq i \leq m, \quad 1 \leq j \leq n, \tag{2}$$

for certain $a_{ij} \in k \setminus \{0\}$, $b_{ij} \in k$, $c_{ij} \in A$. If the c_{ij} are of total degree at most 1 in the x_i's, then a non-commutative version of Buchberger's algorithm terminates for any monomial order on $x_1, \ldots, x_n, \partial_1, \ldots, \partial_m$, and its result is a Gröbner basis with respect to the given monomial order.

In what follows, we shall use differential elimination techniques based on Gröbner bases.

Definition 5 [1] Let D be the polynomial ring over $\mathbb{Q}$ with indeterminates $x_1, \ldots,$ $x_n, y_1, \ldots, y_m$. Assume that monomial orders $<_x$ and $<_y$ on the monomials that only contain respectively the x_i's and the y_i's are given. An *elimination order* is then defined by

$$m_1 \cdot n_1 < m_2 \cdot n_2 \iff m_1 <_x m_2 \quad \text{or} \quad m_1 = m_2 \quad \text{and} \quad n_1 <_y n_2,$$

where m_1, m_2 (resp., n_1, n_2) are monomials containing only the x_i's (resp., y_i's).

Intuitively, an elimination order serves to eliminate the x_i's.

The elimination order which we shall use in this paper is the one induced by the *degree reverse lexicographical orders* on $x_1, \ldots, x_n$ and $y_1, \ldots, y_m$. This is a very common order called *lexdeg* in the Maple package *Groebner*.

Example 5 Given a left ideal I of D, we obtain a Gröbner basis of the left ideal $I \cap k[y_1, \ldots, y_m]$ by computing the Gröbner basis G of I with respect to an elimination order and intersecting G with $k[y_1, \ldots, y_m]$ (which merely amounts to omit all polynomials in G that involve any x_i).

3 Module theory over Ore algebras

Let us give a motivation for the use of modules. Let D be a domain and let us consider a system of equations

$$\sum_{j=1}^{p} R_{ij}\, y_j = 0, \quad 1 \leq i \leq q, \tag{3}$$

where $R_{ij} \in D$, $p, q \in \mathbb{N} = \{1, 2, \ldots\}$. By collecting the coefficients R_{ij}, we obtain a matrix $R \in D^{q \times p}$ which, multiplied by the vector $y = (y_1 : \ldots : y_p)^T$, yields system (3) again.

We shall use the convention that elements of $D^{1\times p}$ are row vectors whereas those of D^p are column vectors. Let us consider the following left D-morphism (i.e., D-linear map):

$$\begin{aligned} D^{1\times q} &\xrightarrow{.R} D^{1\times p}, \\ (P_1 : \ldots : P_q) &\longmapsto (P_1 : \ldots : P_q)\, R. \end{aligned}$$

Then, $\operatorname{im}(.R) = D^{1\times q} R$ is the left D-module generated by the left D-linear combinations of the rows of R (namely, the ring D acts on the elements of $\operatorname{im}(.R) = D^{1\times q} R$ from the left). Let us show how system (3) is associated with the left D-module $M = D^{1\times p}/(D^{1\times q} R)$.

We denote by $\{e_i\}_{1\leq i\leq p}$ (resp., $\{f_j\}_{1\leq j\leq q}$) the standard basis of $D^{1\times p}$ (resp., $D^{1\times q}$), namely e_i is the row vector with 1 in i^{th} position and 0 elsewhere. Let us define by $\kappa : D^{1\times p} \to M$ the left D-morphism which maps any element of $D^{1\times p}$ onto its residue class in M, i.e., modulo $D^{1\times q} R$. In particular, we have

$\kappa(\lambda_1) = \kappa(\lambda_2)$ iff there exists $\mu \in D^{1\times q}$ such that $\lambda_1 = \lambda_2 + \mu\, R$. Then, for $j = 1, \ldots, q$, we have:

$$f_j\, R = (R_{j1} : \ldots : R_{jp}) = \sum_{i=1}^{p} R_{ji}\, e_i \in D^{1\times q}\, R.$$

Therefore, we obtain:

$$\kappa(f_j\, R) = \kappa\left(\sum_{i=1}^{p} R_{ji}\, e_i\right) = \sum_{i=1}^{p} R_{ji}\, \kappa(e_i) = 0.$$

Hence, if we denote by $y_i = \kappa(e_i)$ the residue class of e_i in M, then the left D-module M is defined by

$$\sum_{i=1}^{p} R_{ji}\, y_i = 0, \quad 1 \le j \le q \quad \Leftrightarrow \quad R\, y = 0, \quad y = (y_1 : \ldots : y_p)^T,$$

as well as by the left D-linear combinations of its equations.

The left D-module $M = D^{1\times p}/(D^{1\times q}\, R)$ is said to be *finitely generated* as any element $m \in M$ can be written as $m = \sum_{i=1}^{p} P_i\, y_i$ where $P_i \in D$.

Definition 6 The left D-module $M = D^{1\times p}/(D^{1\times q}\, R)$ is said to be *associated with* (3).

If D is a commutative ring, then we recall that a left D-module M is also a right D-module and conversely. In this case, we shall only say that M is a D-module.

Example 6 Let us consider the Ore algebra defined by

$$D_h = \mathbb{R}(a,\, k,\, \zeta,\, \omega)[\partial; \sigma_1, \delta_1][\delta_h; \sigma_2, \delta_2]$$

of the type of Example 3 and revisit the wind tunnel model defined in [19]

$$\begin{cases} \dot{x}_1(t) = -a\, x_1(t) + k\, a\, x_2(t-h), \\ \dot{x}_2(t) = x_3(t), \\ \dot{x}_3(t) = -\omega^2\, x_2(t) - 2\,\zeta\,\omega\, x_3(t) + \omega^2\, u(t), \end{cases} \tag{4}$$

where $a,\ k,\ \zeta$ and ω are real constants. System (4) gives rise to the matrix

$$R = \begin{pmatrix} \partial + a & -k\, a\, \delta_h & 0 & 0 \\ 0 & \partial & -1 & 0 \\ 0 & \omega^2 & \partial + 2\,\zeta\,\omega & -\omega^2 \end{pmatrix} \in D_h^{3\times 4}, \tag{5}$$

and thus, system (4) corresponds to the D_h-module $M = D_h^{1\times 4}/(D_h^{1\times 3}\, R)$.

In Section 5, we shall develop a dictionary between the properties of the left D-module $M = D^{1\times p}/(D^{1\times q}\, R)$ and the properties of the system $R\, y = 0$. But, at first, let us introduce some definitions of module theory and homological algebra [4,36].

Definition 7 1. A family $(M_i)_{i\in\mathbb{Z}}$ of (left/right) D-modules together with a family $(d_i)_{i\in\mathbb{Z}}$ of (left/right) D-module morphisms $d_i : M_i \to M_{i-1}$ is called a *complex* if we have $d_i \circ d_{i+1} = 0$ for all $i \in \mathbb{Z}$. We write:

$$\dots \xrightarrow{d_{i+2}} M_{i+1} \xrightarrow{d_{i+1}} M_i \xrightarrow{d_i} M_{i-1} \xrightarrow{d_{i-1}} \dots \tag{6}$$

2. The *defect of exactness of (6) at M_i* is defined by:

$$H(M_i) = \ker d_i / \operatorname{im} d_{i+1}.$$

The complex (6) is said to be *exact at M_i* if the defect of exactness $H(M_i)$ is reduced to 0 or, equivalently, if $\ker d_i = \operatorname{im} d_{i+1}$. More generally, the complex (6) is called *exact* if it is exact at every position.

3. We call *short exact sequence* an exact sequence of the form

$$0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0, \tag{7}$$

i.e., where f is injective, g is surjective and $\ker g = \operatorname{im} f$.

We recall some properties of modules that will play important roles in what follows.

Definition 8 [4,36] Let D be a domain which is a left Ore ring and M a finitely generated left D-module. Then, we have:

1. M is *free* if it is isomorphic to $D^{1\times r}$ for a certain $r \in \mathbb{Z}_+ = \{0, 1, \dots\}$.
2. M is *stably free* if there exist $r, s \in \mathbb{Z}_+$ such that we have:

$$M \oplus D^{1\times s} \cong D^{1\times r}.$$

3. M is *projective* if there exist a left D-module N and $r \in \mathbb{Z}_+$ such that:

$$M \oplus N \cong D^{1\times r}.$$

4. M is *reflexive* if the canonical map defined by

$$\varepsilon_M : M \longrightarrow \hom_D(\hom_D(M, D), D), \quad \varepsilon_M(m)(f) = f(m),$$

for all $m \in M$, $f \in \hom_D(M, D)$, is an isomorphism, where $\hom_D(M, D)$ denotes the right D-module of all D-morphisms from M to D.

5. M is *torsion-free* if the left submodule of M defined by

$$t(M) = \{m \in M \mid \exists\, 0 \neq P \in D : \; P\, m = 0\}$$

is reduced to 0. $t(M)$ is called the *torsion submodule* of M and the elements of $t(M)$ are the *torsion elements* of M.

6. M is *torsion* if $t(M) = M$, i.e., every element of M is a torsion element.

Similar definitions hold for right modules over a right Ore domain D.

We note that the fact that $t(M)$ is a (left/right) D-submodule of M implies that we have the following short exact sequence

$$0 \longrightarrow t(M) \xrightarrow{i} M \xrightarrow{\rho} M/t(M) \longrightarrow 0,$$

where i (resp., ρ) denotes the canonical injection (resp., projection).

Proposition 5 *[4,36] Let D be a domain which is a (left/right) Ore ring and M be a finitely generated (left/right) D-module. Then, we have the following implications among these concepts:*

$$\textit{free} \Rightarrow \textit{stably free} \Rightarrow \textit{projective} \Rightarrow \textit{reflexive} \Rightarrow \textit{torsion-free}.$$

Theorem 2 *1. [17,36] If D is a* left hereditary ring *– namely, every left ideal of D is a left projective D-module – (e.g., a commutative Dedekind domain, $A_1(k)$), then every finitely generated torsion-free left D-module is projective. Moreover, if D is a* left principal ideal domain *– namely, every left ideal of D is principal – (e.g., $k[x]$ or $k(t)[\partial; \mathrm{id}_{k(t)}, \frac{d}{dt}]$, where k is a field of constants), then every finitely generated torsion-free left D-module is free.*

2. *[17] If $D = A[\partial_1; \sigma_1, \delta_1] \ldots [\partial_m; \sigma_m, \delta_m]$ is an Ore algebra where σ_i is an automorphism, $i = 1, \ldots, m$, then, every finitely generated projective left D-module is stably free.*
3. *[17,36] Every projective module over a commutative polynomial ring with coefficients in a field is free (Quillen-Suslin theorem).*
4. *[17] If k is a field of characteristic 0, then every stably free left $A_n(k)$-module M with $\mathrm{rank}_{A_n(k)} M \geq 2$ is free.*

Definition 9 [4,36] A short exact sequence of (left/right) D-modules

$$0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0 \tag{8}$$

is called a *split exact sequence* if one of the following equivalent conditions is satisfied:

- There exists a D-morphism $h : M'' \longrightarrow M$ such that $g \circ h = \mathrm{id}_{M''}$.
- There exists a D-morphism $k : M \longrightarrow M'$ such that $k \circ f = \mathrm{id}_{M'}$.
- The (left/right) D-module M is isomorphic to the direct sum of M' and M'', which is denoted by $M \cong M' \oplus M''$, or equivalently, there exist two D-morphisms

$$\begin{cases} \phi = \begin{pmatrix} k \\ g \end{pmatrix} : M \longrightarrow M' \oplus M'', \\ \psi = (f \; : \; h) : M' \oplus M'' \longrightarrow M, \end{cases}$$

satisfying $\phi \circ \psi = \mathrm{id}_{M' \oplus M''}$ and $\psi \circ \phi = \mathrm{id}_M$.

Proposition 6 *[4,36] If M'' is a (left/right) projective D-module and M' and M are (left/right) D-modules, then the short exact sequence (8) splits.*

In the following sections, we shall develop effective algorithms based on Gröbner bases in order to check whether or not a finitely generated left D-module M associated with a linear system (e.g., the differential time-delay system (4)) is torsion-free, reflexive, projective, stably free or free.

4 Syzygy modules and free resolutions

If $R \in D^{q\times p}$ and M is the left D-module $M = D^{1\times p}/(D^{1\times q}\, R)$, then we have the following exact sequence:

$$0 \longrightarrow \ker(.R) \longrightarrow D^{1\times q} \xrightarrow{.R} D^{1\times p} \xrightarrow{\kappa} M \longrightarrow 0. \tag{9}$$

In this section, we show how to extend this exact sequence on the left in order to obtain a long exact sequence involving only M and free left D-modules.

Let M be a finitely generated left module over a left noetherian ring D. We can reformulate the fact that M is finitely generated by saying that there exists a surjective D-morphism $f : D^{1\times p} \to M$ which maps the i^{th} vector of the standard basis $\{e_i\}_{1\le i\le p}$ of $D^{1\times p}$ to some $m_i \in M$. Then, we have the following exact sequence:

$$\begin{array}{ccccc} D^{1\times p} & \xrightarrow{f} & M & \longrightarrow & 0, \\ e_i & \longmapsto & m_i. & & \end{array}$$

This map can fail to be injective since there can be linear relations among the $\{m_i\}_{1\le i\le p}$:

$$\begin{aligned} \ker f = \{P = (P_1 : \ldots : P_p) \in D^{1\times p} \mid f(P) &= \sum_{i=1}^{p} P_i\, f(e_i) \\ &= \sum_{i=1}^{p} P_i\, m_i = 0\}. \end{aligned} \tag{10}$$

Definition 10 [4,36] The D-linear relations among the $m_1, \ldots, m_p$ form the left D-module $S(M)$ defined by (10) which is called the *syzygy module* of M (this module is uniquely defined up to *projective equivalence* [4,36]). A similar definition exists for right D-modules.

Since D is a left noetherian ring, $S(M)$ is a finitely generated left D-module. Therefore, we can again find a suitable free D-module $D^{1\times q}$ and a D-morphism g sending the standard basis vectors of $D^{1\times q}$ to a family of generators of $S(M)$ and we have the following exact sequence:

$$D^{1\times q} \xrightarrow{g} D^{1\times p} \xrightarrow{f} M \longrightarrow 0.$$

This exact sequence is called a *finite presentation* of the left D-module M and M is said to be *finitely presented*. Let us notice that, with respect to the standard bases of $D^{1\times q}$ and $D^{1\times p}$, g is defined by multiplication on the right with the matrix whose i^{th} row corresponds to the i^{th} generator of $S(M)$. Finally, iterating the preceding construction, we get the definition of a *free resolution* of the left D-module M.

Definition 11 1. [4,36] An exact sequence of the form

$$\ldots \xrightarrow{d_3} P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} M \longrightarrow 0 \tag{11}$$

is called a (left/right) *projective resolution* of the (left/right) D-module M if the D-modules P_i are projective (left/right) D-modules. The (left/right) D-module $S_i(M) = \ker d_i$ is called the *i^{th} syzygy (left/right) D-module of M.*

2. [4,36] If the P_i in (11) are free, then (11) is called a *free resolution* of M.
3. [4,36] Let $0 \longrightarrow P_n \xrightarrow{d_n} P_{n-1} \xrightarrow{d_{n-1}} \dots \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} M \longrightarrow 0$ be a projective resolution of M. Then, the *length* of this resolution is n.
4. [4,36] The minimal length of the (left/right) projective resolutions of the (left/right) D-module M is called the *(left/right) projective dimension* $\mathrm{pd}_D({}_D M)$ (resp., $\mathrm{pd}_D(M_D)$) of M. If no such integer exists, then we set $\mathrm{pd}_D({}_D M) = \infty$ (resp., $\mathrm{pd}_D(M_D) = \infty$).
5. [4,36] We set $\mathrm{lgld}\, D = \sup\{\mathrm{pd}_D({}_D M) \mid M \text{ a left } D\text{-module}\} \in \mathbb{Z}_+ \cup \{\infty\}$, which is called the *left global dimension* of D. Similarly for the *right global dimension* $\mathrm{rgld}\, D$. If D is commutative, then we write:

$$\mathrm{gld}\, D = \mathrm{lgld}\, D = \mathrm{rgld}\, D.$$

We now describe the computational tools for the construction of free resolutions. The methods to compute syzygy modules use Gröbner bases and elimination techniques (see Section 6.1 of [2]). Let D be an Ore algebra which satisfies (2) and L a finitely generated left D-module which is a submodule of a free D-module $D^{1\times p}$, $p \in \mathbb{N}$. Thus, a set of generators of L consists of row vectors in $D^{1\times p}$.

Algorithm 1
Input: Set of generators $\{R_1, \dots, R_q\} \subset D^{1\times p}$ of the left D-module L, where $R_i = (R_{i1} : \dots : R_{ip})$, $i = 1, \dots, q$.
Output: $S \in D^{r\times q}$ such that the left D-module $D^{1\times r}\, S$ is the syzygy module $S(L)$ of L.
SYZYGIES $(R_1, \dots, R_q)$
Introduce the indeterminates $\lambda_1, \dots, \lambda_p, \mu_1, \dots, \mu_q$ over D.
$\mathrm{P} \leftarrow \left\{\sum_{j=1}^{p} R_{ij}\, \lambda_j - \mu_i \mid i = 1, \dots, q\right\}$.
Compute the Gröbner basis G of P in the free left D-module generated by $\lambda_1, \dots, \lambda_p, \mu_1, \dots, \mu_q$, namely $\bigoplus_{i=1}^{p} D\,\lambda_i \oplus \bigoplus_{i=1}^{q} D\,\mu_i$, with respect to an order which eliminates the λ_i's.
$S = (S_{ij}) \in D^{r\times q} \leftarrow G \cap \bigoplus_{i=1}^{q} D\,\mu_i = \left\{\sum_{j=1}^{q} S_{ij}\,\mu_j \mid i = 1, \dots, r\right\}$.

Remark 3 Let us suppose that we have $R \in D^{q\times p}$ and the left D-module $M = D^{1\times p}/(D^{1\times q}\, R)$. Then, we can apply the preceding algorithm to the set formed by

$$R_i = (R_{i1} : \dots : R_{ip}) \in L = D^{1\times q}\, R \subseteq D^{1\times p}, \quad i = 1, \dots, q,$$

in order to obtain a matrix $S = (S_{ij}) \in D^{r\times q}$ such that

$$S(D^{1\times q}\, R) = \ker(.R) = \left\{(P_1 : \dots : P_q) \in D^{1\times q} \mid \sum_{i=1}^{q} P_i\, R_i = 0\right\} = D^{1\times r}\, S$$

and we obtain the following exact sequence:

$$D^{1\times r} \xrightarrow{.S} D^{1\times q} \xrightarrow{.R} D^{1\times p} \xrightarrow{\kappa} M \longrightarrow 0.$$

Iterating the process, we obtain a free resolution of the left D-module M.

For further developments and optimization of the technique, see [14].

Example 7 Let D_h be the differential time-delay Ore algebra introduced in Example 6, $R \in D_h^{3\times 4}$ defined by (5) and the D_h-module $L = D_h^{1\times 4}\, R^T$ generated by the rows of the matrix:

$$R^T = \begin{pmatrix} \partial + a & 0 & 0 \\ -k\, a\, \delta_h & \partial & \omega^2 \\ 0 & -1 & \partial + 2\,\zeta\,\omega \\ 0 & 0 & -\omega^2 \end{pmatrix} \in D_h^{4\times 3}.$$

We shall see later that the transposed matrix R^T plays an important role in the characterization of the properties of the D_h-module $M = D_h^{1\times 4}/(D_h^{1\times 3}\, R)$. Let us compute the syzygy module of L. The Gröbner basis of

$$P = \{(\partial + a)\,\lambda_1 - \mu_1, \quad -k\,a\,\delta_h\,\lambda_1 + \partial\,\lambda_2 + \omega^2\,\lambda_3 - \mu_2,$$
$$-\lambda_2 + (\partial + 2\,\zeta\,\omega)\,\lambda_3 - \mu_3, \quad -\omega^2\,\lambda_3 - \mu_4\}$$

with respect to the elimination order induced by the degree reverse lexicographical orders on $\lambda_1 > \lambda_2 > \lambda_3$ and $\mu_1 > \mu_2 > \mu_3 > \mu_4 > \delta_h > \partial$ respectively is (see Appendix 10.3):

$$G = \{\omega^2\,\lambda_2 + \partial\,\mu_4 + \omega^2\,\mu_3 + 2\,\zeta\,\omega\,\mu_4, \quad \omega^2\,k\,a\,\delta_h\,\lambda_1 + \omega^2\,\mu_2 + \omega^2\,\partial\,\mu_3 +$$
$$(\partial^2 + 2\,\zeta\,\omega\,\partial + \omega^2)\,\mu_4, \quad \omega^2\,k\,a\,\delta_h\,\mu_1 + (\omega^2\,\partial + \omega^2\,a)\,\mu_2 +$$
$$(\omega^2\,\partial^2 + \omega^2\,a\,\partial)\,\mu_3 + (\partial^3 + (2\,\zeta\,\omega + a)\,\partial^2 + (\omega^2 + 2\,a\,\zeta\,\omega)\,\partial + a\,\omega^2)\,\mu_4,$$
$$(\partial + a)\,\lambda_1 - \mu_1, \quad \omega^2\,\lambda_3 + \mu_4\}.$$

Intersecting G with $\bigoplus_{i=1}^4 D_h\,\mu_i$ we get

$$S = \{\omega^2\,k\,a\,\delta_h\,\mu_1 + (\omega^2\,\partial + \omega^2\,a)\,\mu_2 + (\omega^2\,\partial^2 + \omega^2\,a\,\partial)\,\mu_3 +$$
$$(\partial^3 + (2\,\zeta\,\omega + a)\,\partial^2 + (\omega^2 + 2\,a\,\zeta\,\omega)\,\partial + a\,\omega^2)\,\mu_4\}.$$

If we denote by Q^T the following row vector with entries in D_h

$$Q^T = (\omega^2\,k\,a\,\delta_h \,:\, \omega^2\,\partial + \omega^2\,a \,:\, \omega^2\,\partial^2 + \omega^2\,a\,\partial \,:$$
$$\partial^3 + (2\,\zeta\,\omega + a)\,\partial^2 + (\omega^2 + 2\,a\,\zeta\,\omega)\,\partial + a\,\omega^2),$$

then we obtain the free resolution of the D_h-module $N = D_h^{1\times 3}/(D_h^{1\times 4}\, R^T)$:

$$0 \longrightarrow D_h \xrightarrow{.Q^T} D_h^{1\times 4} \xrightarrow{.R^T} D_h^{1\times 3} \xrightarrow{\kappa} N \longrightarrow 0. \tag{12}$$

We conclude this section by giving examples of left (right) global dimensions and by stating that free resolutions of finite length exist for every finitely generated module over Ore algebras $D = A[\partial_1; \sigma_1, \delta_1] \ldots [\partial_m; \sigma_m, \delta_m]$, where all σ_i are automorphisms. Note that every Ore algebra that we consider is of this kind (see e.g., Examples 1, 2, and 3).

Proposition 7 *[17] Let A be a domain with a finite left (resp., right) global dimension* lgld A *(resp.,* rgld A*) and σ an automorphism of A.*

1. *The left (resp., right) global dimension of* $A[\partial; \sigma, \delta]$ *satisfies:*

$$\begin{array}{c} \operatorname{lgld} A \leq \operatorname{lgld} A[\partial; \sigma, \delta] \leq \operatorname{lgld} A + 1 \\ (\operatorname{rgld} A \leq \operatorname{rgld} A[\partial; \sigma, \delta] \leq \operatorname{rgld} A + 1). \end{array}$$

If $\delta = 0$*, then we have:*

$$\operatorname{lgld} A[\partial; \sigma, 0] = \operatorname{lgld} A + 1 \quad (\operatorname{rgld} A[\partial; \sigma, 0] = \operatorname{rgld} A + 1).$$

2. *If* k *is a field, then we have* $\operatorname{lgld} k[x_1, \ldots, x_n] = \operatorname{rgld} k[x_1, \ldots, x_n] = n$.
3. *The Weyl algebra* $A_n(k)$ *satisfies* $\operatorname{lgld} A_n(k) = \operatorname{rgld} A_n(k) = n$ *if* k *is a field of characteristic* 0 *and* $\operatorname{lgld} A_n(k) = \operatorname{rgld} A_n(k) = 2n$ *if* k *is a field of characteristic* $p > 0$.

Example 8 By Proposition 7, the Ore algebra S_h of Example 2 has left (resp., right) global dimension 2 because $\operatorname{lgld} k[t] = \operatorname{rgld} k[t] = 1$. For the Ore algebra D_h of Example 3, where $\mathbb{Q} \subseteq k$, we have $\operatorname{lgld} D_h = \operatorname{rgld} D_h = 2$ since $k[t][\partial; \sigma_1, \delta_1] = A_1(k)$ has left (resp., right) global dimension 1 and $\delta_2 = 0$.

Proposition 8 *[17,36] Let* $D = A[\partial_1; \sigma_1, \delta_1] \ldots [\partial_m; \sigma_m, \delta_m]$ *be an Ore algebra, where* σ_i *is an automorphism,* $i = 1, \ldots, m$*, and* M *a finitely generated (left/right)* D*-module. Then, there is a* free *resolution of* M *of length less than or equal to* $\operatorname{lgld} D + 1$ *(resp.,* $\operatorname{rgld} D + 1$*).*

Proof Iteratively applying Corollary 12.3.3 of [17] according to the construction of D from the field k, it follows that every finitely generated projective (left/right) D-module is stably free. Since D has finite (left/right) global dimension according to Proposition 7, M has a projective resolution (11) of length less than or equal to $\operatorname{lgld} D$ (resp., $\operatorname{rgld} D$), where all P_i are finitely generated projective, i.e., stably free (left/right) D-modules. Now, Lemma 9.40 of [36] states that, if M has a *stably free resolution* of length n, namely a resolution of the form (11) where every P_i is a stably free (left/right) D-module, then M has a free resolution of length less than or equal to $n + 1$. Hence, we conclude that M has a free resolution of length less than or equal to $\operatorname{lgld} D + 1$ (resp., $\operatorname{rgld} D + 1$). □

Let us notice that the previous result is a reminiscence of the concept of a *Janet sequence* developed in the theory of differential operators [25].

5 System interpretations of module properties

In this section, we give system interpretations of some module properties.

5.1 A functorial approach to behaviours of linear systems

We firstly need to introduce the concept of *extension modules* [4,35].

Definition 12 [4,36] Let D be a left noetherian ring, M a finitely generated left D-module, $\mathcal{F}$ a left D-module and

$$\ldots \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} M \longrightarrow 0$$

a projective resolution of M. Then, the defects of exactness of the complex

$$\ldots \longleftarrow \hom_D(P_2, \mathcal{F}) \xleftarrow{d_2^*} \hom_D(P_1, \mathcal{F}) \xleftarrow{d_1^*} \hom_D(P_0, \mathcal{F}) \longleftarrow 0,$$

where d_i^* is defined by $d_i^*(f) = f \circ d_i$ for $f \in \hom_D(P_{i-1}, \mathcal{F})$, $i \geq 1$, are defined by the abelian groups:

$$\begin{cases} \mathrm{ext}_D^0(M, \mathcal{F}) = \ker d_1^* = \hom_D(M, \mathcal{F}), \\ \mathrm{ext}_D^i(M, \mathcal{F}) = \ker d_{i+1}^* / \operatorname{im} d_i^*, \quad i \geq 1. \end{cases}$$

The next proposition shows that $\mathrm{ext}_D^i(M, \mathcal{F})$ is well-defined.

Proposition 9 *[4,36] The abelian group $\mathrm{ext}_D^i(M, \mathcal{F})$, $i \in \mathbb{Z}_+$, only depends on M and $\mathcal{F}$ up to group isomorphism, i.e., we can choose any projective resolution of M to compute it.*

If we consider a free resolution of M computed as indicated in Section 4,

$$\ldots \xrightarrow{.R_4} D^{1\times l_3} \xrightarrow{.R_3} D^{1\times l_2} \xrightarrow{.R_2} D^{1\times l_1} \xrightarrow{.R_1} D^{1\times l_0} \xrightarrow{\kappa} M \longrightarrow 0,$$

then, by Definition 12, we obtain the following complex

$$\ldots \xleftarrow{R_4.} \mathcal{F}^{l_3} \xleftarrow{R_3.} \mathcal{F}^{l_2} \xleftarrow{R_2.} \mathcal{F}^{l_1} \xleftarrow{R_1.} \mathcal{F}^{l_0} \longleftarrow 0, \tag{13}$$

where $R_i. : \mathcal{F}^{l_{i-1}} \to \mathcal{F}^{l_i}$ is defined by $(R_i.)(\eta) = R_i\, \eta$ for all $\eta \in \mathcal{F}^{l_{i-1}}$. Then, using Proposition 9, we obtain that:

$$\begin{cases} \mathrm{ext}_D^0(M, \mathcal{F}) \cong \ker_{\mathcal{F}}(R_1.) = \{\eta \in \mathcal{F}^{l_0} \mid R_1\, \eta = 0\}, \\ \mathrm{ext}_D^i(M, \mathcal{F}) \cong \ker_{\mathcal{F}}(R_{i+1}.)/(R_i\, \mathcal{F}^{l_{i-1}}), \quad i \geq 1. \end{cases}$$

Hence, the abelian group of all $\mathcal{F}$-solutions of the linear system $R_1\, \eta = 0$ is intrinsically defined by $\mathrm{ext}_D^0(M, \mathcal{F}) = \hom_D(M, \mathcal{F})$. In particular, the corresponding isomorphism is defined by

$$\begin{aligned} \hom_D(M, \mathcal{F}) \;&\cong\; \ker_{\mathcal{F}}(R_1.) \\ \varphi \;&\longmapsto\; \eta = (\varphi(y_1), \ldots, \varphi(y_{l_0}))^T, \\ \varphi : \{\varphi(y_i) = \eta_i\}_{i=1,\ldots,l_0} \;&\longleftarrow\!\shortmid\; \eta = (\eta_1, \ldots, \eta_{l_0})^T, \end{aligned} \tag{14}$$

where $y_i = \kappa(e_i)$ denotes the residue class of the i^{th} vector e_i of the standard basis of $D^{1\times l_0}$ in $M = D^{1\times l_0}/(D^{1\times l_1}\, R_1)$.

We note that $\ker_{\mathcal{F}}(R_1.) = \{\eta \in \mathcal{F}^{l_0} \mid R_1\, \eta = 0\}$ is called the *behaviour* of the left D-module M in the *behavioural approach* developed in control theory [18,22, 23,29,39,40]. In this approach, the left D-module $\mathcal{F}$ is called the *signal space*. Hence, a behaviour is intrinsically defined by $\hom_D(M, \mathcal{F})$.

The duality between module theory and behaviour theory was firstly developed by U. Oberst in [18] based on B. Malgrange's ideas. It has been largely studied since in the literature (e.g., see [22,29,39] and the references therein). See also [30] for the introduction of homological algebra in the behavioural approach (e.g., extension and torsion functors, homotopic and projective equivalences).

Let us explain why $\operatorname{ext}_D^1(M, \mathcal{F})$ corresponds to the obstruction of the solvability of the inhomogeneous linear system $R_1\,\eta = \zeta$. As (13) is a complex, a necessary condition for the existence of $\eta \in \mathcal{F}^{l_0}$ satisfying the linear system $R_1\,\eta = \zeta$ for a fixed $\zeta \in \mathcal{F}^{l_1}$ is $R_2\,\zeta = R_2\,(R_1\,\eta) = 0$, i.e., $\zeta \in \ker_{\mathcal{F}}(R_2.)$. Moreover, if there exists $\eta \in \mathcal{F}^{l_0}$ such that $R_1\,\eta = \zeta$, then we have $\zeta \in R_1\,\mathcal{F}^{l_0}$. Hence, the inhomogeneous linear system $R_1\,\eta = \zeta$ is solvable iff the residue class of ζ in $\operatorname{ext}_D^1(M, \mathcal{F}) \cong \ker_{\mathcal{F}}(R_2.)/(R_1\,\mathcal{F}^{l_0})$ is 0.

Definition 13 1. [4,36] A left D-module $\mathcal{F}$ is said to be *injective* if, for every left D-module M, then we have $\operatorname{ext}_D^i(M, \mathcal{F}) = 0$ for $i \geq 1$.
2. [4,36] A left D-module $\mathcal{F}$ is *cogenerator* if, for every left D-module M and $0 \neq m \in M$, then there exists $\varphi \in \hom_D(M, \mathcal{F})$ such that $\varphi(m) \neq 0$.

Intuitively, an injective cogenerator left D-module $\mathcal{F}$ corresponds to a sufficiently rich space of solutions for which a good duality between algebra and functional analysis holds. More precisely, an equivalent definition of an injective left D-module $\mathcal{F}$ is that, for every short exact sequence (7), we have the following exact sequence:

$$0 \longleftarrow \hom_D(M', \mathcal{F}) \xleftarrow{f^\star} \hom_D(M, \mathcal{F}) \xleftarrow{g^\star} \hom_D(M'', \mathcal{F}) \longleftarrow 0. \qquad (15)$$

Moreover, if $\mathcal{F}$ is a cogenerator left D-module, then $\hom_D(M, \mathcal{F}) = 0$ implies $M = 0$. Finally, we can prove that $\mathcal{F}$ is an *injective cogenerator* left D-module if the exactness of any complex of the form (15) is equivalent to the exactness of the complex (7) [18]. Hence, if $\mathcal{F}$ is an injective cogenerator left D-module, then the exactness of $\mathcal{F}^{l_2} \xleftarrow{R_2.} \mathcal{F}^{l_1} \xleftarrow{R_1.} \mathcal{F}^{l_0}$ is equivalent to the one of $D^{1\times l_2} \xrightarrow{.R_2} D^{1\times l_1} \xrightarrow{.R_1} D^{1\times l_0}$. In particular, if R_2 generates the syzygy module of $D^{1\times l_1}\,R_1$, i.e., $\ker(.R_1) = D^{1\times l_2}\,R_2$, then a necessary and sufficient condition for the existence of $\eta \in \mathcal{F}^{l_0}$ satisfying the system $R_1\,\eta = \zeta$ for $\zeta \in \mathcal{F}^{l_1}$ fixed is given by $R_2\,\zeta = 0$ (*fundamental principle* [18]). Such a condition can be computed using Algorithm 1.

Proposition 10 *[36] For every ring D, there exists an injective cogenerator left D-module $\mathcal{F}$.*

Let us give a few examples of injective cogenerator modules.

Example 9 1. [18,22,39] If Ω is an open convex subset of $\mathbb{R}^n$, then the space $C^\infty(\Omega)$ (resp., $\mathcal{D}'(\Omega)$) of smooth functions (resp., distributions) on Ω is an injective cogenerator over the ring $D = k[\partial_1; \sigma_1, \delta_1] \ldots [\partial_n; \sigma_n, \delta_n]$ of the differential operators with coefficients in $k = \mathbb{R}$, $\mathbb{C}$ (i.e., $\sigma_i = \mathrm{id}$, $\delta_i = \partial/\partial x_i$).
2. [41] If $\mathcal{F}$ denotes the set of all functions that are smooth on $\mathbb{R}$ except for a finite number of points, then $\mathcal{F}$ is an injective cogenerator left $D = \mathbb{R}(t)[\partial; \sigma, \delta]$-module where $\delta = \mathrm{id}$ and $\delta = d/dt$.

3. [18,39] If $D = k[\partial_1; \sigma_1, \delta_1] \ldots [\partial_n; \sigma_n, \delta_n]$ denotes the ring of forward shift operators defined with coefficients in $k = \mathbb{R}, \mathbb{C}$ (see Example 4), then $\mathcal{F} = k^{\mathbb{N}^n}$ is an injective cogenerator D-module.

In what follows, we shall only consider solutions of linear systems over an Ore algebra D in an injective cogenerator left D-module $\mathcal{F}$.

5.2 System properties − Module properties

We now introduce properties of linear systems over Ore algebras obtained by generalizing some definitions commonly used in the literature for some particular classes of systems (e.g., ODEs, PDEs, differential time-delay equations). In the following definition and the next theorem, the references point to the particular classes of systems for which these concepts have been firstly introduced.

Definition 14 Let D be a left noetherian Ore algebra, $R \in D^{q \times p}$, $\mathcal{F}$ an injective cogenerator left D-module and $\mathcal{B} = \{\eta \in \mathcal{F}^p \mid R\,\eta = 0\}$ the behaviour defined by R and $\mathcal{F}$. Then, we have the following definitions:

- [24,25,39] An *observable* of $\mathcal{B}$ is a left D-linear combination of the system variables η_i (i.e., of the system variables including inputs, states, outputs). An observable $\psi(\eta)$ is called *autonomous* if it satisfies some D-linear equation by itself, namely, $P\,\psi(\eta) = 0$ for some $0 \neq P \in D$. An observable is said to be *free* if it is not autonomous.
- [12,24,25,39] A behaviour $\mathcal{B}$ is said to be *controllable* if every observable of $\mathcal{B}$ is free.
- [24,25] A behaviour $\mathcal{B}$ is *parametrizable* if there exists a matrix $Q \in D^{p \times m}$ such that $\mathcal{B} = Q\,\mathcal{F}^m$, i.e., for every $\eta \in \mathcal{B}$, there exists $\phi \in \mathcal{F}^m$ such that $\eta = Q\,\phi$. Then, Q is called a *parametrization* of $\mathcal{B}$ and ϕ a *potential*.
- [12,20,31] Let D be a commutative polynomial ring and $\pi \in D \setminus \{0\}$. A parametrizable behaviour $\mathcal{B}$ is then called π*-free* if there exist $Q \in D^{p \times m}$, $T \in D^{m \times p}$ and $\alpha \in \mathbb{Z}_+$ such that $\mathcal{B} = Q\,\mathcal{F}^m$ and $T\,Q = \pi^\alpha\, I_m$, where I_m denotes the $m \times m$ identity matrix.
- [12,20] A behaviour $\mathcal{B}$ is said to be *flat* if there exists a parametrization $Q \in D^{p \times m}$ which admits a left-inverse $T \in D^{m \times p}$, i.e., $T\,Q = I_m$. In other words, $\mathcal{B}$ is flat if it is parametrizable and every component ϕ_i of the corresponding potential ϕ is an observable of the system. The potential ϕ is then called a *flat output* of $\mathcal{B}$.

The following theorem gives system interpretations of the module properties introduced in Definition 8.

Theorem 3 *Let D be a noetherian Ore algebra, $R \in D^{q \times p}$, $\mathcal{F}$ an injective cogenerator left D-module, the behaviour $\mathcal{B} = \{\eta \in \mathcal{F}^p \mid R\,\eta = 0\}$ defined by R and $\mathcal{F}$ and the left D-module $M = D^{1 \times p}/(D^{1 \times q}\,R)$. Then, we have:*

1. *[24–26,39] The observables of $\mathcal{B}$ are in a one-to-one correspondence with the elements of the left D-module M.*
2. *[12,22,24–26,39] The autonomous elements of $\mathcal{B}$ are in a one-to-one correspondence with the torsion elements of M.*

3. *[12,20,22,24–26] $\mathcal{B}$ is controllable iff M is torsion-free.*
4. *[24–27] The behaviour $\mathcal{B}$ is parametrizable iff there exists $Q \in D^{p\times m}$ such that $M = D^{1\times p}/(D^{1\times q}\, R) \cong D^{1\times p}\, Q$.*
5. *[12,20,31] Let D be a commutative polynomial ring and $\pi \in D \setminus \{0\}$. Then, a parametrizable behaviour $\mathcal{B}$ is π-free iff the module*

$$D_\pi \otimes_D M = \left\{m/a \mid m \in M,\ a = \pi^n,\ n \in \mathbb{Z}_+\right\}$$

is free over the ring $D_\pi = \{b/a \mid b \in D,\ a = \pi^n,\ n \in \mathbb{Z}_+\}$.

6. *[12,20,26] $\mathcal{B}$ is flat iff M is a free left D-module. Then, the bases of M are in a one-to-one correspondence with the flat outputs of $\mathcal{B}$.*

Proof 1. Let $m \in M$ and let us define $f \in \hom_D(D, M)$ by $f(1) = m$. Applying the functor $\hom_D(\cdot, \mathcal{F})$ to $D \xrightarrow{f} M$, we obtain the following morphism:

$$\begin{array}{ccc} \hom_D(D, \mathcal{F}) & \xleftarrow{f^\star} & \hom_D(M, \mathcal{F}) \\ \varphi \circ f & \longleftarrow\!\!\dashv & \varphi. \end{array}$$

If we denote by y_i the residue class of the i^{th} vector e_i of the standard basis of $D^{1\times p}$ in M, then there exist $a_i \in D$, $i = 1, \dots, p$, such that we have $m = \sum_{i=1}^p a_i\, y_i$. Then, the isomorphism $\hom_D(M, \mathcal{F}) \cong \ker_{\mathcal{F}}(R.)$ defined by (14) holds. Moreover, we have the trivial isomorphism $\hom_D(D, \mathcal{F}) \cong \mathcal{F}$ defined by evaluating an element of $\hom_D(D, \mathcal{F})$ at $1 \in D$. Hence, we obtain

$$(\varphi \circ f)(1) = \varphi(m) = \sum_{i=1}^p a_i\, \varphi(y_i) = \sum_{i=1}^p a_i\, \eta_i,$$

where $\eta_i = \varphi(y_i)$, $i = 1, \dots, p$ (see (14)). Therefore, we obtain an observable $\psi : \mathcal{B} = \ker_{\mathcal{F}}(R.) \to \mathcal{F}$ defined by $\psi(\eta) = \sum_{i=1}^p a_i\, \eta_i$. Let us check that the observable ψ does not depend on the choice of the a_i but only on $m \in M$. Let us consider $m = \sum_{i=1}^p b_i\, y_i$, $b_i \in D$, $i = 1 \dots p$. Then, we have

$$v = (a_1 - b_1, \dots, a_p - b_p) \in D^{1\times q}\, R \Rightarrow \exists\, L \in D^{1\times q} :\ v = L\, R,$$

and thus, we obtain $\sum\limits_{i=1}^p a_i\, \eta_i - \sum\limits_{i=1}^p b_i\, \eta_i = L\,(R\,\eta) = 0$.

Conversely, let us consider an observable $\psi : \mathcal{B} = \ker_{\mathcal{F}}(R.) \to \mathcal{F}$ defined by $\psi(\eta) = \sum_{i=1}^p a_i\, \eta_i$ for $a_i \in D$, $i = 1, \dots, p$. Then, we can associate with ψ the element $m = \sum_{i=1}^p a_i\, y_i \in M$. Let us prove that this map is well-defined. Let us suppose that we have $\sum_{i=1}^p a_i \eta_i = \sum_{i=1}^p b_i \eta_i$ for all $\eta \in \mathcal{B}$. Then, we have $\sum_{i=1}^p (a_i - b_i)\, \eta_i = 0$ on $\mathcal{B}$. Hence, if we define

$$\mathcal{B}^\natural = \{\lambda \in D^{1\times p} \mid \forall\, \eta \in \mathcal{B} :\ \lambda\, \eta = 0\},$$

then we obtain $(a_1 - b_1, \dots, a_p - b_p) \in \mathcal{B}^\natural$. We easily check that we have $D^{1\times q}\, R \subseteq \mathcal{B}^\natural$. Using the fact that $\mathcal{B}^\natural \subseteq D^{1\times p}$ and D is a left noetherian domain, then we obtain that $\mathcal{B}^\natural$ is a finitely generated left D-module, and thus, there exists $R' \in D^{q'\times p}$ such that $\mathcal{B}^\natural = D^{1\times q'}\, R'$. Let us define the left D-module $M' = D^{1\times p}/(D^{1\times q'}\, R')$ and $\mathcal{B}' = \ker_{\mathcal{F}}(R'.) \cong \hom_D(M', \mathcal{F})$. Then, by definition of $\mathcal{B}'$, we have the trivial inclusion $\mathcal{B} \subseteq \mathcal{B}'$. Now, applying the functor $\hom(\cdot, \mathcal{F})$ to the standard short exact sequence [4,36]

$$0 \longrightarrow \mathcal{B}^\natural/(D^{1\times q}\, R) \longrightarrow M \longrightarrow M' \longrightarrow 0,$$

and using the fact that $\mathcal{F}$ is an injective left D-module, we then obtain the following short exact sequence:

$$0 \longleftarrow \hom_D(\mathcal{B}^\natural/(D^{1\times q}\, R), \mathcal{F}) \longleftarrow \hom_D(M, \mathcal{F}) \longleftarrow \hom_D(M', \mathcal{F}) \longleftarrow 0.$$

In particular, we have $\mathcal{B}' \subseteq \mathcal{B}$, and thus, we obtain $\mathcal{B}' = \mathcal{B}$. Hence, we have $\hom_D(\mathcal{B}^\natural/(D^{1\times q}\, R), \mathcal{F}) = 0$ and, using the fact that $\mathcal{F}$ is a cogenerator left D-module, we obtain that $\mathcal{B}^\natural/(D^{1\times q}\, R) = 0$, i.e., $\mathcal{B}^\natural = D^{1\times q}\, R$. Hence, we have $(a_1 - b_1, \dots, a_p - b_p) \in \mathcal{B}^\natural = D^{1\times q}\, R$, and thus, the residue class of $(a_1 - b_1, \dots, a_p - b_p)$ in $M = D^{1\times p}/(D^{1\times q}\, R)$ vanishes, which proves that $m = \sum_{i=1}^p a_i\, y_i = \sum_{i=1}^p b_i\, y_i$.

2 follows from 1 and the definitions of a torsion element of M and an autonomous observable of $\mathcal{B}$. Then, 3 directly follows from 2.

4. $\mathcal{B}$ is parametrizable iff there exists $Q \in D^{p\times m}$ such that $\mathcal{B} = Q\,\mathcal{F}^m$, i.e., iff we have the following exact sequence $\mathcal{F}^q \xleftarrow{R.} \mathcal{F}^p \xleftarrow{Q.} \mathcal{F}^m$. Using the fact that $\mathcal{F}$ is an injective cogenerator left D-module, we know that the previous sequence is exact iff the sequence $D^{1\times q} \xrightarrow{.R} D^{1\times p} \xrightarrow{.Q} D^{1\times m}$ is exact, i.e., iff $M = D^{1\times p}/(D^{1\times q}\, R) \cong D^{1\times p}\, Q$.

5. If the parametrizable behaviour $\mathcal{B}$ is π-free, then, using 4, there exist $Q \in D^{p\times m}$, $T \in D^{m\times p}$ and $\alpha \in \mathbb{Z}_+$ such that $M \cong D^{1\times p}\, Q$ and $T\, Q = \pi^\alpha\, I_m$. Let us define the ring $D_\pi = \{b/a \mid b \in D,\ a = \pi^n,\ n \in \mathbb{Z}_+\}$. Using the fact that D_π is a *flat* D-module [4,36], we obtain $D_\pi \otimes_D M \cong D_\pi^{1\times p}\, Q$. Moreover, $(1/\pi^\alpha)\, T$ is a left-inverse of Q over the ring D_π which easily implies that $D_\pi \otimes_D M \cong D_\pi^{1\times p}\, Q = D_\pi^{1\times m}$.

Conversely, let us suppose that there exists $0 \neq \pi \in D$ such that we have $D_\pi \otimes_D M \cong D_\pi^{1\times m}$. Using the fact that $\mathcal{B}$ is parametrizable, by 4, we obtain that there exists $Q \in D^{p\times m}$ such that $M \cong D^{1\times p}\, Q$. Since D_π is a flat D-module [4,36], we obtain $D_\pi \otimes_D M \cong D_\pi^{1\times p}\, Q$, and thus, $D_\pi^{1\times p}\, Q \cong D_\pi^{1\times m}$. Therefore, there exists $S \in D_\pi^{m\times p}$ such that $S\, Q = I_m$. Finally, writing $S = (1/\pi^\alpha)\, T$, where $T \in D^{m\times p}$, we obtain $T\, Q = \pi^\alpha\, I_m$.

6. A behaviour $\mathcal{B}$ is flat iff there exist $Q \in D^{p\times m}$ and $T \in D^{m\times p}$ such that $\mathcal{B} = Q\,\mathcal{F}^m$ and $T\, Q = I_m$, i.e., iff we have the split exact sequence:

$$0 \longleftarrow R\,\mathcal{F}^p \xleftarrow{R.} \mathcal{F}^p \xleftarrow{Q.} \mathcal{F}^m \longleftarrow 0. \tag{16}$$

Now, the injective cogenerator property of $\mathcal{F}$ implies that (16) is a split exact sequence iff we have the split exact sequence

$$0 \longrightarrow D^{1\times q}\, R \longrightarrow D^{1\times p} \xrightarrow{.Q} D^{1\times m} \longrightarrow 0,$$

and thus, iff $M \cong D^{1\times p}\, Q = D^{1\times m}$, i.e., iff M is a free left D-module. □

We refer to [33] for a new characterization of controllability in the time-varying case: An analytic time-varying controllable linear system is a projection of a flat system.

We recall the following well-known concepts of primeness developed in the literature of multidimensional systems [40]. These concepts allow us to classify the multidimensional linear systems with constant coefficients.

Definition 15 [12,18,27,40] Let $D = \mathbb{R}[x_1, \dots, x_n]$ be a commutative polynomial ring, R a *full row rank matrix* (namely, its rows are D-linearly independent) in $D^{q \times p}$, J the ideal generated by the $q \times q$ minors of R and

$$V(J) = \{\xi \in \mathbb{C}^n \mid \forall\, P \in J : \ P(\xi) = 0\}$$

the algebraic variety defined by J. Then, we have the following definitions:

- R is called *minor left-prime* if $\dim_{\mathbb{C}} V(J) \leq n - 1$, i.e., the greatest common divisor of all the $q \times q$ minors of R is 1.
- R is called *weakly zero left-prime* if $\dim_{\mathbb{C}} V(J) \leq 0$, i.e., all the $q \times q$ minors of R may only vanish simultaneously in a finite number of points of $\mathbb{C}^n$.
- R is called *zero left-prime* if $\dim_{\mathbb{C}} V(J) = -1$, i.e., all the $q \times q$ minors of R do not vanish simultaneously in $\mathbb{C}^n$.

Theorem 4 *[12,18,27] Let $D = \mathbb{R}[x_1, \dots, x_n]$ be a commutative polynomial ring, $R \in D^{q \times p}$ a full row rank matrix and $M = D^{1 \times p}/(D^{1 \times q} R)$. Then,*

1. *R is minor left-prime iff the D-module M is torsion-free.*
2. *If $n = 3$, then R is weakly zero left-prime iff the D-module M is reflexive.*
3. *R is zero left-prime iff the D-module M is projective, i.e., free by the Quillen-Suslin theorem (see 3 of Theorem 2).*

Hence, some of the concepts defined in Definition 8 generalize to non-commutative polynomial rings and non-full row rank matrices the well-known concepts of primeness developed for multidimensional systems [27]. Moreover, Theorem 3 shows that they play fundamental roles in control theory. Therefore, the next sections are dedicated to giving effective algorithms which check the module properties and compute parametrizations and flat outputs.

6 Constructive computation of $\mathrm{ext}^i_D(M, D)$

In the previous section, we have shown how some structural properties of linear systems (behaviours) are translated into module properties via a dictionary developed in Theorem 3. In Section 7, we shall prove that these module properties can be checked by computing certain extension modules of the form $\mathrm{ext}^i_D(N, D)$, $i \geq 1$. Hence, we focus in this section on the constructive computation of such extension modules.

6.1 Involutions

Definition 16 Let k be a field and D a (non-commutative) k-algebra. An *involution* θ of D is a k-linear map $\theta : D \to D$ satisfying

$$\begin{aligned} \forall\, a_1, a_2 \in D, \quad \theta(a_1 \cdot a_2) &= \theta(a_2) \cdot \theta(a_1), \\ \theta \circ \theta &= \mathrm{id}_D, \end{aligned} \tag{17}$$

i.e., θ is an *anti-automorphism* of order two of the k-algebra D.

Proposition 11 *By means of an involution θ of D, left D-modules can be turned into right D-modules and vice versa: Let D be a k-algebra, M a right D-module and θ an involution of D, then we can define the left D-module $\widetilde{M}$, which is equal to M as a set and which is endowed with the same addition as M, but with the following left action of D:*

$$a\,m = m\,\theta(a), \quad m \in \widetilde{M}, \quad a \in D.$$

Property (17) of θ ensures that $\widetilde{M}$ is a well-defined left D-module.

Example 10 1. Let D be a commutative ring (e.g., $D = k[x_1, \ldots, x_n]$). Then, $\theta = \mathrm{id}_D$ is a trivial involution of D.
2. Let $A_n(k) = k[x_1, \ldots, x_n][\partial_1; \sigma_1, \delta_1] \ldots [\partial_n; \sigma_n, \delta_n]$ be the Weyl algebra (see Example 1). An involution θ of $A_n(k)$ can be defined by:

$$x_i \longmapsto x_i, \quad \partial_i \longmapsto -\partial_i, \quad 1 \leq i \leq n.$$

3. Let $S_h = k[t][\delta_h; \sigma_h,\ \delta]$ be the shift Ore algebra (see Example 2). An involution θ of S_h can be defined by $t \longmapsto -t, \quad \delta_h \longmapsto \delta_h$.
4. Let $H_h = k[t][\partial; \sigma_1, \delta_1][\delta_h; \sigma_2,\ \delta_2][\tau_h; \sigma_3,\ \delta_3]$ be the k-algebra of differential time-delay and advance operators (see Example 3 for more details). An involution θ of H_h can be defined by

$$t \longmapsto t, \quad \partial \longmapsto -\partial, \quad \delta_h \longmapsto \tau_h, \quad \tau_h \longmapsto \delta_h.$$

5. We can prove that there is no non-trivial involution for the Ore algebra D_h of differential time-delay operators (see Example 3). Therefore, we need to embed D_h into the k-algebra H_h of differential time-delay and advance operators and use the involution defined in 4.

Definition 17 Let D be an Ore algebra with an involution θ, $R \in D^{q \times p}$ and $M = D^{1 \times p}/(D^{1 \times q}\, R)$ a left D-module.

- The *dual* M^* of M is the right D-module defined by $\hom_D(M, D)$.
- The *transposed module* of M is the right D-module defined by:

$$N = D^q/(R\, D^p).$$

  If D is a commutative ring, then we have $N = D^{1\times q}/(D^{1\times p}\, R^T)$, which explains the terminology.
- The *adjoint module* of M is the left D-module defined by

$$\widetilde{N} = D^{1\times q}/(D^{1\times p}\, \theta(R)),$$

  where $\theta(R)$ is the transpose of the matrix obtained by applying θ to each of its entries, i.e.:

$$\theta(R) = (\theta(R_{ij}))^T_{1\leq i\leq q, 1\leq j\leq p}.$$

  Hence, the left D-module $\widetilde{N} = D^{1\times q}/(D^{1\times p}\, \theta(R))$ corresponds to the linear system $\theta(R)\, z = 0$, where $z = (z_1 : \ldots : z_q)^T$ is the set of generators of $\widetilde{N}$ obtained by taking the residue classes of the standard basis vectors of $D^{1\times q}$ in $\widetilde{N}$.

Example 11 1. If D is a commutative ring, then $\theta(R)$ is just the transposed matrix, namely $\theta(R) = R^T$, and the transposed D-module is defined by $\widetilde{N} = D^{1\times q}/(D^{1\times p}\, R^T) = D^q/(R\, D^p) = N$.
2. Let us consider the k-algebra $H_h = k[t][\partial;\sigma_1,\delta_1][\delta_h;\sigma_2,\delta_2][\tau_h;\sigma_3,\delta_3]$ defined in Example 3 and $R = (t\,\partial \,:\, -t^2\,\delta_h) \in H_h^{1\times 2}$. Then, using 4 of Example 10, we obtain:

$$\theta(R) = \begin{pmatrix} -\partial\, t \\ -\tau_h\, t^2 \end{pmatrix} = \begin{pmatrix} -t\,\partial - 1 \\ -(t+h)^2\,\tau_h \end{pmatrix} \in H_h^{2\times 1}.$$

Finally, it is proved in [28] that N is uniquely defined by M up to a projective equivalence [4,36]. In particular, this result shows that $\mathrm{ext}^i_D(N,\mathcal{F})$, $i \geq 1$, only depends on M and on the right D-module $\mathcal{F}$ and not on a particular presentation of M.

6.2 Computation of extension modules

We only consider here extension modules of the form $\mathrm{ext}^i_D(M,D)$, $i \in \mathbb{Z}_+$. It is important to notice that $\mathrm{ext}^i_D(M,D)$, $i \in \mathbb{Z}_+$, inherit a right module structure from the right action of D.

The next algorithm gives a description of the left D-module $\widetilde{\mathrm{ext}^1_D(M,D)}$, which corresponds to the right D-module $\mathrm{ext}^1_D(M,D)$ (see Proposition 11).

Algorithm 2
Input: Ore algebra D satisfying (2) with an involution θ and $R = (R_1^T : \ldots : R_q^T)^T \in D^{q\times p}$.
Output: A list $L = (L_1,\ L_2)$ of two matrices such that:
$L_1 \in D^{m\times q}$ is such that $\widetilde{\mathrm{ext}^1_D(M,D)} = (D^{1\times m}\, L_1)/(D^{1\times p}\,\theta(R))$, where $M = D^{1\times p}/(D^{1\times q}\, R)$,
$L_2 \in D^{q\times r}$ is such that $L_1 = \textsc{Syzygies}\,(L_2)$.

$\textsc{AdjExt1}(R)$
$R_2 \leftarrow \textsc{Syzygies}\,(R)$,
$L_2 \leftarrow \theta(R_2)$,
$L_1 \leftarrow \textsc{Syzygies}\,(L_2)$,
$L \leftarrow (L_1,\ L_2)$.

Proof The module M is given as the cokernel $D^{1\times p}/(D^{1\times q}\, R)$ of the D-morphism $D^{1\times q} \xrightarrow{.R} D^{1\times p}$. Computing the second syzygy left D-module of M, we obtain a matrix $R_2 \in D^{r\times q}$ such that the sequence

$$D^{1\times r} \xrightarrow{.R_2} D^{1\times q} \xrightarrow{.R} D^{1\times p}$$

is exact. Upon dualization, we get the complex

$$D^r \xleftarrow{R_2.} D^q \xleftarrow{R.} D^p$$

with defect of exactness $\mathrm{ext}^1_D(M,D) = \ker(R_2.)/(R\, D^p)$ at D^q. Taking adjoints and writing $L_2 = \theta(R_2) \in D^{q\times r}$, we obtain the complex

$$D^{1\times r} \xleftarrow{.L_2} D^{1\times q} \xleftarrow{.\theta(R)} D^{1\times p}$$

with defect of exactness $\widetilde{\mathrm{ext}^1_D(M,\,D)}$. Computing the syzygy left D-module of $D^{1\times q}\,L_2$ yields the exact sequence

$$D^{1\times r} \xleftarrow{.L_2} D^{1\times q} \xleftarrow{.L_1} D^{1\times m}$$

such that $\widetilde{\mathrm{ext}^1_D(M,\,D)} = \ker(.L_2)/(D^{1\times p}\,\theta(R)) = (D^{1\times m}\,L_1)/(D^{1\times p}\,\theta(R))$. $\square$

Example 12 We compute the extension D_h-modules $\mathrm{ext}^i_{D_h}(N,\,D_h)$, $i = 1, 2$, of the D_h-module $N = D_h^{1\times 3}/(D_h^{1\times 4}\,R^T)$ defined in Example 7. In Example 7, we have already computed the free resolution (12) of N. Thus, we have $\ker(.R^T) = D_h\,Q^T$, where Q^T is defined in Example 7. Then, using the fact that $D_h = \mathbb{R}(a,\,k,\,\zeta,\,\omega)[\partial;\,\sigma_1,\,\delta_1][\delta_h;\,\sigma_2,\,\delta_2]$ is a commutative polynomial ring, we obtain that $\theta(Q^T) = Q$ (see 1 of Example 11). Hence, we have the following complex $0 \longleftarrow D_h \xleftarrow{.Q} D_h^{1\times 4} \xleftarrow{.R} D_h^{1\times 3} \longleftarrow 0$ and its defects of exactness are defined by:

$$\mathrm{ext}^1_{D_h}(N,\,D_h) = \ker(.Q)/(D_h^{1\times 3}\,R), \tag{18}$$

$$\mathrm{ext}^2_{D_h}(N,\,D_h) = D_h/(D_h^{1\times 4}\,Q). \tag{19}$$

Following Algorithm 2, we need to compute the syzygy module of $D_h^{1\times 4}\,Q$. The Gröbner basis of

$$P = \{(\omega^2\,k\,a\,\delta_h)\,\lambda - \mu_1, \quad (\omega^2\,\partial + \omega^2\,a)\,\lambda - \mu_2, \quad (\omega^2\,\partial^2 + \omega^2\,a\,\partial)\,\lambda - \mu_3,$$
$$(\partial^3 + 2\,\zeta\,\omega\,\partial^2 + a\,\partial^2 + \omega^2\,\partial + 2\,a\,\zeta\,\omega\,\partial + a\,\omega^2)\,\lambda - \mu_4\}$$

with respect to the elimination order induced by the degree reverse lexicographical orders on λ and $\mu_1 > \mu_2 > \mu_3 > \mu_4 > \delta_h > \partial$ respectively is:

$$G = \{(\partial + a)\,\mu_1 - k\,a\,\delta\,\mu_2, \quad \omega^2\,\mu_2 + (\partial + 2\,\zeta\,\omega)\,\mu_3 - \omega^2\,\mu_4, \quad \partial\,\mu_2 - \mu_3,$$
$$\omega^2\,(\partial + a)\,\lambda - \mu_2, \quad \omega^2\,k\,a\,\delta\,\lambda - \mu_1\}.$$

See Appendix 10.3. Therefore, we obtain that the syzygy module of $D_h^{1\times 4}\,Q$ is defined by the matrix

$$L = \begin{pmatrix} \partial + a & -k\,a\,\delta_h & 0 & 0 \\ 0 & \omega^2 & \partial + 2\,\zeta\,\omega & -\omega^2 \\ 0 & \partial & -1 & 0 \end{pmatrix} \in D_h^{3\times 4}, \tag{20}$$

and thus, we have $\mathrm{ext}^1_{D_h}(N,\,D_h) = (D_h^{1\times 3}\,L)/(D_h^{1\times 3}\,R)$. Finally, using (19), the D_h-module $\mathrm{ext}^2_{D_h}(N,\,D_h)$ corresponds to the linear system defined by $Q\,z = 0$, namely:

$$\begin{cases} (\omega^2\,k\,a\,\delta_h)\,z = 0, \\ (\omega^2\,\partial + \omega^2\,a)\,z = 0, \\ (\omega^2\,\partial^2 + \omega^2\,a\,\partial)\,z = 0, \\ (\partial^3 + (2\,\zeta\,\omega + a)\,\partial^2 + (\omega^2 + 2\,a\,\zeta\,\omega)\,\partial + a\,\omega^2)\,z = 0. \end{cases} \tag{21}$$

We note that $\mathrm{ext}^2_{D_h}(N,\,D_h) \neq 0$ because, otherwise, in G, we would have had $\lambda - \sum_{i=1}^4 P_i\,\mu_i$, for some $P_i \in D_h$.

The quotient module $\widetilde{\mathrm{ext}^1_D(M, D)} = (D^{1\times m}\, L_1)/(D^{1\times p}\, \theta(R))$ can be computed using elimination techniques similar to Algorithm 1.

Algorithm 3
Input: $T \in D^{a\times b}$, $S = (S_1^T : \ldots : S_c^T)^T \in D^{c\times b}$ s.t. $D^{1\times a}\, T \subseteq D^{1\times c}\, S$
Output: A set C of generating equations satisfied by the residue class z_i of $S_i = (S_{i1} : \ldots : S_{ib})$ in the left module $(D^{1\times c}\, S)/(D^{1\times a}\, T)$.
QUOTIENT(S, T)
Introduce the indeterminates $\lambda_1, \ldots, \lambda_b$ and $\mu_1, \ldots, \mu_c$ over D.
for $i = 1, \ldots, c$ **do**

$$L \leftarrow \left\{ \sum_{j=1}^{b} S_{ij}\, \lambda_j - \mu_i \right\} \cup \left\{ \sum_{j=1}^{b} T_{kj}\, \lambda_j \ \mid\ k = 1, \ldots, a \right\}$$

Compute the Gröbner basis G_i of L in $\bigoplus_{j=1}^{b} D\, \lambda_j \oplus D\, \mu_i$ with respect to an order which eliminates the λ_j's.
endfor
$C \leftarrow \bigcup_{i=1}^{c} (G_i \cap D\, \mu_i)$.

Remark 4 In the result of the preceding algorithm, each $G_i \cap D\, \mu_i$ is a generating set of the relations fulfilled by z_i. Let us notice that every polynomial in $G_i \cap D\, \mu_i$ has the form $P\, \mu_i$, for a certain $P \in D$.

Example 13 In Example 12, we proved that

$$\mathrm{ext}^1_{D_h}(N, D_h) = (D_h^{1\times 3}\, L)/(D_h^{1\times 3}\, R),$$

where R (resp., L) is defined by (5) (resp., (20)), $N = D_h^{1\times 3}/(D_h^{1\times 4}\, R^T)$ and $D_h = \mathbb{R}(a, k, \zeta, \omega)[\partial; \sigma_1, \delta_1][\delta_h; \sigma_2, \delta_2]$. In order to check whether or not $\mathrm{ext}^1_{D_h}(N, D_h)$ is equal to 0, we apply QUOTIENT(L, R) described in Algorithm 3. If we denote $R = (R_1^T : R_2^T : R_3^T)^T$, $L = (l_1^T : l_2^T : l_3^T)^T$, then we easily check that we have $G_i \cap D_h\, \mu_i = \{\mu_i\}$, for $i = 1, \ldots, 3$, because we have $l_1 = R_1$, $l_2 = R_3$, $l_3 = R_2$. Hence, we have $D_h^{1\times 3}\, L = D_h^{1\times 3}\, R$, which shows that $\mathrm{ext}^1_{D_h}(N, D_h) = (D_h^{1\times 3}\, L)/(D_h^{1\times 3}\, R) = 0$.

The next algorithm gives a description of $\widetilde{\mathrm{ext}^1_D(M, D)}$ in which the quotient is explicitly computed.

Algorithm 4
Input: An Ore algebra D satisfying (2) with an involution θ and $R = (R_1^T : \ldots : R_q^T)^T \in D^{q\times p}$, $M = D^{1\times p}/(D^{1\times q}\, R)$.
Output: A tuple $L = (L_0, L_1, L_2)$ such that
$\widetilde{\mathrm{ext}^1_D(M, D)} = (D^{1\times m}\, L_1)/(D^{1\times p}\, \theta(R))$,
$L_2 \in D^{q\times r}$, $L_1 = (l_1^T : \ldots : l_m^T)^T = $ SYZYGIES$(L_2) \in D^{m\times q}$ and
L_0 is a generating set for the relations satisfied by the residue class z_i of l_i in the left D-module $(D^{1\times m}\, L_1)/(D^{1\times p}\, \theta(R))$.
EXT1(R)
$(L_1, L_2) \leftarrow$ ADJEXT(R),
$L_0 \leftarrow$ QUOTIENT$(L_1, \theta(R))$,
$L \leftarrow (L_0, L_1, L_2)$.

Remark 5 In our implementation of EXT1 in OREMODULES [8], the first matrix L_0 is actually a matrix in block diagonal form, where each block consists of only one column and possibly several rows. The entries of the i^{th} block form a Gröbner basis of the annihilator of the i^{th} row of L_1 in the left D-module $\widetilde{\text{ext}_D^1(M, D)}$. Hence, L_0 is an identity matrix iff $\text{ext}_D^1(M, D) = 0$.

Finally, let us notice that while using EXT1, we start to compute a free resolution of the left D-module M (see Algorithm 2). If we denote this free resolution by

$$\ldots \xrightarrow{.R_3} D^{1\times p_2} \xrightarrow{.R_2} D^{1\times p_1} \xrightarrow{.R_1} D^{1\times p_0} \xrightarrow{\kappa} M \longrightarrow 0,$$

$p_0 = p$, $p_1 = q$ and $R_1 = R$, then $\text{EXT1}(R_i)$ computes $\widetilde{\text{ext}_D^i(M, D)}$, $i \geq 1$.

7 Characterization of module properties

The main motivation for introducing the concept of extension functor is explained in the next theorems which give an effective way to check the module properties defined in Section 3, and thus, the structural properties of the corresponding linear system (see Section 5).

Definition 18 Let A and B be two rings and M an abelian group. Then, M is said to be an *A-B-bimodule*, denoted ${}_A M_B$, if M is a left A-module and a right B-module and the two actions satisfy the following associative law:

$$\forall\, a \in A, \quad \forall\, b \in B, \quad \forall\, m \in M, \quad a\,(m\,b) = (a\,m)\,b.$$

In this section, D is supposed to be a left and a right noetherian domain. Then, by Proposition 1, we know that D satisfies the left and right Ore properties. We recall a classical result in homological algebra saying that a short exact sequence of bimodules gives rise to a long exact sequence of the cohomology right modules.

Lemma 1 *[4,36] Let* $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ *be an exact sequence of D-D-bimodules and N a left D-module. Then, we have the following exact sequence of right D-modules:*

$$\begin{array}{l} 0 \longrightarrow \hom_D(N, A) \longrightarrow \hom_D(N, B) \longrightarrow \hom_D(N, C) \\ \longrightarrow\ \text{ext}_D^1(N, A) \ \longrightarrow\ \text{ext}_D^1(N, B) \ \longrightarrow\ \text{ext}_D^1(N, C) \\ \longrightarrow\ \text{ext}_D^2(N, A) \ \longrightarrow \qquad \ldots \end{array}$$

Let us state the first main result of this section.

Theorem 5 *[29] Let M be a left D-module defined by the finite presentation*

$$F_1 \xrightarrow{d_1} F_0 \xrightarrow{\kappa} M \longrightarrow 0, \tag{22}$$

and N the transposed right D-module of M (see Definition 17) defined by:

$$0 \longleftarrow N \longleftarrow F_1^* \xleftarrow{d_1^*} F_0^* \longleftarrow M^* \longleftarrow 0. \tag{23}$$

Then, we have $t(M) \cong \text{ext}_D^1(N, D)$. In particular, M is a torsion-free left D-module iff $\text{ext}_D^1(N, D) = 0$.

Proof Let X be a D-D-bimodule. Then, applying the right exact functor $X \otimes_D \cdot$ [4,36] to the exact sequence (22) gives the exact sequence

$$X \otimes_D F_1 \xrightarrow{\mathrm{id}_X \otimes d_1} X \otimes_D F_0 \xrightarrow{\mathrm{id}_X \otimes \kappa} X \otimes_D M \longrightarrow 0,$$

whereas applying the left exact functor $\hom(\cdot, X)$ to the exact sequence (23) yields the exact sequence:

$$0 \longrightarrow \hom_D(N, X) \longrightarrow \hom_D(F_1^*, X) \longrightarrow \hom_D(F_0^*, X).$$

Using that the F_i are finitely generated free left D-modules, we have [4,36]

$$\hom_D(F_i^*, X) \cong X \otimes_D F_i, \quad i = 1, 2,$$

and thus, we finally obtain the following exact sequence:

$$0 \longrightarrow \hom_D(N, X) \longrightarrow X \otimes_D F_1 \longrightarrow X \otimes_D F_0 \longrightarrow X \otimes_D M \longrightarrow 0. \quad (24)$$

Now, let $K = Q(D)$ be the *left* and *right quotient field* of D [17] with its bimodule structure and consider the exact sequence of D-D-bimodules:

$$0 \longrightarrow D \longrightarrow K \longrightarrow K/D \longrightarrow 0. \quad (25)$$

Using Lemma 1 with $A = D$, $B = K$, $C = K/D$, we get the exact sequence of right D-modules

$$\begin{aligned} 0 \longrightarrow \hom_D(N, D) \longrightarrow \hom_D(N, K) \longrightarrow \hom_D(N, K/D) \\ \longrightarrow \mathrm{ext}_D^1(N, D) \longrightarrow 0, \end{aligned} \quad (26)$$

because $\mathrm{ext}_D^1(N, K) = 0$ since K is a left *injective* D-module [4,36].

Moreover, by applying the right exact functor $\cdot \otimes_D M$ to the short exact sequence (25), we obtain the exact sequence of left D-modules

$$M \xrightarrow{i_K} K \otimes_D M \longrightarrow (K/D) \otimes_D M \longrightarrow 0,$$

which extends to the following exact sequence

$$0 \longrightarrow t(M) \longrightarrow M \xrightarrow{i_K} K \otimes_D M \longrightarrow (K/D) \otimes_D M \longrightarrow 0, \quad (27)$$

where the kernel of the D-morphism i_K, defined by

$$\ker i_K = \{m \in M \mid i_K(m) = 1 \otimes m = 0\},$$

is the left submodule of M formed by all elements $m \in M$ for which there exists $0 \neq d \in D$ such that $d\, m = 0$, i.e., the torsion submodule $t(M)$ of M.

Using the fact that the F_i are finitely generated free left D-modules, and thus, *flat* left D-modules [4,36], by applying $\cdot \otimes_D F_i$ to the exact sequence (25), we obtain the exact sequence:

$$0 \longrightarrow F_i \longrightarrow K \otimes_D F_i \longrightarrow (K/D) \otimes_D F_i \longrightarrow 0. \quad (28)$$

Finally, combining the exact sequences (24) for $X = D$, K and K/D, (26), (27) and (28) yields the commutative exact diagram:

$$\begin{array}{ccccccccc}
 & & & & & & & & 0 \\
 & & & & & & & & \downarrow \\
 & & 0 & & 0 & & 0 & & t(M) \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \hom_D(N, D) & \longrightarrow & F_1 & \longrightarrow & F_0 & \longrightarrow & M & \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \hom_D(N, K) & \longrightarrow & K \otimes_D F_1 & \longrightarrow & K \otimes_D F_0 & \longrightarrow & K \otimes_D M & \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \hom_D(N, K/D) & \longrightarrow & (K/D) \otimes_D F_1 & \longrightarrow & (K/D) \otimes_D F_0 & \longrightarrow & (K/D) \otimes_D M & \longrightarrow 0. \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 & & \mathrm{ext}^1_D(N, D) & & 0 & & 0 & & 0 \\
 & & \downarrow & & & & & & \\
 & & 0 & & & & & &
\end{array}$$

Then, an easy chase in the previous commutative exact diagram proves that:

$$t(M) \cong \mathrm{ext}^1_D(N, D).$$

□

Lemma 2 *[4] If* $M_1 \xrightarrow{\alpha_1} M_2 \xrightarrow{\alpha_2} M_3 \xrightarrow{\alpha_3} M_4$ *is a complex of (left/right) D-modules, then we have an exact sequence*

$$0 \longrightarrow H(M_2) \longrightarrow \mathrm{coker}\,\alpha_1 \longrightarrow \ker \alpha_3 \longrightarrow H(M_3) \longrightarrow 0,$$

where $H(M_i)$ denotes the defect of exactness of the complex at M_i, namely, $H(M_i) = \ker \alpha_i / \mathrm{im}\, \alpha_{i-1}$, $i = 2, 3$, *and* $\mathrm{coker}\,\alpha_1 = M_2/\mathrm{im}\,\alpha_1$.

We are now in position to state the second main result.

Theorem 6 *Let M be a finitely presented left D-module, i.e., defined by the finite presentation (22), and N the transposed module of M defined by the finite presentation (23). Then, we have the following exact sequence:*

$$0 \longrightarrow \mathrm{ext}^1_D(N, D) \longrightarrow M \xrightarrow{\varepsilon_M} M^{**} \longrightarrow \mathrm{ext}^2_D(N, D) \longrightarrow 0. \tag{29}$$

Hence, M is a reflexive left D-module iff $\mathrm{ext}^i_D(N, D) = 0$ *for* $i = 1, 2$.

Proof Let (22) be a finite presentation of M. Then, its transposed module is defined by $N = \mathrm{coker}\, d_1^*$. We extend this finite presentation to an exact sequence by computing the second and the third syzygy module of N. We obtain the following exact sequence:

$$0 \longleftarrow N \longleftarrow F_1^* \xleftarrow{d_1^*} F_0^* \xleftarrow{d_0^*} F_{-1}^* \xleftarrow{d_{-1}^*} F_{-2}^*. \tag{30}$$

Dualizing the exact sequence (30) and using the fact that F_i is canonically isomorphic to F_i^{**} because the F_i are finitely generated free left D-modules, we obtain the following complex:

$$F_1 \xrightarrow{d_1} F_0 \xrightarrow{d_0} F_{-1} \xrightarrow{d_{-1}} F_{-2}. \tag{31}$$

Now, by applying Lemma 2 to the complex (31) and using the definitions of $\mathrm{ext}_D^1(N, D)$ and $\mathrm{ext}_D^2(N, D)$, we get the exact sequence:

$$0 \longrightarrow \mathrm{ext}_D^1(N, D) \longrightarrow \mathrm{coker}\, d_1 \longrightarrow \ker d_{-1} \longrightarrow \mathrm{ext}_D^2(N, D) \longrightarrow 0.$$

According to the presentation of M, we have $M = \mathrm{coker}\, d_1$. Now, using the exact sequences (23) and (30), we obtain the following exact sequence:

$$0 \longleftarrow M^* = \ker d_1^* \longleftarrow F_{-1}^* \xleftarrow{d_{-1}^*} F_{-2}^*. \tag{32}$$

Applying the left exact functor $\hom_D(\cdot, D)$ to (32) [4,36] yields the following exact sequence

$$0 \longrightarrow M^{**} \longrightarrow F_{-1} \xrightarrow{d_{-1}} F_{-2},$$

proving that $M^{**} = \ker d_{-1}$. Finally, we have the exact sequence (29), which proves that $M \cong M^{**}$ iff $\mathrm{ext}_D^i(N, D) = 0$ for $i = 1, 2$. □

Corollary 1 *Let M be a finitely generated left D-module. Then M is a torsion left D-module iff* $\hom_D(M, D) = 0$.

Proof $\Rightarrow$ If M is a torsion left D-module, then, for every $m \in M$, there exists $0 \neq P \in D$ such that $P\, m = 0$. Therefore, for every $f \in \hom_D(M, D)$, we have $P\, f(m) = f(P\, m) = 0$. Hence, using the fact that $f(m) \in D$, we obtain $f(m) = 0$, i.e., $f = 0$, and thus, $\hom_D(M, D) = 0$.

$\Leftarrow$ If $M^* = \hom_D(M, D) = 0$, then $M^{**} = \hom_D(M^*, D) = 0$. Then, using the exact sequence (29), we obtain that $\mathrm{ext}_D^1(N, D) = \ker \varepsilon_M = M$. Finally, using Theorem 5, we conclude that $t(M) = \mathrm{ext}_D^1(N, D) = M$, i.e., M is a torsion left D-module. □

Finally, we give the last important result of this section.

Theorem 7 *Let us suppose that* $\mathrm{rgld}\, D = n < \infty$ *and let M be a left D-module defined by the finite presentation (22) and N its transposed module defined by the finite presentation (23). Then, M is a projective left D-module iff* $\mathrm{ext}_D^i(N, D) = 0$, $i = 1, \ldots, \mathrm{rgld}\, D$.

Proof $\Rightarrow$ Let us suppose that M is a projective left D-module. In particular, M is reflexive, and thus, by Theorem 6, we have $\mathrm{ext}_D^i(N, D) = 0$, $i = 1, 2$.

From the exact sequence (23), we deduce [4,36]:

$$\mathrm{ext}_D^{i+2}(N, D) \cong \mathrm{ext}_D^i(M^*, D), \quad i \geq 1.$$

Now, using the fact that M is a projective left D-module, we obtain that M^* is a projective right D-module [4,36]. Therefore, we have $\mathrm{ext}_D^i(M^*, D) = 0$, $i \geq 1$ [4,36] which implies that $\mathrm{ext}_D^i(N, D) = 0$, $i \geq 3$, and finally proves the first implication.

$\Leftarrow$ Let us suppose that $\mathrm{ext}_D^i(N, D) = 0$, $i = 1, \ldots, \mathrm{rgld}\, D$. We consider a projective resolution of length $\mathrm{rgld}\, D$ of the right D-module N:

$$0 \longleftarrow N \longleftarrow P_0 \xleftarrow{r_1} P_1 \xleftarrow{r_2} P_2 \xleftarrow{r_3} \ldots \xleftarrow{r_{\mathrm{rgld}\, D}} P_{\mathrm{rgld}\, D} \longleftarrow 0. \tag{33}$$

Dualizing (33) and using the fact that $\mathrm{ext}^i_D(N, D) = 0$, $i = 1, \ldots, \mathrm{rgld}\, D$, we obtain that the following complex

$$0 \longrightarrow N^* \longrightarrow P_0^* \xrightarrow{r_1^*} P_1^* \xrightarrow{r_2^*} P_2^* \xrightarrow{r_3^*} \ldots \xrightarrow{r^*_{\mathrm{rgld}\, D}} P^*_{\mathrm{rgld}\, D} \longrightarrow 0 \tag{34}$$

is an exact sequence. The dual of a projective D-module is also a projective D-module [4,36], and thus, P_i^* is a projective left D-module for $i = 0, \ldots, r_{\mathrm{rgld}\, D}$. Moreover, the exact sequence (34) ends with a projective left D-module $P^*_{\mathrm{rgld}\, D}$, and thus, (34) is a split exact sequence (see Proposition 6). Therefore, by induction, we deduce that $\mathrm{im}\, r_i^*$ is a projective left D-module for $i = 1, \ldots, \mathrm{rgld}\, D$, and thus, we obtain the split exact sequence

$$0 \longrightarrow N^* \longrightarrow P_0^* \longrightarrow \mathrm{im}\, r_1^* \longrightarrow 0,$$

which shows that $P_0^* \cong N^* \oplus \mathrm{im}\, r_1^*$, i.e., N^* is a projective left D-module. Hence, the right D-module N^{**} is also projective. Dualizing again the split exact sequence (34) and using the fact that the P_i are projective D-modules, and thus, reflexive, namely $P_i^{**} \cong P_i$, we obtain the split exact sequence

$$0 \longleftarrow N^{**} \longleftarrow P_0 \xleftarrow{r_1} P_1 \xleftarrow{r_2} P_2 \xleftarrow{r_3} \ldots \xleftarrow{r_{\mathrm{rgld}\, D}} P_{\mathrm{rgld}\, D} \longleftarrow 0,$$

from which we deduce that $N \cong N^{**}$, and thus, N is a projective right D-module. The exact sequence (23) ends with the projective right D-module N, and thus, (23) is a split exact sequence. Thus, we deduce that M^* is a projective right D-module, which implies that M^{**} is a projective left D-module. Finally, using the fact that $\mathrm{ext}^i_D(N, D) = 0$, $i = 1, 2$, by Theorem 6, we obtain that $M \cong M^{**}$, and thus, M is a projective left D-module. □

If D satisfies the properties stated in 2 of Theorem 2, then Theorem 7 gives a constructive way to check stably freeness.

Corollary 2 *Let $M = D^{1\times p}/(D^{1\times q}\, R)$ be a finitely presented left D-module, $\mathcal{F}$ an injective cogenerator left D-module and $\mathcal{B} = \ker_{\mathcal{F}}(R.)$. Then, $\mathcal{B}$ is parametrizable iff $t(M) = 0$, i.e., iff $\mathcal{B}$ is controllable. In this case, the matrix L_2 in* Ext1(R) *is a parametrization of $\mathcal{B}$.*

Proof ⇒ Let us suppose that $\mathcal{B}$ is parametrizable. By 4 of Theorem 3, there exists $Q \in D^{p\times m}$ such that $M = D^{1\times p}/(D^{1\times q}\, R) \cong D^{1\times p}\, Q \subseteq D^{1\times m}$. Therefore, the left D-module M is isomorphic to a left submodule $\phi(M)$ of $D^{1\times m}$, where ϕ is the isomorphism from M to $D^{1\times p}\, Q$. Let us suppose that there exists a non-zero torsion element $z \in M$ satisfying $d\, z = 0$, where $0 \neq d \in D$. Then, we have $\phi(d\, z) = 0$, and thus, using the fact that ϕ is a D-morphism, then we obtain $d\, \phi(z) = 0$, i.e., $\phi(z)$ is a torsion element of $D^{1\times m}$. But, $D^{1\times m}$ is a free left D-module, and thus, $t(D^{1\times m}) = 0$, which proves that $t(M) = 0$. By 3 of Theorem 3, we conclude that $\mathcal{B}$ is controllable.

⇐ If $\mathcal{B}$ is controllable, then, by 3 of Theorem 3, we obtain that $t(M) = 0$. Let us consider the following finite free resolution of $N = D^q/(R\, D^p)$:

$$0 \longleftarrow N \longleftarrow D^q \xleftarrow{R.} D^p \xleftarrow{Q.} D^m.$$

Using Theorem 5 and $t(M) = 0$, we obtain $t(M) \cong \mathrm{ext}^1_D(N, D) = 0$, and thus, we have the following exact sequence $D^{1\times q} \xrightarrow{.R} D^{1\times p} \xrightarrow{.Q} D^{1\times m}$. Therefore, we have $M = D^{1\times p}/(D^{1\times q}\, R) \cong D^{1\times p}\, Q$, i.e., by 4 of Theorem 3, $\mathcal{B}$ is parametrizable. □

Example 14 Let us check whether or not the differential time-delay system defined by (4) is controllable and parametrizable. By 3 of Theorem 3, we know that (4) is controllable iff the D_h-module $M = D_h^{1\times 4}/(D_h^{1\times 3}\, R)$ is torsion-free, where R is defined by (5). By Theorem 5, this is equivalent to $\mathrm{ext}^1_{D_h}(N, D_h) = 0$, where $N = D_h^{1\times 3}/(D_h^{1\times 4}\, R^T)$ (see 1 of Example 11). Using Example 13, we find that (4) is controllable and, by Corollary 2, we deduce that a parametrization of (4) is given by Q defined in Example 7:

$$\begin{cases} x_1(t) = (\omega^2\, k\, a\, \delta_h)\, z(t), \\ x_2(t) = (\omega^2\, \partial + \omega^2\, a)\, z(t), \\ x_3(t) = (\omega^2\, \partial^2 + \omega^2\, a\, \partial)\, z(t), \\ u(t) = (\partial^3 + (2\,\zeta\,\omega + a)\,\partial^2 + (\omega^2 + 2\,a\,\zeta\,\omega)\,\partial + a\,\omega^2)\, z(t). \end{cases} \tag{35}$$

Finally, the fact that $\mathrm{ext}^2_{D_h}(N, D_h) \neq 0$ implies that $M = D_h^{1\times 4}/(D_h^{1\times 3}\, R)$ is not a projective, and thus, is not a free D_h-module (see 3 of Theorem 2). The obstruction for M to be free is given by (21): the fact that system (21) is not equivalent to $z = 0$ means that it is not possible to express z in terms of a D_h-linear combination of x_1, x_2, x_3 and u. Therefore, by 6 of Theorem 3, (4) is not a flat differential time-delay system.

A more general concept of parametrizability can be defined if we allow the integration of autonomous observables. For more details, we refer to [32].

8 π-freeness, minimal parametrizations and flatness

8.1 Left-inverses & π-freeness

Projectiveness is a necessary condition for flatness. Therefore, by Theorem 7, we need to compute $\mathrm{ext}^i_D(N, D)$ for $i = 1, \ldots, \mathrm{rgld}\, D$. However, if the system is defined by a full row rank matrix R, then it is easy to see that the left D-module $M = D^{1\times p}/(D^{1\times q}\, R)$ is projective iff M is stably free. Then, the next proposition gives a more economic way to check stably freeness.

Proposition 12 *If R has full row rank, namely, $S(D^{1\times q}\, R) = 0$, then the following assertions are equivalent:*

1. *M is a stably free left D-module,*
2. *$\exists\, S \in D^{p\times q} :\ R\, S = I_q$,*
3. *$N = \mathrm{ext}^1_D(M, D) = 0$.*

Proof Using the fact that M is defined by a full row rank matrix, we have the exact sequence:

$$0 \longrightarrow D^{1\times q} \xrightarrow{.R} D^{1\times p} \longrightarrow M \longrightarrow 0. \tag{36}$$

Dualizing this exact sequence, we obtain the following exact sequence

$$0 \longleftarrow N \longleftarrow D^q \stackrel{R.}{\longleftarrow} D^p \longleftarrow M^* \longleftarrow 0, \tag{37}$$

where we easily check that $N = \mathrm{ext}^1_D(M, D)$.

$1 \Rightarrow 2$. Let us suppose that M is a stably free left D-module. Therefore, (36) is a split exact sequence because it ends with a projective left D-module (see Proposition 6), i.e., there exists $S \in D^{p\times q}$ such that $R\,S = I_q$.

$2 \Rightarrow 3$. Let us suppose that there exists $S \in D^{p\times q}$ such that $R\,S = I_q$. Hence, for all $\lambda \in D^q$, we have $\lambda = R\,\mu$ where $\mu = S\,\lambda \in D^p$. Therefore, the D-morphism $R. : D^p \to D^q$ is surjective, and thus, $N = \mathrm{coker}(R.) = 0$.

$3 \Rightarrow 1$. Let us suppose that $N = 0$. Then, (37) ends with a free D-module, and thus, (37) splits (see Proposition 6), i.e., there exists $S \in D^{p\times q}$ such that $R\,S = I_q$. Therefore, (36) is also a split exact sequence, and thus, we have $D^{1\times p} \cong D^{1\times q} \oplus M$, which proves that M is a stably free left D-module. □

Hence, an effective test of stably freeness checks whether or not there exists a right-inverse of R. If Gröbner bases for left modules are available, then the computation of left-inverses is more convenient.

Algorithm 5
Input: An Ore algebra D satisfying (2) and a matrix $R \in D^{q\times p}$.
Output: A matrix $S \in D^{p\times q}$ satisfying $SR = I_p$ if it exists and [ ] else.
LEFT-INVERSE(R)
Introduce indeterminates $\lambda_j,\ j = 1, \ldots, p$ and $\mu_i,\ i = 1, \ldots, q$, over D.
$P \leftarrow \{\sum_{j=1}^{p} R_{ij}\,\lambda_j - \mu_i \mid i = 1, \ldots, q\}$.
Compute the Gröbner basis G of P in $\bigoplus_{i=1}^{p} D\,\lambda_i \oplus \bigoplus_{i=1}^{q} D\,\mu_i$
with respect to an order which eliminates the λ_i's.
Remove from G the elements which do not contain any λ_i
and call G' this new set.
Write G' in the form $Q_1 \cdot (\lambda_1 : \ldots : \lambda_p)^T - Q_2 \cdot (\mu_1 : \ldots : \mu_q)^T$.
If Q_1 is invertible over D, return $S = Q_1^{-1}\,Q_2 \in D^{p\times q}$, else return [ ].

Now, using an involution θ of D, we can compute a right-inverse $S \in D^{p\times q}$ of $R \in D^{q\times p}$ ($R\,S = I_q$), when such an inverse exists, by defining

$$\text{RIGHT-INVERSE}(R) = \theta(\text{LEFT-INVERSE}(\theta(R))).$$

Therefore, if $R \in D^{q\times p}$ has full row rank, by 2 of Proposition 12, the left D-module $M = D^{1\times p}/(D^{1\times q}\,R)$ is stably free iff RIGHT-INVERSE(R) $\neq$ [ ].

Example 15 Let us consider again the differential time-delay system defined by (4). Applying Algorithm 5 to $\theta(R) = R^T$, where R is defined by (5), we are led to the Gröbner basis G given in Example 7. We easily check that G does not contain any relation of the form $\lambda_i - \sum_{j=1}^{4} S_{ij}\,\mu_j$, where $S_{ij} \in D_h$, for $i = 1, 2, 3$. Therefore, $M = D_h^{1\times 4}/(D_h^{1\times 3}\,R)$ is not a projective D_h-module, and thus, (4) is not a flat system [20] (see also Example 14).

If $D = k[x_1, \dots, x_n]$ is a commutative polynomial ring over a field k, then we can study the obstructions for a system to be free (i.e., projective by the Quillen-Suslin theorem stated in 3 of Theorem 2). In this case, they are given by polynomials containing a certain number of variables x_i which depends on the properties of the corresponding D-module M.

Definition 19 Let $D = k[x_1, \dots, x_n]$, $R \in D^{q\times p}$ be a full row rank matrix, $M = D^{1\times p}/(D^{1\times q}\, R)$ and $N = D^{1\times 9}/(D^{1\times p}\, R^T)$. Then, we define:

$$i(M) = \min_{i\geq 1}\{i - 1 \mid \operatorname{ext}^i_D(N, D) \neq 0\}.$$

If no such integer exists, then we set $i(M) = \infty$. The index $i(M)$ is called the *torsion-free degree* of M.

We note that the torsion-free degree of the module $M = D^{1\times p}/(D^{1\times q}\, R)$ over $D = k[x_1, \dots, x_n]$ measures how far M is from being projective. In particular, $i(M) = 0$ means that M has some torsion elements, whereas $i(M) = 1$ means that M is a torsion-free but not reflexive D-module, ..., and $i(M) = \infty$ means that for all $i \geq 1$, we have $\operatorname{ext}^i_D(N, D) = 0$, i.e., M is a projective, i.e., free D-module (see Theorem 7).

In the next proposition, we shall need the following lemma.

Lemma 3 *[4,36] Let D be a commutative noetherian ring, M a finitely generated D-module and S a* multiplicatively closed subset *of D (namely, $1 \in S$ and for all $s_1,\ s_2 \in S$, we have $s_1\, s_2 \in S$). Let us introduce the commutative noetherian ring $S^{-1}\, D = \{a/s \mid a \in D,\ s \in S\}$ and the $S^{-1}\, D$-module*

$$S^{-1}\, D \otimes_D M = \{m/s \mid m \in M,\ s \in S\}.$$

Then, we have:

$$S^{-1}\, D \otimes_D \operatorname{ext}^j_D(M, D) \cong \operatorname{ext}^j_{S^{-1}\, D}(S^{-1}\, D \otimes_D M, S^{-1}\, D), \quad j \geq 0.$$

Proposition 13 *[31] Let $D = k[x_1, \dots, x_n]$, $R \in D^{q\times p}$ be a full row rank matrix and the D-modules, $M = D^{1\times p}/(D^{1\times q}\, R)$ and $N = D^{1\times q}/(D^{1\times p}\, R^T)$. For every partition of $X = \{x_1, \dots, x_n\}$ into two disjoint subsets X_1 and X_2 with respectively $n - i(M)$ elements and $i(M)$ elements, there exists*

$$\begin{cases} \pi_{n-i(M)} \in k[X_1], & \text{if} \quad 0 \leq i(M) \leq n-1, \\ \pi_{n-i(M)} \in k, & \text{if} \quad i(M) = \infty, \end{cases}$$

such that the module over $D_{\pi_{n-i(M)}} = \{d/a \mid d \in D,\ a = \pi^j_{n-i(M)},\ j \in \mathbb{Z}_+\}$

$$D_{\pi_{n-i(M)}} \otimes_D M = \left\{m/a \mid m \in M,\ a = \pi^j_{n-i(M)},\ j \in \mathbb{Z}_+\right\}$$

is free. Hence, there exist $Q \in D^{p\times(p-q)}$, $S \in D^{p\times q}$, $T \in D^{(p-q)\times p}$ and $\nu \in \mathbb{Z}_+$ such that:

$$(S:\ Q)\begin{pmatrix} R \\ T \end{pmatrix} = \pi^\nu_{n-i(M)}\, I_p, \quad \begin{pmatrix} R \\ T \end{pmatrix}(S:\ Q) = \pi^\nu_{n-i(M)}\, I_p. \tag{38}$$

Proof If $i(M) = \infty$, then, using 2 of Proposition 7, we have:

$$\mathrm{ext}^i_D(N, D) = 0, \quad i = 1, \ldots, \mathrm{gld}\, D = n.$$

So, by Proposition 7, M is a projective D-module, and thus, by 3 of Theorem 2, M is a free D-module. Hence, we can choose $\pi_{n-\infty} = 1 \in k \setminus \{0\}$ and we have $D_{\pi_{n-\infty}} = D$. Using the fact that R has full row rank, then we have the following exact sequence:

$$0 \longrightarrow D^{1\times q} \xrightarrow{.R} D^{1\times p} \longrightarrow M \longrightarrow 0. \tag{39}$$

We apply the functor $K \otimes_D \cdot$ to the exact sequence (39), where $K = Q(D)$ denotes the quotient field of D. Using the fact that K is a *flat* D-module [4,36], then we obtain the exact sequence

$$0 \longrightarrow K^{1\times q} \xrightarrow{.R} K^{1\times p} \longrightarrow K \otimes_D M \longrightarrow 0,$$

which shows that:

$$\mathrm{rank}_D(M) = \dim_K(K \otimes_D M) = \dim_K K^{1\times p} - \dim_K K^{1\times q} = p - q.$$

Hence, since M is free, it is then isomorphic to D^{p-q}. We obtain the following split exact sequence

$$0 \longrightarrow D^{1\times q} \underset{\xleftarrow{.S}}{\xrightarrow{.R}} D^{1\times p} \underset{\xleftarrow{.T}}{\xrightarrow{.Q}} D^{1\times(p-q)} \longrightarrow 0,$$

which gives identities (38).

Now, let us suppose that we have $0 \leq i(M) \leq n-1$. Then, we have:

$$\mathrm{ext}^j_D(N, D) = 0, \quad 0 \leq j \leq i(M).$$

Let us consider the multiplicatively closed subset $S = k[X_1] \setminus \{0\}$ of D and $S^{-1}\, D = k(X_1)[X_2]$. By 2 of Proposition 7, we have $\mathrm{gld}\, S^{-1}\, D = i(M)$, and thus, for the $S^{-1}\, D$-module $S^{-1}\, D \otimes_D N$, we have [35]:

$$\forall\, j \geq i(M) + 1, \quad \mathrm{ext}^j_{S^{-1}\, D}(S^{-1}\, D \otimes_D N, S^{-1}\, D) = 0.$$

Then, by Lemma 3, we conclude that, for all $j \geq i(M) + 1$, we have:

$$S^{-1}\, D \otimes_D \mathrm{ext}^j_D(N, D) \cong \mathrm{ext}^j_{S^{-1}\, D}(S^{-1}\, D \otimes_D N, S^{-1}\, D) = 0,$$

Let us consider $i(M)+1 \leq j \leq n$. One can prove that a non-zero $\mathrm{ext}^j_D(N, D)$ is a torsion D-module for $j \geq 1$ (see [25,29] for more details). So, if $\mathrm{ext}^j_D(N, D) \neq 0$, then:

$$\mathrm{ann}(\mathrm{ext}^j_D(N, D)) = \{P \in D \mid \forall\, m \in \mathrm{ext}^j_D(N, D),\;\; P\, m = 0\} \neq 0.$$

If $i_{S^{-1}D} : \mathrm{ext}^j_D(N, D) \to S^{-1}\, D \otimes \mathrm{ext}^j_D(N, D)$ denotes the canonical D-morphism (see (27)), then, for $0 \neq z \in \mathrm{ext}^j_D(N, D)$, we have:

$$i_{S^{-1}D}(z) = 0 \Leftrightarrow \{P \in D \mid P\, z = 0\} \cap S \neq \emptyset.$$

Therefore, using the fact that we have $S^{-1}\, D \otimes \mathrm{ext}_D^j(N, D) = 0$, we obtain:

$$\mathrm{ann}(\mathrm{ext}_D^j(N, D)) \cap (k[X_1] \setminus \{0\}) \neq \emptyset.$$

For $i(M) + 1 \leq j \leq n$, if $\mathrm{ext}_D^j(N, D) \neq 0$, then we can take any element $p_j \in \mathrm{ann}(\mathrm{ext}_D^j(N, D)) \cap (k[X_1] \setminus \{0\})$, else, we take $p_j = 1$. Let us denote by $\pi_{n-i(M)}$ the product of the p_j for $i(M) + 1 \leq j \leq n$. By construction, we have $\pi_{n-i(M)} \in k[X_1]$ and:

$$\pi_{n-i(M)}\ \mathrm{ext}_D^j(N, D) = 0, \quad 0 \leq j \leq n.$$

If we denote by $S_{\pi_{n-i(M)}} = \{1, \pi_{n-i(M)}, \pi_{n-i(M)}^2, \ldots\}$ the multiplicatively closed subset of D defined by $\pi_{n-i(M)}$ and

$$D_{\pi_{n-i(M)}} \triangleq S_{\pi_{n-i(M)}}^{-1}\, D = \{d/a \mid d \in D,\ a = (\pi_{n-i(M)})^m,\ m \in \mathbb{Z}_+\},$$

then, using Lemma 3, from $D_{\pi_{n-i(M)}} \otimes_D \mathrm{ext}_D^j(N, D) = 0,\ 0 \leq j \leq n$, we conclude that:

$$\mathrm{ext}^j_{D_{\pi_{n-i(M)}}}(D_{\pi_{n-i(M)}} \otimes_D N, D_{\pi_{n-i(M)}}) = 0, \quad 0 \leq j \leq n.$$

Since the transposed module of $M' \triangleq D_{\pi_{n-i(M)}} \otimes_D M$ is $D_{\pi_{n-i(M)}} \otimes_D N$ (see [31] for more details), by Theorem 7, it follows that M' is a projective $D_{\pi_{n-i(M)}}$-module, and thus, by 3 of Theorem 2, M' is a free $D_{\pi_{n-i(M)}}$-module.

Finally, using the fact that M' is a free $D_{\pi_{n-i(M)}}$-module of rank $p-q$, there exist $\overline{Q} \in D_{\pi_{n-i(M)}}^{p\times(p-q)}$, $\overline{S} \in D_{\pi_{n-i(M)}}^{p\times q}$ and $\overline{T} \in D_{\pi_{n-i(M)}}^{(p-q)\times p}$ such that we have the following split exact sequence

$$0 \longrightarrow D_{\pi_{n-i(M)}}^{1\times q} \underset{\xleftarrow{.\overline{S}}}{\xrightarrow{.R}} D_{\pi_{n-i(M)}}^{1\times p} \underset{\xleftarrow{.\overline{T}}}{\xrightarrow{.\overline{Q}}} D_{\pi_{n-i(M)}}^{1\times(p-q)} \longrightarrow 0,$$

or, equivalently, the following Bézout identities:

$$(\overline{S}:\ \overline{Q}) \begin{pmatrix} R \\ \overline{T} \end{pmatrix} = I_p, \quad \begin{pmatrix} R \\ \overline{T} \end{pmatrix} (\overline{S}:\ \overline{Q}) = I_p.$$

Let us write $\overline{Q} = \pi_{n-i(M)}^{-\alpha}\, Q'$, $\overline{S} = \pi_{n-i(M)}^{-\beta}\, S'$ and $\overline{T} = \pi_{n-i(M)}^{-\gamma}\, T'$, where the entries of Q', S' and T' belong to D and $\alpha, \beta, \gamma \in \mathbb{Z}_+$. Let us define $\nu = \max(\alpha+\gamma, \beta)$, $\theta = \nu - (\alpha+\gamma) \in \mathbb{Z}_+$, $\sigma = \nu - \beta \in \mathbb{Z}_+$, and the following matrices:

$$Q = \pi_{n-i(M)}^{\theta}\, Q' \in D^{p\times(p-q)}, \quad T = T' \in D^{(p-q)\times p}, \quad S = \pi_{n-i(M)}^{\sigma}\, S' \in D^{p\times q}.$$

Then, using the previous Bézout identities, we finally obtain (38). □

In order to compute such a polynomial $\pi_{n-i(M)}$, we can follow the next algorithm.

Algorithm 6

Input: A commutative polynomial ring $D = k[x_1, \ldots, x_n]$, where k is a field, a left D-module $M = D^{1\times p}/(D^{1\times q}\, R)$, $i(M)$ the torsion-free degree of M, a partition of $X = \{x_1, \ldots, x_n\}$ into disjoint subsets X_1 and X_2 with resp. $n - i(M)$ and $i(M)$ elements.

Output: An ideal J of $k[X_1]$ such that any element $\pi \in J$ satisfies that $D_\pi \otimes_D M$ is a free D_π-module.

π-POLYNOMIAL(R, X_1)
  $R_1 \leftarrow R^T$.
  for $i = i(M) + 1, \ldots, n$ **do**
    $L_i = [L_{i0},\ L_{i1},\ L_{i2}] \leftarrow$ EXT1 (R_i).
    Introduce the indeterminates $\mu_j,\ j = 1, \ldots, \mathrm{rowdim}(L_{i1})$ over D.
    $P_{ij} \leftarrow \{d \in D \mid d \cdot \mu_j \in L_{i0}\}, \quad j = 1, \ldots, \mathrm{rowdim}(L_{i1})$.
    $R_{i+1} \leftarrow$ SYZYGIES (R_i).
  endfor
  $I \leftarrow \bigcap_{i,j} \langle P_{ij} \rangle$.
  $J \leftarrow I \cap k[X_1]$.

In π-POLYNOMIAL, the set $\{d \in D \mid d \cdot \mu_j \in L_{i0}\}$ is obtained in the step QUOTIENT of EXT1(R_i). See Remark 4 for more details. Moreover, in the last step of π-POLYNOMIAL, the intersection of a left ideal with $k[X_1]$ can be computed by means of elimination techniques (see [2] and Example 5).

Example 16 In Examples 14 and 15, we proved that the system defined by (4) is not flat, i.e., the associated D_h-module $M = D_h^{1\times 4}/(D_h^{1\times 3}\, R)$ is not free. Let us find a polynomial $\pi \in \mathbb{R}(a,\ k,\ \zeta,\ \omega)[\delta_h; \sigma_2, \delta_2]$ such that the $(D_h)_\pi$-module $(D_h)_\pi \otimes_{D_h} M$ is free. In Examples 12 and 13, we saw that $\mathrm{ext}^1_{D_h}(N, D_h) = 0$ and $\mathrm{ext}^2_{D_h}(N, D_h) \neq 0$. Therefore, the torsion-free degree $i(M)$ of M is 1. Applying Algorithm 6 and using (21), we find that:

$$I = (\omega^2\, k\, a\, \delta_h,\ \omega^2\, \partial + \omega^2\, a,\ \omega^2\, \partial^2 + \omega^2\, a\, \partial,\ \partial^3 + (2\, \zeta\, \omega + a)\, \partial^2 \\ +(\omega^2 + 2\, a\, \zeta\, \omega) + a\, \omega^2).$$

Thus, we have $I \cap \mathbb{R}(a,\ k,\ \zeta,\ \omega)[\delta_h; \sigma_2, \delta_2] = (\delta_h)$, which shows that (4) is δ_h-free. See [20] for applications to the motion planning problem.

See [31] for an extension of Proposition 13 to non-full row rank matrices.

8.2 Minimal parametrizations and flatness

The next theorem generalizes a result of [26] obtained for systems of partial differential equations. We assume that D is a left/right noetherian domain.

Theorem 8 *[30] Let M be a torsion-free left D-module defined by the finite presentation (22). Then, there exists a left D-morphism $d_0' : F_0 \longrightarrow F_{-1}'$, where F_{-1}' is a finitely generated free left D-module, such that we have the exact sequence $F_1 \xrightarrow{d_1} F_0 \xrightarrow{d_0'} F_{-1}'$ and $M_{-1}' = \operatorname{coker} d_0'$ is either 0 or a torsion left D-module. Such a morphism d_0' is then called a* minimal parametrization.

Proof Let us consider the beginning of a free resolution of the transposed right D-module N of M, i.e., we have the following exact sequence

$$0 \longleftarrow N \longleftarrow F_1^* \xleftarrow{d_1^*} F_0^* \xleftarrow{d_0^*} F_{-1}^* \longleftarrow L \longleftarrow 0,$$

where F_{-1}^* is a finitely generated free right D-module and $L = \ker d_0^*$.

If we have $L = 0$, dualizing the previous exact sequence and using the fact that M is a torsion-free left D-module, and thus, $\mathrm{ext}_D^1(N, D) = 0$ by Theorem 5, then the following complex

$$F_1 \xrightarrow{d_1} F_0 \xrightarrow{d_0} F_{-1} \longrightarrow 0$$

is exact in F_0. Then, we have $\mathrm{ext}_D^2(N, D) = \operatorname{coker} d_0$. Hence, $\mathrm{ext}_D^2(N, D)$ is either 0 if M is a reflexive left D-module (and thus, $M \cong \operatorname{im} d_0 = F_{-1}$ is a free left D-module) or, by Corollary 1, $\mathrm{ext}_D^2(N, D)$ is a non-zero torsion left D-module because we have $\hom_D(\mathrm{ext}_D^2(N, D), D) = 0$. Hence, $d_0' = d_0$ and $F_0' = F_0$ satisfy the assertions and d_0 is a minimal parametrization of M.

Now, let us suppose that $L \neq 0$. Then, we have the following exact sequence $0 \longleftarrow \operatorname{im} d_0^* \longleftarrow F_{-1}^* \longleftarrow L \longleftarrow 0$, from which we deduce that:

$$l' \triangleq \operatorname{rank}_D \operatorname{im} d_0^* = \operatorname{rank}_D F_{-1}^* - \operatorname{rank}_D L \geq 1. \tag{40}$$

Indeed, if $\operatorname{rank}_D \operatorname{im} d_0^* = \dim_K(K \otimes_D \operatorname{im} d_0^*) = 0$, then $K \otimes_D \operatorname{im} d_0^* = 0$, i.e., $\operatorname{im} d_0^*$ is a torsion right D-module. But, $\operatorname{im} d_0^* \subseteq F_0^*$ and F_0^* is a free, and thus, torsion-free right D-module. Hence, $\operatorname{im} d_0^* = 0$, and thus, $\ker d_1^* = \operatorname{im} d_0^* = 0$, i.e., $F_{-1}^* = 0$ which contradicts the hypothesis that $L \neq 0$.

From the matrix R_0 which represents d_0^* in the standard bases of F_{-1}^* and F_0^*, let us extract a $((\operatorname{rank}_D F_0^*) \times l')$-submatrix R_0' composed by D-linear independent columns of R_0, which, in turn, defines a right D-morphism $d_0'^* : F_{-1}'^* \longrightarrow F_0^*$ in the standard bases, where $F_{-1}'^*$ is a free right D-module of rank l'. Using the fact that R_0' has full column rank, then $d_0'^*$ is injective. Thus, we have the following commutative exact diagram

$$\begin{tikzcd}
 & & & 0 & & \\
 & & 0 & \operatorname{coker}\phi \arrow[u] & & \\
0 & N_1 \arrow[l] & F_0^* \arrow[l] \arrow[u] & F_{-1}^* \arrow[l, "d_0^*"'] \arrow[u] & L \arrow[l] & 0 \arrow[l] \\
0 & N_1' \arrow[l] & F_0^* \arrow[l] \arrow[u, equal] & F_{-1}'^* \arrow[l, "d_0'^*"'] \arrow[u, "\phi"'] & 0, \arrow[l] & \\
 & & 0 \arrow[u] & 0 \arrow[u] & &
\end{tikzcd} \tag{41}$$

where $d_0'^* = d_0^* \circ \phi$, $N_1 = \operatorname{coker} d_{-1}^*$ and $N_1' = \operatorname{coker} d_{-1}'^*$. Using the snake lemma [4,36], we obtain the following exact sequence:

$$0 \longleftarrow N_1 \xleftarrow{\psi} N_1' \longleftarrow \operatorname{coker}\phi \longleftarrow L \longleftarrow 0.$$

Using (40), we have:

$$\begin{aligned}\operatorname{rank}_D N_1 - \operatorname{rank}_D N_1' &= \operatorname{rank}_D L - \operatorname{rank}_D \operatorname{coker}\phi \\ &= \operatorname{rank}_D L - \operatorname{rank}_D F_{-1}^* + \operatorname{rank}_D F_{-1}'^* \\ &= -\operatorname{rank}_D \operatorname{im} d_0^* + \operatorname{rank}_D F_{-1}'^* = 0.\end{aligned}$$

Therefore, we obtain $\operatorname{rank}_D N_1 = \operatorname{rank}_D N_1'$, and thus, $\operatorname{rank}_D \ker\psi = 0$ because $0 \longleftarrow N_1 \xleftarrow{\psi} N_1' \longleftarrow \ker\psi \longleftarrow 0$ is an exact sequence. Hence, $\ker\psi$ is a torsion right D-module. This fact implies that $\hom_D(\ker\psi, D) = 0$ (see Corollary 1) and, dualizing the previous exact sequence, we obtain

$$\psi^*(N_1^*) = N_1'^*,$$

with the notations $N_1^* = \hom_D(N_1, D)$ and $N_1'^* = \hom_D(N_1', D)$. Dualizing the commutative exact diagram (41), we obtain the commutative exact diagram:

$$\begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \\ 0 \longrightarrow & & N_1^* & \xrightarrow{\sigma} & F_0 & \xrightarrow{d_0} & F_{-1} \\ & & \downarrow \psi^* & & \| & & \downarrow \phi^* \\ 0 \longrightarrow & & N_1'^* & \xrightarrow{\tau} & F_0 & \xrightarrow{d_0'} & F_{-1}'. \\ & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & \end{array}$$

Therefore, we have $\sigma = \tau \circ \psi^*$, $\ker d_0 = \sigma(N_1^*) = \tau(\psi^*(N_1^*)) = \tau(N_1'^*)$ and $\ker d_0' = \tau(N_1'^*)$. Hence, we obtain $\ker d_0 = \ker d_0'$. Finally, the defect of exactness of the following complex

$$F_1 \xrightarrow{d_1} F_0 \xrightarrow{d_0'} F_{-1}' \tag{42}$$

at F_0 is defined by $H(F_0) = \ker d_0' / \operatorname{im} d_1 = \ker d_0' / \ker d_0$, because we have $\operatorname{ext}_D^1(N, D) = 0$, i.e., $\ker d_0 = \operatorname{im} d_1$. Therefore, we obtain that $H(F_0) = 0$, and thus, (42) is an exact sequence and d_0' is a parametrization of M. Finally, we have $\hom_D(\operatorname{coker} d_0', D) = \ker d_0'^* = 0$, and thus, by Corollary 1, $\operatorname{coker} d_0'$ is a torsion left D-module. □

Algorithm 7
Input: An Ore algebra D satisfying (2) with an involution θ, a torsion-free left D-module $M = D^{1\times p}/(D^{1\times q}\, R)$.
Output: A minimal parametrization $Q' \in D^{p\times m}$ of M.
MINIMAL-PARAMETRIZATION(R)
  $L \leftarrow$ SYZYGIES($\theta(R)$).
  $m \leftarrow \operatorname{rank}_D(L)$.
  Select m D-linearly independent rows of L and form a $(m \times p)$-matrix L' with them.
  $Q' \leftarrow \theta(L')$.

Example 17 Let us consider the first set of *Maxwell equations* [15], namely,

$$\begin{cases} \dfrac{\partial \mathbf{B}}{\partial t} + \nabla \wedge \mathbf{E} = 0, \\ \nabla . \mathbf{B} = 0, \end{cases} \tag{43}$$

where $\mathbf{B}$ (resp., $\mathbf{E}$) denotes the *magnetic* (resp., *electric*) *field*. We shall only consider smooth fields on an open convex subset of $\mathbb{R}^4$ (see 1 of Example 9). In electromagnetism, it is well-known that (43) is parametrized by

$$\begin{cases} \nabla \wedge \mathbf{A} = \mathbf{B}, \\ -\nabla V - \dfrac{\partial \mathbf{A}}{\partial t} = \mathbf{E}, \end{cases} \tag{44}$$

where $(\mathbf{A}, V)$ is called *quadri-potential* [15]. In terms of module theory, this result means that the D-module $M = D^{1\times 6}/(D^{1\times 4}\, R)$ is torsion-free, where $D = \mathbb{R}[\partial_1; id, \delta_1] \cdots [\partial_4; id, \delta_4]$, $\delta_i = \frac{\partial}{\rho x_i}$, $x_4 = t$ the time variable, and R is the matrix of differential operators defining (43), namely:

$$R = \begin{pmatrix} \partial_4 & 0 & 0 & 0 & -\partial_3 & \partial_2 \\ 0 & \partial_4 & 0 & \partial_3 & 0 & -\partial_1 \\ 0 & 0 & \partial_4 & -\partial_2 & \partial_1 & 0 \\ \partial_1 & \partial_2 & \partial_3 & 0 & 0 & 0 \end{pmatrix} \in D^{4\times 6}.$$

We easily check that a free resolution of the D-module M is defined by

$$0 \longrightarrow D \xrightarrow{.R_2} D^{1\times 4} \xrightarrow{.R} D^{1\times 6} \xrightarrow{\kappa} M \longrightarrow 0, \tag{45}$$

where $R_2 = (\partial_1 \;:\; \partial_2 \;:\; \partial_3 \;:\; -\partial_4) \in D^{1\times 4}$. Hence, we obtain $\operatorname{rank}_D(M) = 6 - 4 + 1 = 3$ or, in other words, system (43) is defined by $4 - 1 = 3$ D-linearly independent equations in 6 unknowns $(\mathbf{B}, \mathbf{E})$, and thus, its solutions depend on 3 arbitrary functions of x_1, x_2, x_3, x_4. But, the parametrization (44) of the torsion-free D-module M depends on 4 arbitrary potentials, namely $(\mathbf{A}, V)$. Hence, by Theorem 8, we know that system (43) admits some parametrizations with only 3 arbitrary potentials. Let us compute such parametrizations following Algorithm 7. Using Algorithm 1, the D-module $N = D^{1\times 4}/(D^{1\times 6}\, R^T)$ admits the following free resolution

$$0 \longleftarrow N \longleftarrow D^{1\times 4} \xleftarrow{.R^T} D^{1\times 6} \xleftarrow{.R_{-1}^T} D^{1\times 4} \xleftarrow{.R_{-2}^T} D \longleftarrow 0, \tag{46}$$

where:

$$R_{-1}^T = \begin{pmatrix} 0 & -\partial_3 & \partial_2 & \partial_4 & 0 & 0 \\ \partial_3 & 0 & -\partial_1 & 0 & \partial_4 & 0 \\ -\partial_2 & \partial_1 & 0 & 0 & 0 & \partial_4 \\ 0 & 0 & 0 & -\partial_1 & -\partial_2 & -\partial_3 \end{pmatrix}, \quad R_{-2}^T = (\partial_1 \;:\; \partial_2 \;:\; \partial_3 \;:\; \partial_4).$$

Let us point out that

$$R_{-1}^T \begin{pmatrix} \mathbf{H} \\ -\mathbf{D} \end{pmatrix} = \begin{pmatrix} \jmath \\ \rho \end{pmatrix} \quad \Leftrightarrow \quad \begin{cases} \nabla \wedge \mathbf{H} - \dfrac{\partial \mathbf{D}}{\partial t} = \jmath, \\ \nabla . \mathbf{D} = \rho, \end{cases}$$

is exactly the second set of Maxwell equations, where $\jmath$ denotes the *density of current*, ρ the *density of electric charge*, **H** the *magnetic induction* and **D** the *electric induction* [15]. Then, the conservation of current is given by:

$$R_{-2}^T \begin{pmatrix} \jmath \\ \rho \end{pmatrix} = 0 \quad \Leftrightarrow \quad \nabla . \jmath + \frac{\partial \rho}{\partial \mathrm{t}} = 0.$$

If we denote by $S(R^T)$ the syzygy module of R^T, from the free resolution (46) of N, we obtain that $\mathrm{rank}_D\, S(R^T) = \mathrm{rank}_D\, D^{1\times 4}\, R_{-1}^T = 4-1 = 3$. Hence, if we select 3 distinct rows of R_{-1}^T and transpose the corresponding matrix, we obtain a parametrization of (43) with only 3 potentials. In particular, we have the following 4 minimal parametrizations of (43) with only 3 arbitrary potentials $\xi = (\xi_1 : \xi_2 : \xi_3)^T$:

$$\begin{cases} \partial_3\, \xi_2 = B_1, \\ -\partial_3\, \xi_1 = B_2, \\ \partial_2\, \xi_1 - \partial_1\, \xi_2 = B_3, \\ \partial_4\, \xi_1 - \partial_1\, \xi_3 = E_1, \\ \partial_4\, \xi_2 - \partial_2\, \xi_3 = E_2, \\ -\partial_3\, \xi_3 = E_3, \end{cases} \qquad \begin{cases} -\partial_2\, \xi_2 = B_1, \\ -\partial_3\, \xi_1 + \partial_1\, \xi_2 = B_2, \\ \partial_2\, \xi_1 = B_3, \\ \partial_4\, \xi_1 - \partial_1\, \xi_3 = E_1, \\ -\partial_2\, \xi_3 = E_2, \\ \partial_4\, \xi_2 - \partial_3\, \xi_3 = E_3, \end{cases} \qquad \begin{cases} \partial_3\, \xi_1 - \partial_2\, \xi_2 = B_1, \\ \partial_1\, \xi_2 = B_2, \\ -\partial_1\, \xi_1 = B_3, \\ -\partial_1\, \xi_3 = E_1, \\ \partial_4\, \xi_1 - \partial_2\, \xi_3 = E_2, \\ \partial_4\, \xi_2 - \partial_3\, \xi_3 = E_3, \end{cases}$$

$$\begin{cases} -\nabla \wedge \xi = \mathbf{B}, \\ \dfrac{\partial\, \xi}{\partial t} = \mathbf{E}. \end{cases}$$

Equivalently, these minimal parametrizations can be obtained by setting one component of the quadri-potential $(\mathbf{A}, \mathbf{V})$ to 0.

To finish, let us state a result which will allow us to compute some bases of a π-free linear system with constant coefficients.

Proposition 14 *Let $D = k[x_1, \ldots, x_n]$ be a commutative polynomial ring over a field k and $M = D^{1\times p}/(D^{1\times q}\, R)$ a torsion-free D-module. By Theorem 8, there exists a minimal parametrization $Q \in D^{p\times m}$ of M, i.e., a parametrization Q of M such that $L = D^{1\times m}/(D^{1\times p}\, Q)$ is a torsion D-module. Then, for all $0 \neq \pi \in \mathrm{ann}(L) = \{P \in D \mid \forall\, l \in L :\ P\, l = 0\}$, the D_π-module $D_\pi \otimes_D M$ is free. In particular, there exists $S \in D_\pi^{m\times p}$ such that $S\, Q = I_m$ and $\{\kappa(S_i)\}_{1\leq i\leq m}$ is a basis of the D_π-module $D_\pi \otimes_D M$, where S_i denotes the i^{th} row of S and κ the canonical projection $D^{1\times p} \to M$.*

Proof By hypothesis, we have the following exact sequence:

$$D^{1\times q} \xrightarrow{.R} D^{1\times p} \xrightarrow{.Q} D^{1\times m} \longrightarrow L \longrightarrow 0. \tag{47}$$

Moreover, the fact that the D-module L is torsion implies that $\mathrm{ann}(L) \neq 0$. If $0 \neq \pi \in \mathrm{ann}(L)$, then the following sequence obtained by applying the right exact functor $D_\pi \otimes_D \cdot$ to the exact sequence (47) is exact because D_π is a flat D-module [36]:

$$D_\pi^{1\times q} \xrightarrow{.R} D_\pi^{1\times p} \xrightarrow{.Q} D_\pi^{1\times m} \longrightarrow D_\pi \otimes_D L \longrightarrow 0. \tag{48}$$

But, by definition of π, we have $D_\pi \otimes_D L = 0$, and thus, we obtain

$$D_\pi \otimes_D M = D_\pi^{1\times p}/(D_\pi^{1\times q}\, R) \cong D_\pi^{1\times p}\, Q = D_\pi^{1\times m},$$

i.e., the D_π-module $D_\pi \otimes_D M$ is free of rank m. Moreover, by Proposition 6, the exact sequence (48) splits, and thus, there exists $S \in D_\pi^{m\times p}$ such that $S\,Q = I_m$. Finally, if we denote by $\{f_i\}_{1\le i\le m}$ the standard basis of $D_\pi^{1\times m}$, then we obtain that $\{\kappa(f_i\, S)\}_{1\le i\le m}$ is a basis of $D_\pi \otimes_D M$. □

From Proposition 14, we obtain the following algorithm: using Algorithm 7, we compute a minimal parametrization Q of the torsion-free D-module M and then find $\mathrm{ann}(D^{1\times m}/(D^{1\times p}\, Q))$. After choosing a non-zero polynomial $\pi \in \mathrm{ann}(L)$, using Algorithm 5 over D_π, we obtain a left-inverse $S = (S_1^T \,:\, \cdots \,:\, S_m^T)^T$ of Q, which gives the basis $\{\kappa(S_i)\}_{1\le i\le m}$ of the D_π-module $D_\pi \otimes_D M$.

Example 18 In Example 14, we proved that (35) is a parametrization of system (4). We easily check that (35) is also a minimal parametrization of system (4) as the D_h-module $L = D_h/(D_h^{1\times 4}\, Q) = \mathrm{ext}^2_{D_h}(N, D_h)$ is torsion. From the definition (21) of $\mathrm{ext}^2_{D_h}(N, D_h)$, we obtain $\mathrm{ann}(L) = (\delta_h,\ \partial + a)$. Hence, if we choose $\pi = \delta_h$, then we easily check that

$$S = (\pi^{-1}/(\omega^2\, k\, a) \,:\, 0 \,:\, 0 \,:\, 0) \in D_\pi^{1\times 4}$$

is a left-inverse of the parametrization Q defined in Example 7. Hence, we obtain that $z = \pi^{-1}\, x_1$, and thus, x_1 are bases of the $(D_h)_\pi$-module $(D_h)_\pi \otimes_D M$.

If the characteristic of the field k is 0, then we refer to [34] and the references therein for constructive algorithms which compute bases of a stably free left $A_n(k)$-module M with $\mathrm{rank}_{A_n(k)} M \ge 2$.

9 Conclusion

We hope that we have convinced the reader that the simultaneous use of module theory, homological algebra, effective algebra and computational methods allows us to study effectively the structural properties of linear multidimensional systems (systems of ODEs, systems of PDEs, (differential) time-delay systems, discrete systems, convolutional codes ... with constant or variable coefficients). In particular, in this unified mathematical framework, we presented effective algorithms which check controllability/parametrizability/flatness/π-freeness . . and computed (minimal) parametrizations/autonomous elements/flat outputs/π-polynomials. Certain of these problems were still open for some classes of linear multidimensional systems [12,20,39,40].

The Maple package *OreModules*, based on *Mgfun* [6], as well as Maple worksheets containing the explicit examples of the appendix are available at

`http://wwwb.math.rwth-aachen.de/OreModules`.

We hope that OREMODULES will become in the future a platform for the implementation of different algorithms obtained in the literature of multidimensional linear systems (see e.g., [11,23,25–27,30,35,39,40] and the references therein).

10 Appendix: Examples

In this appendix, we give some Maple worksheets which decide controllability, parametrizability, flatness and π-freeness of some linear time-invariant/ time-varying OD systems, differential time-delay systems and systems of PDE with constant or variable coefficients. These results have been obtained using the Maple package *OreModules* [8] which is based on the library *Mgfun* [6] (e.g. Ore algebras and non-commutative Gröbner bases are developed in *Mgfun*). In these examples, the most time-consuming computation is that of the $\mathrm{ext}^i_D(N, D)$. We give some timings for these operations. All examples were run on a Pentium II, 450 MHz with 512 MB RAM using Maple 8 (*OreModules* is available for Maple V release 5, Maple 6, Maple 8, and Maple 9). Finally, we refer the reader to [8, 10] for a description of *OreModules* and a library of examples illustrating other functions.

10.1 Two pendula mounted on a cart

The first example that we consider is a time-invariant OD system describing the linearization around the vertical of a system formed by two pendula mounted on a cart. See for more details Examples 5.2.1 and 5.2.12 in [23].

```
> with(OreModules):
```

After loading the Maple package OREMODULES, the first step is to define the Ore algebra

$$D = \mathbb{Q}(m_1, m_2, M, L_1, L_2, g)[t][\partial; \sigma, \delta],$$

where $\sigma = \mathrm{id}_{\mathbb{R}(m_1,m_2,M,L_1,L_2,g)[t]}$ and $\delta = \frac{d}{dt}$. This algebra is also denoted by *Alg*.

```
> Alg:=DefineOreAlgebra(diff=[Dt,t], polynom=[t],
>  comm=[m1,m2,M, L1, L2,g]):
```

In *Alg*, we need to declare the constants m_1, m_2, M, L_1, L_2 and g which occur in the system. Then, we define the matrix $R \in D^{3\times 4}$ corresponding to the system.

```
> R:=evalm([[m1*L1*Dt^2, m2*L2*Dt^2, -1, (M+m1+m2)*Dt^2],
>  [m1*L1^2*Dt^2-m1*L1*g, 0, 0, m1*L1*Dt^2],
>  [0, m2*L2^2*Dt^2-m2*L2*g, 0, m2*L2*Dt^2]]);
```

$$R := \begin{bmatrix} m1\ L1\ Dt^2 & m2\ L2\ Dt^2 & -1 & (M + m1 + m2)\ Dt^2 \\ m1\ L1^2\ Dt^2 - m1\ L1\ g & 0 & 0 & m1\ L1\ Dt^2 \\ 0 & m2\ L2^2\ Dt^2 - m2\ L2\ g & 0 & m2\ L2\ Dt^2 \end{bmatrix}$$

Then, we define $R_{\mathrm{adj}} = \theta(R) \in D^{4\times 3}$ using the involution θ given in 2 of Example 10.

```
> R_adj:=Involution(R, Alg);
```

$$R\_adj := \begin{bmatrix} m1\ L1\ Dt^2 & m1\ L1^2\ Dt^2 - m1\ L1\ g & 0 \\ m2\ L2\ Dt^2 & 0 & m2\ L2^2\ Dt^2 - m2\ L2\ g \\ -1 & 0 & 0 \\ Dt^2\ M + Dt^2\ m1 + Dt^2\ m2 & m1\ L1\ Dt^2 & m2\ L2\ Dt^2 \end{bmatrix}$$

Let us compute $\mathrm{ext}^1_D(N, D)$, where $N = D^{1\times 4}/(D^{1\times 3}\, R_{\mathrm{adj}})$, using the procedure $\mathrm{EXT1}(R_{\mathrm{adj}})$.

```
> st:=time(): Ext1:=Exti(R_adj, Alg, 1): time()-st;
```

$$1.220$$

The computation of $\mathrm{ext}^1_D(N, D)$ only takes 1.220 s. Let us notice that all the computations are done generically. In other words, the results are valid for almost all values of the parameters (e.g., outside an algebraic hypersurface).

```
> Ext1[1];
```

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Ext1[1] gives the matrix L_0 returned by $\mathrm{EXT1}(R)$ and defined in Algorithm 4. In our case, L_0 is the identity matrix which shows that $\mathrm{ext}^1_D(N, D) = 0$ (see Remark 5). Therefore, the system is controllable, and thus, parametrizable (see Theorems 3 and 5). A parametrization of the system is given by the matrix L_2 =*Ext1[3]* of $\mathrm{EXT1}(R)$ (see Algorithm 4).

```
> map(collect, Ext1[3], Dt);
```

$$\begin{aligned} &[-L2\ Dt^4 + Dt^2\ g] \\ &[-Dt^4\ L1 + Dt^2\ g] \\ &[L2\ Dt^6\ L1\ M + (-L2\ g\ m1 - L2\ g\ M - g\ L1\ M - L1\ g\ m2)\ Dt^4 \\ &\ + (g^2\ M + g^2\ m1 + m2\ g^2)\ Dt^2] \\ &[L2\ Dt^4\ L1 + (-L2\ g - g\ L1)\ Dt^2 + g^2] \end{aligned}$$

Therefore, we obtain the parametrization $(x_1 : x_2 : x_3 : u)^T = L_2\, z$ of the system $R\,(x_1 : x_2 : x_3 : u)^T = 0$. Since the system is time-invariant, then, by 1 of Theorem 2, we know that the D-module $M = D^{1\times 4}/(D^{1\times 3}\, R)$ is free, and thus, the system defined by R is flat.

```
> LeftInverse(Ext1[3], Alg);
```

$$\begin{bmatrix} \dfrac{L1^2}{g^2\,(L1 - L2)} & -\dfrac{L2^2}{g^2\,(L1 - L2)} & 0 & -\dfrac{L2 - L1}{g^2\,(L1 - L2)} \end{bmatrix}$$

$\mathrm{LEFT\text{-}INVERSE}(Ext1[3], Alg)$ computes a left-inverse of $L_2 = Ext1[3]$. We deduce that $z = (L_1^2\, x_1 - L_2^2\, x_2 + (L_1 - L_2)\, u)/(g^2\,(L_1 - L_2))$ is a flat output of the system.

Let us notice that the difference of the pendula lengths $L_1 - L_2$ appears in the denominator of the flat output of the system. Thus, we need to study the non-generic case where $L_1 = L_2$.

```
> Rmod:=subs(L2=L1, evalm(R));
```

$$Rmod := \begin{bmatrix} m1\ L1\ Dt^2 & m2\ L1\ Dt^2 & -1 & (M + m1 + m2)\ Dt^2 \\ m1\ L1^2\ Dt^2 - m1\ L1\ g & 0 & & m1\ L1\ Dt^2 \\ 0 & m2\ L1^2\ Dt^2 - m2\ L1\ g & 0 & 0 \\ & & & m2\ L1\ Dt^2 \end{bmatrix}$$

```
> st:=time(): Ext1mod:=Exti(Involution(Rmod, Alg),
Alg, 1): time()-st;
```

$$0.959$$

```
> Ext1mod;
```

$$\left[\begin{bmatrix} L1\ Dt^2 - g & 0 & 0 \\ 0 & L1\ Dt^2 - g & 0 \\ 0 & 0 & L1\ Dt^2 - g \end{bmatrix},\right.$$

$$\begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & m1\ g + m2\ g & -1 & Dt^2\ M \\ 0 & M\ L1\ Dt^2 - g\ M - m1\ g - m2\ g & 1 & 0 \end{bmatrix},$$

$$\left.\begin{bmatrix} -Dt^2 \\ -Dt^2 \\ L1\ Dt^4\ M - Dt^2\ m1\ g - g\ M\ Dt^2 - Dt^2\ m2\ g \\ L1\ Dt^2 - g \end{bmatrix}\right]$$

The first matrix of *Ext1mod* is not an identity matrix, and thus, we know that $\mathrm{ext}^1_D(N_{\mathrm{mod}}, D) \neq 0$, where $N_{\mathrm{mod}} = D^{1\times 3}/(D^{1\times 4}\,\theta(R_{\mathrm{mod}}))$ and R_{mod} corresponds to R where $L_2 = L_1$ (see Remark 5). Thus, the system is not controllable (see Theorems 3 and 5). The second matrix of *Ext1mod* gives a family of generators of the torsion elements of $M_{\mathrm{mod}} = D^{1\times 4}/(D^{1\times 3}\,R_{\mathrm{mod}})$, i.e., of the non-controllable elements (autonomous observables) of the system. The initial rows in the first two matrices of *Ext1mod* show that $d = x_1 - x_2$ satisfies $(L_1\,\partial^2 - g)\,d = 0$ (non-controllable element). The third matrix of *Ext1mod* gives a parametrization of the torsion-free left D-module $M_{\mathrm{mod}}/t(M_{\mathrm{mod}})$, i.e., of the controllable part of the system [32]. We note that it has only taken 0.959 s to obtain all this information.

Finally, using the integration of the autonomous observables, we can express all solutions of the system defined by R_{mod} in terms of these integrals and one arbitrary function. See [8,32] for more details.

10.2 Linear differential algebraic equations

Let us consider the following example of a time-varying linear OD system which corresponds to a linear system of differential algebraic equations (DAEs) studied in [16]. Therefore, we introduce the Weyl algebra $D = A_1(\mathbb{Q})$ (see Example 1).

```
> Alg:=DefineOreAlgebra(diff=[Dt,t], polynom=[t]):
```

The time-varying linear system is defined by means of the following matrix.

```
> R:=evalm([[-t*Dt+1, t^2*Dt, -1, 0],
>  [-Dt, t*Dt+1, 0, -1]]);
```

$$R := \begin{bmatrix} -t\,Dt+1 & t^2\,Dt & -1 & 0 \\ -Dt & t\,Dt+1 & 0 & -1 \end{bmatrix}$$

Then, the system is described by $R\,(x_1 : x_2 : u_1 : u_2)^T = 0$ and we introduce the finitely presented left D-module $M = D^{1\times 4}/(D^{1\times 2}\,R)$.

```
> R_adj:=Involution(R, Alg);
```

$$R\_adj := \begin{bmatrix} t\,Dt+2 & Dt \\ -t^2\,Dt-2\,t & -t\,Dt \\ -1 & 0 \\ 0 & -1 \end{bmatrix}$$

Using the procedure EXTI, we compute $\widetilde{\mathrm{ext}^1_D(N, D)}$, where $N = D^{1\times 2}/(D^{1\times 4}\,R_{\mathrm{adj}})$ is the finitely presented left D-module associated with $R_{\mathrm{adj}} = \theta(R) \in D^{4\times 2}$.

```
> st:=time(): Ext1:=Exti(R_adj, Alg, 1): time()-st;
>  Ext1[1];
```

$$0.301$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

This shows that the left D-module $M = D^{1\times 4}/(D^{1\times 2}\,R)$ is torsion-free (see Remark 5), and thus, a projective left D-module (see 1 of Theorem 2). Then, a parametrization L_2 of the system is given by the following matrix of operators.

```
> Ext1[3];
```

$$\begin{bmatrix} t & 1 \\ 1 & 0 \\ 0 & -t\,Dt+1 \\ 0 & -Dt \end{bmatrix}$$

Therefore, $(x_1 : x_2 : x_3 : u)^T = L_2\,z$ is a (minimal) parametrization of the system $R\,(x_1 : x_2 : x_3 : u)^T = 0$.

Now, if the parametrization L_2 admits a left-inverse, then the left D-module $M = D^{1\times 4}/(D^{1\times 2}\;R)$ is free, i.e., the system defined by R is flat. Let us check whether or not L_2 admits a left-inverse.

```
> LeftInverse(Ext1[3], Alg);
```

$$P := \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -t \end{bmatrix}$$

Hence, we obtain that the system defined by R is flat and a flat output (i.e., a basis of the left D-module M) is defined by $(z_1 : z_2)^T = P\,(x_1 : x_2 : x_3 : u)^T$.

10.3 Wind tunnel model

In this example, we consider the linear differential time-delay system (4) defined in Example 6. We first define the algebra $D_h = \mathbb{Q}(a, k, \zeta, \omega)[t, s][\partial; \sigma_1, \delta_1][\delta_h; \sigma_2, \delta_2]$ of differential time-delay operators.

```
> Alg:=DefineOreAlgebra(diff=[Dt,t],
>  dual_shift=[delta,s], polynom=[t,s],
>  comm=[a,omega,zeta,k]):
```

The linear differential time-delay system is defined by matrix (5).

```
> R:=evalm([[Dt+a, -k*a*delta, 0, 0],
>  [0, Dt, -1, 0],
>  [0, omega^2, Dt+2*zeta*omega, -omega^2]]);
```

$$R := \begin{bmatrix} Dt + a & -k\,a\,\delta & 0 & 0 \\ 0 & Dt & -1 & 0 \\ 0 & \omega^2 & Dt + 2\,\zeta\,\omega & -\omega^2 \end{bmatrix}$$

```
> R_adj:=linalg[transpose](R);
```

$$R\_adj := \begin{bmatrix} Dt + a & 0 & 0 \\ -k\,a\,\delta & Dt & \omega^2 \\ 0 & -1 & Dt + 2\,\zeta\,\omega \\ 0 & 0 & -\omega^2 \end{bmatrix}$$

The Gröbner basis G defined in Example 7 can be computed as follows.

```
> Integrability(R_adj, Alg);
```

$$\begin{aligned}&[\omega^2\,k\,a\,\delta\,\mu_1 + \omega^2\,Dt\,\mu_2 + \omega^2\,a\,\mu_2 + \omega^2\,Dt^2\,\mu_3 + \omega^2\,a\,Dt\,\mu_3 + Dt^3\,\mu_4 \\ &+ 2\,Dt^2\,\zeta\,\omega\,\mu_4 + a\,Dt^2\,\mu_4 + Dt\,\omega^2\,\mu_4 + 2\,a\,Dt\,\zeta\,\omega\,\mu_4 + a\,\omega^2\,\mu_4, \\ &\lambda_3\,\omega^2 + \mu_4,\ \omega^2\,\lambda_2 + Dt\,\mu_4 + \omega^2\,\mu_3 + 2\,\zeta\,\omega\,\mu_4,\ \lambda_1\,Dt + \lambda_1\,a - \mu_1, \\ &\omega^2\,\lambda_1\,k\,a\,\delta + Dt^2\,\mu_4 + \omega^2\,Dt\,\mu_3 + 2\,Dt\,\zeta\,\omega\,\mu_4 + \omega^2\,\mu_2 + \omega^2\,\mu_4]\end{aligned}$$

The syzygy module of $D_h^{1\times 4}\,R^T$ is obtained by SYZYGIES(R_{adj}) (see Example 7).

```
> Q_transp:=SyzygyModule(R_adj, Alg);
```

$$\begin{aligned}Q\_transp := \Big[&\omega^2\,k\,a\,\delta\,,\ Dt\,\omega^2 + a\,\omega^2\,,\ \omega^2\,Dt^2 + \omega^2\,a\,Dt \\ &Dt\,\omega^2 + a\,\omega^2 + Dt^3 + 2\,Dt^2\,\zeta\,\omega + a\,Dt^2 + 2\,a\,Dt\,\zeta\,\omega\Big]\end{aligned}$$

The Gröbner basis G defined in Example 12 can be computed as follows.

```
> Integrability(linalg[transpose](Q_transp), Alg);
```

$$\begin{aligned}&[-\omega^2\,\mu_4 + \omega^2\,\mu_2 + Dt\,\mu_3 + 2\,\zeta\,\omega\,\mu_3,\ -\mu_3 + Dt\,\mu_2,\ -k\,a\,\delta\,\mu_2 \\ &+Dt\,\mu_1 + a\,\mu_1, \lambda_1\,\omega^2\,k\,a\,\delta - \mu_1, \lambda_1\,Dt\,\omega^2 + \lambda_1\,a\,\omega^2 - \mu_2]\end{aligned}$$

The syzygy module $D_h^{1\times 4}\, L$ is given by SYZYGIES(Q) (see Example 12).

```
> L:=SyzygyModule(linalg[transpose](Q_transp), Alg);
```

$$L := \begin{bmatrix} Dt + a & -k\,a\,\delta & 0 & 0 \\ 0 & \omega^2 & Dt + 2\,\zeta\,\omega & -\omega^2 \\ 0 & Dt & -1 & 0 \end{bmatrix}$$

Finally, we compute QUOTIENT(L, R) (see Example 13) as follows.

```
> Quotient(L, R, Alg);
```

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

We deduce that $\mathrm{ext}^1_{D_h}(N, D_h) = 0$, where $N = D_h^{1\times 3}/(D_h^{1\times 4}\, R^T)$ (see Remark 5), and thus, system (4) is controllable and parametrizable (see Theorems 3 and 5). We can directly compute $\mathrm{ext}^1_{D_h}(N, D_h)$ using EXT1(R_{adj}) defined in Algorithm 4.

```
> st:=time(): Ext1:=Exti(Involution(R, Alg), Alg, 1);
>  time()-st;
```

$$Ext1 := \left[\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} Dt + a & -k\,a\,\delta & 0 & 0 \\ 0 & \omega^2 & Dt + 2\,\zeta\,\omega & -\omega^2 \\ 0 & Dt & -1 & 0 \end{bmatrix}, \right.$$

$$\left. \begin{bmatrix} -\omega^2\,k\,a\,\delta \\ -Dt\,\omega^2 - a\,\omega^2 \\ -\omega^2\,Dt^2 - \omega^2\,a\,Dt \\ -Dt^3 - 2\,Dt^2\,\zeta\,\omega - a\,Dt^2 - Dt\,\omega^2 - 2\,a\,Dt\,\zeta\,\omega - a\,\omega^2 \end{bmatrix} \right]$$

$$0.819$$

The first two matrices in *Ext1* are the identity matrix and L. Moreover, *Ext1[3]* gives (up to the sign) the parametrization (35) of system (4), i.e., we have

$$(x_1 : x_2 : x_3 : u)^T = Ext1[3]\, z,$$

where z is an arbitrary function of t.

We compute $\mathrm{ext}^2_{D_h}(N, D_h)$ in order to check whether or not system (4) is flat.

```
> Ext2:=Exti(Involution(R, Alg), Alg, 2);
```

$$Ext2 := \left[\begin{bmatrix} \delta \\ Dt + a \end{bmatrix}, \begin{bmatrix} 1 \end{bmatrix}, \mathrm{SURJ}(1), \mathrm{INJ}(1) \right]$$

Since *Ext2[1]* is not an identity matrix, we know that $\mathrm{ext}^2_{D_h}(N, D_h) \neq 0$ (see Remark 5). Hence, the D_h-module $M = D_h^{1\times 4}/(D_h^{1\times 3}\, R)$ is a torsion-free but not a free D_h-module, and thus, (4) is not a flat system. Finally, let us notice that (21) is equivalent to the reduced system:

$$(\partial + a)\, y = 0, \quad \delta_h\, y = 0.$$

The formal obstructions of flatness are defined by π-POLYNOMIAL$(R_{\text{adj}}, \{\partial,\ \delta_h\})$.

```
> PiPolynomial(R, Alg);
```

$$[\delta,\ Dt + a]$$

The π-polynomial, such that $(D_h)_\pi \otimes_{D_h} M$ is a free $(D_h)_\pi$-module, is defined by the generator of the principal ideal π-POLYNOMIAL$(R_{\text{adj}}, \{\delta_h\})$ of $\mathbb{R}(a,\ k,\ \zeta,\ \omega)$ $[\delta_h; \sigma_2, \delta_2]$.

```
> PiPolynomial(R, Alg, [delta]);
```

$$[\delta]$$

Therefore, we find that $\pi = \delta_h$ and system (4) is π-free (see Example 16) [20].

```
> PiPolynomial(R, Alg, [Dt]);
```

$$[Dt + a]$$

The fact that (4) is not a flat system is coherent with the fact that the full row-rank matrix R, defined by (5), does not admit a right-inverse (R admits a right-inverse iff the D_h-module M is projective, and thus, free by 3 of Theorem 2).

```
> RightInverse(R, Alg);
```

$$[]$$

The fact that (4) is not a flat system is also coherent with the fact that its parametrization (35) does not admit a left-inverse (if a linear system is parametrizable and its parametrization admits a left-inverse, then the system is flat).

```
> LeftInverse(Ext1[3], Alg);
```

$$[]$$

However, we have shown that system (4) is δ_h-flat, and thus, the $(D_h)_\pi$-module $(D_h)_\pi \otimes_{D_h} M$ is free. Let us compute a basis of the free $(D_h)_\pi$-module. In order to do that, we compute a left-inverse S of the parametrization *Ext1[3]* in the commutative polynomial ring $(D_h)_\pi$.

```
> S:=LocalLeftInverse(Ext1[3], [delta], Alg);
```

$$S := \left[\ -\frac{1}{\delta\,\omega^2\,k\,a} \quad 0 \quad 0 \quad 0\ \right]$$

By construction, we have $S\,Ext1[3] = 1$.

```
> Mult(S, Ext1[3], Alg);
```

$$[\ 1\ ]$$

Hence, we obtain that $z = S\,(x_1\ :\ x_2\ :\ x_3\ :\ u)^T = -\delta_h^{-1}\,x_1/(\omega^2\,k\,a)$ is a basis of the $(D_h)_\pi$-module $(D_h)_\pi \otimes_{D_h} M$ because we have $(x_1\ :\ x_2\ :\ x_3\ :\ u)^T = Ext1[3]\,z$ and $z = -\delta_h^{-1}\,x_1/(\omega^2\,k\,a) \in (D_h)_\pi \otimes_{D_h} M$.

Let us finally point out that we can also substitute $z = -\delta_h^{-1}\, x_1/(\omega^2\, k\, a)$ into the parametrization $(x_1 \,:\, x_2 \,:\, x_3 \,:\, u)^T = Ext1[3]\, z$ of system (4) in order to express the system variables in terms of x_1.

```
> T:=Mult(Ext1[3], S, Alg);
```

$$T := \begin{bmatrix} 1 & 0 & 0 & 0 \\ \dfrac{Dt + a}{k\, a\, \delta} & 0 & 0 & 0 \\ \dfrac{Dt\,(Dt + a)}{k\, a\, \delta} & 0 & 0 & 0 \\ \dfrac{Dt\, \omega^2 + a\, \omega^2 + Dt^3 + 2\, Dt^2\, \zeta\, \omega + a\, Dt^2 + 2\, a\, Dt\, \zeta\, \omega}{\omega^2\, k\, a\, \delta} & 0 & 0 & 0 \end{bmatrix}$$

We obtain $(x_1 \,:\, x_2 \,:\, x_3 \,:\, u)^T = T_1\, x_1$, where T_1 denotes the first column of T, and thus, x_1 is also a basis of the module $(D_h)_\pi \otimes_{D_h} M$ over $(D_h)_\pi$, and thus, a flat output of system (4) over $(D_h)_\pi$. To finish, let us notice that, by imposing x_1 to be a desired trajectory $x_{1d}(t)$, we obtain the corresponding open-loop input (motion planning problem) [8, 20]:

$$u_d(t) = \frac{1}{\omega^2\, a\, k} \left(\left(\frac{d}{dt}^{2} + 2\, \zeta\, \omega\, \frac{d}{dt} + \omega^2 \right) \left(\frac{d}{dt} + a \right) \right) x_{1d}(t + h).$$

10.4 An electric transmission line

To finish with linear differential time-delay systems, we study the example of an electric transmission line [37]. We exhibit an explict parametrization of this system. It seems that no parametrization for such a system was previously known [20].

We first introduce $D_h = \mathbb{Q}(a_0, a_1, a_2, a_3, a_4, a_5, b_0)[t, s][\partial; \sigma_1, \delta_1][\delta_h; \sigma_2, \delta_2]$, the Ore algebra of differential time-delay operators.

```
> Alg:=DefineOreAlgebra(diff=[Dt,t],
>  dual_shift=[delta,s], polynom=[t,s],
>  comm=[a[0],a[1],a[2],a[3],a[4],a[5],b[0]]):
```

The electric transmission line is defined by means of the following matrix:

```
> R:=evalm([[Dt+a[0],-(a[4]*Dt+a[0])*delta, -a[0],
>  0, -b[0]*Dt],
>  [-delta*(a[5]*Dt+a[1]), Dt+a[1], 0, a[1], 0],
>  [a[2], -a[2]*a[4]*delta, Dt, 0, -a[2]*b[0]],
>  [a[3]*a[5]*delta, -a[3], 0, Dt, 0]]);
```

$$R := \begin{bmatrix} Dt + a_0 & -(a_4\, Dt + a_0)\,\delta & -a_0 & 0 & -b_0\, Dt \\ -\delta\,(a_5\, Dt + a_1) & Dt + a_1 & 0 & a_1 & 0 \\ a_2 & -a_2\, a_4\, \delta & Dt & 0 & -a_2\, b_0 \\ a_3\, a_5\, \delta & -a_3 & 0 & Dt & 0 \end{bmatrix}$$

```
> R_adj:=Involution(R, Alg):
> st:=time(): Ext1:=Exti(R_adj, Alg, 1): time()-st;
>  Ext1[1];
```

$$10.351$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Therefore, by Remark 5 and Theorems 3 and 5, we find that the electric transmission line is controllable and parametrizable and a parametrization is given by:

```
> Ext1[3];
```

$$\left[-b_0\,Dt^4 - Dt^2\,b_0\,a_1\,a_3 - a_0\,a_2\,b_0\,Dt^2 - a_1\,a_0\,a_2\,b_0\,a_3 - a_0\,a_1\,a_2\,b_0\,Dt\right]$$

$$\left[-\,a_1\,b_0\,Dt^3 - a_5\,b_0\,\delta\,Dt^4 - a_5\,\delta\,Dt^2\,b_0\,a_1\,a_3 - a_1\,b_0\,\delta\,Dt^3 \right.$$
$$\left. -\,a_0\,a_2\,a_5\,b_0\,\delta\,Dt^2 - a_0\,a_1\,a_2\,a_5\,a_3\,\delta\,b_0 - a_0\,a_1\,a_2\,b_0\,\delta\,Dt\right]$$

$$\left[a_0\,a_2\,a_1\,b_0\,a_3\,a_5\,\delta^2 - a_0\,a_2\,b_0\,Dt^2 - a_1\,a_0\,a_2\,b_0\,a_3 + a_0\,a_1\,a_2\,b_0\,\delta^2\,Dt \right.$$
$$\left. -\,a_0\,a_1\,a_2\,b_0\,Dt + a_0\,a_2\,a_5\,\delta^2\,b_0\,Dt^2\right]$$

$$\left[a_0\,a_1\,a_2\,a_5\,a_3\,\delta\,b_0 - \delta\,Dt^2\,b_0\,a_1\,a_3 + a_5\,\delta\,Dt^2\,b_0\,a_1\,a_3 - a_0\,a_1\,a_2\,a_3\,\delta\,b_0\right]$$

$$\left[-\,Dt^4 - a_1\,Dt^3 - a_0\,a_1\,Dt^2 - a_0\,Dt^3 + a_5\,Dt^4\,\delta^2\,a_4 + a_1\,a_5\,Dt^2\,a_3\,a_4\,\delta^2 \right.$$
$$+\,a_1\,Dt^3\,a_4\,\delta^2 + a_0\,a_2\,a_5\,Dt^2\,a_4\,\delta^2 + a_0\,Dt^3\,\delta^2\,a_5 - a_1\,a_3\,a_0\,a_2 - a_0\,a_1\,a_2\,Dt$$
$$-\,a_0\,a_1\,a_3\,Dt + a_0\,a_1\,Dt^2\,\delta^2 + a_0\,a_1\,a_2\,a_5\,a_3\,a_4\,\delta^2 + a_0\,a_1\,a_2\,Dt\,a_4\,\delta^2$$
$$\left. +\,a_0\,a_1\,a_3\,\delta^2\,Dt\,a_5 - a_1\,Dt^2\,a_3 - a_0\,a_2\,Dt^2\right]$$

```
> st:=time(): Ext2:=Exti(R_adj, Alg, 2): time()-st;
>  Ext2[1];
```

$$6.990$$

$$\left[-\,\delta\,{a_1}^2\,a_5\,{a_3}^2 + 2\,a_0\,a_1\,a_2\,a_5\,a_3\,\delta - Dt\,\delta\,a_2\,a_1\,a_0 + \delta\,Dt\,{a_1}^2\,a_3 \right.$$
$$+\,\delta\,a_5\,a_2\,a_0\,Dt\,a_1 - \delta\,{a_1}^2\,a_5\,a_3\,Dt + \delta^3\,{a_1}^2\,a_2\,a_0 - 2\,{a_5}^2\,a_2\,a_0\,a_1\,a_3\,\delta^3$$
$$\left. +\,{a_5}^2\,{a_2}^2\,{a_0}^2\,\delta^3 + {a_1}^2\,{a_5}^2\,{a_3}^2\,\delta^3 - a_5\,{a_2}^2\,{a_0}^2\,\delta - {a_1}^2\,\delta\,a_2\,a_0\right]$$

$$\left[Dt\,\delta^2\,a_1 - Dt^2 + a_1\,a_3\,a_5\,\delta^2 - Dt\,a_1 - a_1\,a_3 - a_5\,\delta^2\,a_2\,a_0\right]$$

$$\left[\delta\,a_2\,a_0 + \delta\,Dt^2\right]$$

$$\left[a_1\,Dt^3 + Dt^2\,{a_1}^2 + a_0\,a_2\,a_5\,Dt^2 - a_5\,Dt^2\,a_1\,a_3 + \delta^2\,{a_1}^2\,a_2\,a_0 \right.$$
$$-\,2\,{a_5}^2\,a_2\,a_0\,a_1\,a_3\,\delta^2 + {a_5}^2\,{a_2}^2\,{a_0}^2\,\delta^2 + {a_1}^2\,{a_5}^2\,{a_3}^2\,\delta^2 + Dt\,{a_1}^2\,a_3$$
$$\left. +\,a_5\,a_2\,a_0\,Dt\,a_1 - {a_1}^2\,a_5\,a_3\,Dt + a_1\,a_5\,a_3\,a_2\,a_0 - {a_1}^2\,a_5\,{a_3}^2\right]$$

Since we have $\mathrm{ext}^2_{D_h}(N, D_h) \neq 0$, where $N = D_h^{1\times 4}/(D_h^{1\times 5}\,R^T)$, the transmission line is not a flat system. Thus, we have $i(M) = 1$, and we can find a polynomial π that contains only Dt or δ such that $(D_h)_\pi \otimes_{D_h} M$ is a free $(D_h)_\pi$-module. The third argument for π-POLYNOMIAL selects the variable for the π-polynomial:

```
> pi:=PiPolynomial(R, Alg, [delta]): factor(pi);
```

$$[\delta (a_5^2\, a_2^2\, a_0^2\, \delta^4 - 2\,\delta^2\, a_5\, a_2^2\, a_0^2 + a_2^2\, a_0^2 - 2\, a_5^2\, a_2\, a_0\, a_1\, a_3\, \delta^4 + 4\,\delta^2\, a_1\, a_5\, a_3\, a_2\, a_0 + a_1^2\, a_2\, a_0 - 2\,\delta^2\, a_1^2\, a_2\, a_0 - 2\, a_1\, a_3\, a_0\, a_2 + \delta^4\, a_1^2\, a_2\, a_0 + a_1^2\, a_5^2\, a_3^2\, \delta^4 - 2\,\delta^2\, a_1^2\, a_5\, a_3^2 + a_1^2\, a_3^2)]$$

We conclude that the system is π-free, where π is the previous polynomial in δ_h, and thus, the $(D_h)_\pi$-module $(D_h)_\pi \otimes_{D_h} M$ is free. Let us compute a basis of the $(D_h)_\pi$-module $(D_h)_\pi \otimes_{D_h} M$.

```
> S:=LocalLeftInverse(Ext1[3], pi, Alg):
> T:=map(collect, S, delta);
```

$$T := \left[-\frac{\delta\,(\delta^4\, a_0\, a_5^2 + (-a_5\, a_0 - a_0\, a_5^2 + a_1 - a_1\, a_5)\,\delta^2 - a_1 + a_1\, a_5 + a_5\, a_0)}{\%2\, a_0\, b_0\,(-1 + a_5)},\right.$$
$$\frac{(a_5\, a_0 - a_1\, a_5\, a_4 + a_1\, a_4)\,\delta^4 + (-a_5\, a_0 + a_1\, a_5\, a_4 - a_1\, a_4 - a_0)\,\delta^2 + a_0}{\%2\, a_0\, b_0\,(-1 + a_5)},$$
$$-\frac{\delta\,((-a_1\, a_5\, a_3 + a_5\, a_2\, a_0)\,\delta^2 - a_0\, a_2 + a_1\, a_3)}{\%2\, a_0\, a_2\, b_0},$$
$$\frac{(a_0\, a_5^2\, a_2 - a_5^2\, a_1\, a_3 + a_1^2)\,\delta^4 + (2\, a_1\, a_5\, a_3 - 2\, a_5\, a_2\, a_0 - 2\, a_1^2)\,\delta^2}{\%2\, a_1\, a_3\, b_0\,(-1 + a_5)}$$
$$\left. + \frac{a_1^2 + a_0\, a_2 - a_1\, a_3}{\%2\, a_1\, a_3\, b_0\,(-1 + a_5)},\ -\frac{\delta\, a_1\,(\delta^2 - 1)}{\%2\, a_0}\right]$$
$$\%1 := a_1^2\, a_2\, a_0$$
$$\%2 := (a_0^2\, a_2^2\, a_5^2 + \%1 - 2\, a_5^2\, a_1\, a_3\, a_2\, a_0 + a_5^2\, a_1^2\, a_3^2)\,\delta^5 + (-2\, a_5\, a_1^2\, a_3^2 + 4\, a_0\, a_1\, a_2\, a_5\, a_3 - 2\, a_1^2\, a_2\, a_0 - 2\, a_0^2\, a_5\, a_2^2)\,\delta^3 + (\%1 - 2\, a_1\, a_3\, a_2\, a_0 + a_2^2\, a_0^2 + a_1^2\, a_3^2)\,\delta$$

We check that $T \in (D_h)_\pi^{1\times 5}$ is a left-inverse of *Ext1[3]*, i.e., $T\ Ext1[3] = 1$.

```
> Mult(T, Ext1[3], Alg);
```

$$[\ 1\ ]$$

Hence, $z = T\,(x_1 : \ldots : x_4 : u)^T$ is a basis of the $(D_h)_\pi$-module $(D_h)_\pi \otimes_{D_h} M$ which satisfies $(x_1 : \ldots : x_4 : u)^T = Ext1[3]\, z$.

10.5 Einstein equations

Let us show that the results exposed in this paper can also be interesting for the study of underdetermined systems of PDEs coming from mathematical physics. We shall study the parametrizability of the *Einstein equations* using the *linearized Ricci equations* in the vacuum [24] (see also [40]).

Let us introduce the Weyl algebra $D = A_4(\mathbb{Q})$ (x_1, x_2, and x_3 stand for the three space components and $x_4 = c\,t$ for the time component t up to the speed of light factor c).

```
> Alg:=DefineOreAlgebra(diff=[D1,x1], diff=[D2,x2],
>  diff=[D3,x3], diff=[D4,x4], polynom=[x1,x2,x3,x4]):
```

The linearized Ricci equations in the vacuum are defined by the following 10×10 matrix of partial differential operators.

```
> R:=evalm(
>  [[D2^2+D3^2-D4^2,D1^2,D1^2,-D1^2,-2*D1*D2,0,0,-2*D1*D3,0,2*D1*D4],
>  [D2^2,D1^2+D3^2-D4^2,D2^2,-D2^2,-2*D1*D2,-2*D2*D3,0,0,2*D2*D4,0],
>  [D3^2,D3^2,D1^2+D2^2-D4^2,-D3^2,0,-2*D2*D3,2*D3*D4,-2*D1*D3,0,0],
>  [D4^2,D4^2,D4^2,D1^2+D2^2+D3^2,0,0,-2*D3*D4,0,-2*D2*D4,-2*D1*D4],
>  [0,0,D1*D2,-D1*D2,D3^2-D4^2,-D1*D3,0,-D2*D3,D1*D4,D2*D4],
>  [D2*D3,0,0,-D2*D3,-D1*D3,D1^2-D4^2,D2*D4,-D1*D2,D3*D4,0],
>  [D3*D4,D3*D4,0,0,0,-D2*D4,D1^2+D2^2,-D1*D4,-D2*D3,-D1*D3],
>  [0,D1*D3,0,-D1*D3,-D2*D3,-D1*D2,D1*D4,D2^2-D4^2,0,D3*D4],
>  [D2*D4,0,D2*D4,0,-D1*D4,-D3*D4,-D2*D3,0,D1^2+D3^2,-D1*D2],
>  [0,D1*D4,D1*D4,0,-D2*D4,0,-D1*D3,-D3*D4,-D1*D2,D2^2+D3^2]]);
```

$$R := \begin{bmatrix}
D2^2+D3^2-D4^2 & D1^2 & D1^2 & -D1^2 & -2\,D1\,D2 & 0 & 0 & -2\,D1\,D3 & 0 & 2\,D1\,D4 \\
D2^2 & D1^2+D3^2-D4^2 & D2^2 & -D2^2 & -2\,D1\,D2 & -2\,D2\,D3 & 0 & 0 & 2\,D2\,D4 & 0 \\
D3^2 & D3^2 & D1^2+D2^2-D4^2 & -D3^2 & 0 & -2\,D2\,D3 & 2\,D3\,D4 & -2\,D1\,D3 & 0 & 0 \\
D4^2 & D4^2 & D4^2 & D1^2+D2^2+D3^2 & 0 & 0 & -2\,D3\,D4 & 0 & -2\,D2\,D4 & -2\,D1\,D4 \\
0 & 0 & D1\,D2 & -D1\,D2 & D3^2-D4^2 & -D1\,D3 & 0 & -D2\,D3 & D1\,D4 & D2\,D4 \\
D2\,D3 & 0 & 0 & -D2\,D3 & -D1\,D3 & D1^2-D4^2 & D2\,D4 & -D1\,D2 & D3\,D4 & 0 \\
D3\,D4 & D3\,D4 & 0 & 0 & 0 & -D2\,D4 & D1^2+D2^2 & -D1\,D4 & -D2\,D3 & -D1\,D3 \\
0 & D1\,D3 & 0 & -D1\,D3 & -D2\,D3 & -D1\,D2 & D1\,D4 & D2^2-D4^2 & 0 & D3\,D4 \\
D2\,D4 & 0 & D2\,D4 & 0 & -D1\,D4 & -D3\,D4 & -D2\,D3 & 0 & D1^2+D3^2 & -D1\,D2 \\
0 & D1\,D4 & D1\,D4 & 0 & -D2\,D4 & 0 & -D1\,D3 & -D3\,D4 & -D1\,D2 & D2^2+D3^2
\end{bmatrix}$$

```
> R_adj:=Involution(R, Alg):
```

Let us study whether or not the linearized Ricci equations are parametrizable. Let us notice that this problem is related to a question posed by J. Wheeler on the existence of potentials for the Einstein equations [24].

```
> st:=time(): Ext1:=Exti(R_adj, Alg, 1): time()-st;
>  Ext1[1];
```

$$86.810$$

$$\begin{bmatrix}
\%1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \%1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \%1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \%1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \%1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \%1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \%1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \%1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \%1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \%1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \%1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \%1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \%1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \%1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \%1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \%1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \%1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \%1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \%1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \%1
\end{bmatrix}$$

$$\%1 := D3^2 + D2^2 - D4^2 + D1^2$$

Therefore, we see that the linearized Ricci equations are not parametrizable because the system, defined by $R \in D^{10\times 10}$, admits a family of 20 torsion elements which generate the torsion submodule of the D-module $M = D^{1\times 10}/(D^{1\times 10}\,R)$. Let us notice that every torsion element of the system satisfies the Dalembertian equation, namely $\left(\triangle - c^2\,\frac{\partial^2}{\partial t^2}\right)\,y = 0$ (travelling wave in space-time).

The list of 20 torsion elements of the system is given by the following matrix.

```
> Ext1[2];
```

$$\left[\begin{array}{c}
D2\,D4\,,\ 0\,,\ 0\,,\ 0\,,\ -D1\,D4\,,\ 0\,,\ 0\,,\ 0\,,\ D1^2\,,\ -D1\,D2\\
D3\,D4\,,\ 0\,,\ 0\,,\ 0\,,\ 0\,,\ 0\,,\ D1^2\,,\ -D1\,D4\,,\ 0\,,\ -D1\,D3\\
D2\,D3\,,\ 0\,,\ 0\,,\ 0\,,\ -D1\,D3\,,\ D1^2\,,\ 0\,,\ -D1\,D2\,,\ 0\,,\ 0\\
D4^2\,,\ 0\,,\ 0\,,\ D1^2\,,\ 0\,,\ 0\,,\ 0\,,\ 0\,,\ 0\,,\ -2\,D1\,D4\\
D3^2\,,\ 0\,,\ D1^2\,,\ 0\,,\ 0\,,\ 0\,,\ 0\,,\ -2\,D1\,D3\,,\ 0\,,\ 0\\
D2^2\,,\ D1^2\,,\ 0\,,\ 0\,,\ -2\,D1\,D2\,,\ 0\,,\ 0\,,\ 0\,,\ 0\,,\ 0\\
0\,,\ D3\,D4\,,\ 0\,,\ 0\,,\ 0\,,\ -D2\,D4\,,\ D2^2\,,\ 0\,,\ -D2\,D3\,,\ 0\\
0\,,\ D4^2\,,\ 0\,,\ D2^2\,,\ 0\,,\ 0\,,\ 0\,,\ 0\,,\ -2\,D2\,D4\,,\ 0\\
0\,,\ D3^2\,,\ D2^2\,,\ 0\,,\ 0\,,\ -2\,D2\,D3\,,\ 0\,,\ 0\,,\ 0\,,\ 0\\
0\,,\ -D1\,D4\,,\ 0\,,\ 0\,,\ D2\,D4\,,\ 0\,,\ 0\,,\ 0\,,\ D1\,D2\,,\ -D2^2\\
0\,,\ -D1\,D3\,,\ 0\,,\ 0\,,\ D2\,D3\,,\ D1\,D2\,,\ 0\,,\ -D2^2\,,\ 0\,,\ 0\\
0\,,\ 0\,,\ D4^2\,,\ D3^2\,,\ 0\,,\ 0\,,\ -2\,D3\,D4\,,\ 0\,,\ 0\,,\ 0\\
0\,,\ 0\,,\ -D2\,D4\,,\ 0\,,\ 0\,,\ D3\,D4\,,\ D2\,D3\,,\ 0\,,\ -D3^2\,,\ 0\\
0\,,\ 0\,,\ -D1\,D4\,,\ 0\,,\ 0\,,\ 0\,,\ D1\,D3\,,\ D3\,D4\,,\ 0\,,\ -D3^2\\
0\,,\ 0\,,\ D1\,D2\,,\ 0\,,\ D3^2\,,\ -D1\,D3\,,\ 0\,,\ -D2\,D3\,,\ 0\,,\ 0\\
0\,,\ 0\,,\ 0\,,\ D2\,D3\,,\ 0\,,\ D4^2\,,\ -D2\,D4\,,\ 0\,,\ -D3\,D4\,,\ 0\\
0\,,\ 0\,,\ 0\,,\ D1\,D3\,,\ 0\,,\ 0\,,\ -D1\,D4\,,\ D4^2\,,\ 0\,,\ -D3\,D4\\
0\,,\ 0\,,\ 0\,,\ D1\,D2\,,\ D4^2\,,\ 0\,,\ 0\,,\ 0\,,\ -D1\,D4\,,\ -D2\,D4\\
0\,,\ 0\,,\ 0\,,\ 0\,,\ D3\,D4\,,\ -D1\,D4\,,\ D1\,D2\,,\ 0\,,\ 0\,,\ -D2\,D3\\
0\,,\ 0\,,\ 0\,,\ 0\,,\ 0\,,\ -D1\,D4\,,\ 0\,,\ D2\,D4\,,\ D1\,D3\,,\ -D2\,D3
\end{array}\right]$$

In fact, we can prove that $t(M)$ can be generated by 10 torsion elements. See [32] for more details.

A parametrization of $M/t(M) = D^{1\times 10}/(D^{1\times 20}\;Ext1[2])$ is defined by:

```
> Ext1[3];
```

$$\left[\begin{array}{cccc}
-2\,D1 & 0 & 0 & 0\\
0 & -2\,D2 & 0 & 0\\
0 & 0 & -2\,D3 & 0\\
0 & 0 & 0 & -2\,D4\\
-D2 & -D1 & 0 & 0\\
0 & -D3 & -D2 & 0\\
0 & 0 & -D4 & -D3\\
-D3 & 0 & -D1 & 0\\
0 & -D4 & 0 & -D2\\
-D4 & 0 & 0 & -D1
\end{array}\right]$$

Therefore, the underdetermined linear system of PDEs $Ext1[2]\,y = 0$, which is associated with the D-module $M/t(M) = D^{1\times 10}/(D^{1\times 20}\;Ext1[2])$, is parametrized by $Ext1[3]$, i.e., we have $y = Ext1[3]\,z$, where $z = (z_1 : z_2 : z_3 : z_4)^T$ are four potentials. Let us check whether or not $M/t(M)$ is a reflexive left D-module.

```
> st:=time(): Ext2:=Exti(R_adj, Alg, 2): time()-st;
>  Ext2[1];
```

$$5.529$$

$$\begin{bmatrix} D1 & 0 & 0 & 0 \\ D4^2 & 0 & 0 & 0 \\ D3\,D4 & 0 & 0 & 0 \\ D2\,D4 & 0 & 0 & 0 \\ D3^2 & 0 & 0 & 0 \\ D2\,D3 & 0 & 0 & 0 \\ D2^2 & 0 & 0 & 0 \\ 0 & D2 & 0 & 0 \\ 0 & D4^2 & 0 & 0 \\ 0 & D3\,D4 & 0 & 0 \\ 0 & D1\,D4 & 0 & 0 \\ 0 & D3^2 & 0 & 0 \\ 0 & D1\,D3 & 0 & 0 \\ 0 & D1^2 & 0 & 0 \\ 0 & 0 & D3 & 0 \\ 0 & 0 & D4^2 & 0 \\ 0 & 0 & D2\,D4 & 0 \\ 0 & 0 & D1\,D4 & 0 \\ 0 & 0 & D2^2 & 0 \\ 0 & 0 & D1\,D2 & 0 \\ 0 & 0 & D1^2 & 0 \\ 0 & 0 & 0 & D4 \\ 0 & 0 & 0 & D3^2 \\ 0 & 0 & 0 & D2\,D3 \\ 0 & 0 & 0 & D1\,D3 \\ 0 & 0 & 0 & D2^2 \\ 0 & 0 & 0 & D1\,D2 \\ 0 & 0 & 0 & D1^2 \end{bmatrix}$$

Therefore, we obtain that $M/t(M)$ is a torsion-free but not a reflexive left D-module. Now, let us compute a free resolution of the linearized Ricci equations.

```
> st:=time(): Resolution(R, Alg, 4); time()-st;
```

$$\mathrm{table}([1 = R,$$

$$2 = \begin{bmatrix} -D4 & -D4 & -D4 & -D4 & 0 & 0 & 2\,D3 & 0 & 2\,D2 & 2\,D1 \\ -D3 & -D3 & D3 & D3 & 0 & 2\,D2 & -2\,D4 & 2\,D1 & 0 & 0 \\ -D2 & D2 & -D2 & D2 & 2\,D1 & 2\,D3 & 0 & 0 & -2\,D4 & 0 \\ D1 & -D1 & -D1 & D1 & 2\,D2 & 0 & 0 & 2\,D3 & 0 & -2\,D4 \end{bmatrix},$$

$$3 = \mathrm{INJ}(4),$$

$$4 = ZERO$$

$$])$$

$$7.540$$

Hence, we obtain that $\mathrm{rank}_D M = 10 - 10 + 4 = 4$. Then, using the fact that $\mathrm{rank}_D M = \mathrm{rank}_D(M/t(M))$, we obtain that $Ext1[3]$ is a minimal parametrization of $M/t(M)$.

We now compute a free resolution of the left D-module $\widetilde{N} = D^{1\times 10}/(D^{1\times 10}\, R_{\mathrm{adj}})$.

```
> st:=time(): Resolution(R_adj, Alg, 4); time()-st;
```

$$\mathrm{table}([1 = R\_adj,\ 2 = \begin{bmatrix} 2\,\mathrm{D1} & 0 & 0 & 0 & \mathrm{D2} & 0 & 0 & \mathrm{D3} & 0 & \mathrm{D4} \\ 0 & 2\,\mathrm{D2} & 0 & 0 & \mathrm{D1} & \mathrm{D3} & 0 & 0 & \mathrm{D4} & 0 \\ 0 & 0 & 2\,\mathrm{D3} & 0 & 0 & \mathrm{D2} & \mathrm{D4} & \mathrm{D1} & 0 & 0 \\ 0 & 0 & 0 & 2\,\mathrm{D4} & 0 & 0 & \mathrm{D3} & 0 & \mathrm{D2} & \mathrm{D1} \end{bmatrix},$$

$$3 = \mathrm{INJ}(4),$$
$$4 = ZERO$$
$$])$$

$$3.360$$

From the previous free resolution of the D-module $N = D^{1\times 10}/(D^{1\times 10}\, R_{\mathrm{adj}})$, we deduce that $\mathrm{ext}^2_D(N, D) = D^{1\times 4}/(D^{1\times 10}\, Ext1[3])$ and $\mathrm{ext}^i_D(N, D) = 0$ for $i \geq 3$.

10.6 Lie-Poisson structures

Finally, we apply OREMODULES to another underdetermined linear system of PDEs with variable coefficients that appears in mathematical physics. Let us give an example coming from the study of *Lie-Poisson structures* [3,38].

```
> Alg:=DefineOreAlgebra(diff=[D1,x1], diff=[D2,x2],
>  diff=[D3,x3], polynom=[x1,x2,x3]):
```

The following example appears in the study of the E_2 *algebra* [3]. The authors of [3] investigated the possibility to parametrize all the solutions of the system of PDEs defined by the following matrix.

```
> R:=evalm([[x1*D3, x2*D3, 0],
>  [-x1*D2+x2*D1, -1, x2*D3],
>  [-1, -x2*D1+x1*D2, x1*D3]]);
```

$$R := \begin{bmatrix} x1\,\mathrm{D3} & x2\,\mathrm{D3} & 0 \\ -x1\,\mathrm{D2} + x2\,\mathrm{D1} & -1 & x2\,\mathrm{D3} \\ -1 & -x2\,\mathrm{D1} + x1\,\mathrm{D2} & x1\,\mathrm{D3} \end{bmatrix}$$

Let $D = A_3(\mathbb{Q})$ be the Weyl algebra (see Example 1) and $M = D^{1\times 3}/(D^{1\times 3}\, R)$ be the left D-module associated with R. The previous problem can be solved by computing $\widetilde{\mathrm{ext}^1_D(N, D)}$, where $\widetilde{N} = D^{1\times 3}/(D^{1\times 3}\, \theta(R))$ is a left D-module and θ is the involution defined in 2 of Example 10.

```
> Ext1:=Exti(Involution(R, Alg), Alg, 1);
```

$$Ext1 := \left[\begin{bmatrix} D3 & 0 & 0 \\ -x2\,D1 + x1\,D2 & 0 & 0 \\ 0 & x2\,D3 & 0 \\ 0 & x1\,D3 & 0 \\ 0 & -x2\,D1 + x1\,D2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \right.$$

$$\left. \begin{bmatrix} x1 & x2 & 0 \\ D1 & D2 & D3 \\ -1 & -x2\,D1 + x1\,D2 & x1\,D3 \end{bmatrix}, \begin{bmatrix} -x2\,D3 \\ x1\,D3 \\ -x1\,D2 + x2\,D1 \end{bmatrix} \right]$$

We find that the system $R\,(F\,:\,G\,:\,H)^T = 0$ is not parametrizable because there exist two torsion elements in the left D-module M

$$\begin{cases} \Phi_1 = x_1\,F + x_2\,G, \\ \Phi_2 = \partial_1\,F + \partial_2\,G + \partial_3\,H, \end{cases}$$

which satisfy the following equations:

$$\begin{cases} \partial_3\,\Phi_1 = 0, \\ (-x_2\,\partial_1 + x_1\,\partial_2)\,\Phi_1 = 0, \end{cases} \qquad \begin{cases} x_2\,\partial_3\,\Phi_2 = 0, \\ x_1\,\partial_3\,\Phi_2 = 0, \\ (-x_2\,\partial_1 + x_1\,\partial_2)\,\Phi_2 = 0. \end{cases}$$

However, the system of PDEs $Ext1[2]\,y = 0$ is parametrized by $y = Ext1[3]\,z$. Up to the mistake underlined in [38] concerning the existence of the torsion elements, we recover the parametrization exhibited in [3].

The previous computation shows that M is not a torsion left D-module even if R is a square matrix. Hence, R does not have full row rank. This result can be checked by computing a free resolution of the left D-module $M = D^{1\times 3}/(D^{1\times 3}\,R)$.

```
> Free:=FreeResolution(R, Alg);
```

$$Free := \text{table}([1 = \begin{bmatrix} x1\,D3 & x2\,D3 & 0 \\ -x1\,D2 + x2\,D1 & -1 & x2\,D3 \\ -1 & -x2\,D1 + x1\,D2 & x1\,D3 \end{bmatrix},$$
$$2 = \begin{bmatrix} -x2\,D1 + x1\,D2 & x1\,D3 & -x2\,D3 \end{bmatrix},$$
$$3 = \text{INJ}(1)$$
$$])$$

In particular, the left D-module $L = D^{1\times 3}/(D\,Free[2])$ is torsion-free as we have $L \cong D^{1\times 3}\,R \subset D^{1\times 3}$ and $D^{1\times 3}$ is a torsion-free left D-module.

```
> ext:=Exti(Involution(Free[2], Alg), Alg, 1);
```

$$ext := [\left[\ 1\ \right], \left[\ -x2\,\mathrm{D1} + x1\,\mathrm{D2} \quad x1\,\mathrm{D3} \quad -x2\,\mathrm{D3}\ \right],$$
$$\left[\begin{array}{ccc} \mathrm{D3} & x1\,\mathrm{D3} & 0 \\ -\mathrm{D2} & -x1\,\mathrm{D2} + x2\,\mathrm{D1} & x2 \\ -\mathrm{D1} & -1 & x1 \end{array}\right]]$$

Therefore, the matrices $R \in D^{3\times 3}$ and *ext[3]* are two parametrizations of the left D-module $L = D^{1\times 3}/(D\,Free[2])$. Let us compute $\mathrm{rank}_D\,L$.

```
> OreRank(Free[2], Alg);
```

$$2$$

Hence, we conclude that $y = R\,z$ and $y = ext[3]\,z$ are not minimal parametrizations of $Free[2]\,(F\,:\,G\,:\,H)^T = 0$ because they depend on three arbitrary functions of $x_1,\ x_2,\ x_3$. Let us compute minimal parametrizations of the system $Free[2]\,(F\,:\,G\,:\,H)^T = 0$.

```
> P:=MinimalParametrizations(Free[2], Alg);
```

$$P := \left[\left[\begin{array}{cc} \mathrm{D3} & x1\,\mathrm{D3} \\ -\mathrm{D2} & -x1\,\mathrm{D2} + x2\,\mathrm{D1} \\ -\mathrm{D1} & -1 \end{array}\right], \left[\begin{array}{cc} \mathrm{D3} & 0 \\ -\mathrm{D2} & x2 \\ -\mathrm{D1} & x1 \end{array}\right], \left[\begin{array}{cc} x1\,\mathrm{D3} & 0 \\ -x1\,\mathrm{D2} + x2\,\mathrm{D1} & x2 \\ -1 & x1 \end{array}\right]\right]$$

Then, we can check that $y = P[1]\,z$, $y = P[2]\,z$ and $y = P[3]\,z$ are three minimal parametrizations of the system $Free[2]\,(F\,:\,G\,:\,H)^T = 0$, where $z = (z_1\,:\,z_2)^T$, and z_1 and z_2 are two arbitrary functions of $x_1,\ x_2$ and x_3.

```
> seq(SyzygyModule(P[i], Alg), i=1..3);
```

$$\%1,\ \%1,\ \%1$$
$$\%1 := \left[\ -x2\,\mathrm{D1} + x1\,\mathrm{D2} \quad x1\,\mathrm{D3} \quad -x2\,\mathrm{D3}\ \right]$$

To finish, we consider $B_3(\mathbb{Q}) = \mathbb{Q}(x_1, x_2, x_3)[\partial_1; \sigma_1, \delta_1][\partial_2; \sigma_2, \delta_2][\partial_3; \sigma_3, \delta_3]$, where σ_i and δ_i are defined as in the last part of Example 1 and we define the left $B_3(\mathbb{Q})$-module $M' = B_3^{1\times 3}(\mathbb{Q})/(B_3^{1\times 3}(\mathbb{Q})\,R)$ associated with R. Since the domain of coefficients of $B_3(\mathbb{Q})$ is the *quotient field* of $\mathbb{Q}[x_1, x_2, x_3]$, we are going to use the Weyl algebra $D = A_3(\mathbb{Q})$ (see Example 1) and allow our algorithms to divide by non-zero polynomials in x_1, x_2, x_3. This is taken into account by EXTIRAT below.

```
> st:=time(): ExtiRat(Involution(R, Alg), Alg, 1);
>  time()-st;
```

$$\left[\left[\begin{array}{cc} \mathrm{D3} & 0 \\ -x1\,\mathrm{D2} + x2\,\mathrm{D1} & 0 \\ 0 & \mathrm{D3} \\ 0 & -x1\,\mathrm{D2} + x2\,\mathrm{D1} \end{array}\right],\right.$$
$$\left.\left[\begin{array}{ccc} x1 & x2 & 0 \\ 0 & -x1^2\,\mathrm{D2} + x1\,x2\,\mathrm{D1} - x2 & -x1^2\,\mathrm{D3} \end{array}\right], \left[\begin{array}{c} x2\,\mathrm{D3} \\ -x1\,\mathrm{D3} \\ -x2\,\mathrm{D1} + x1\,\mathrm{D2} \end{array}\right]\right]$$

$$0.670$$

We find that the system $R\,(F\,:\,G\,:\,H)^T = 0$ is not parametrizable because there exist two torsion elements

$$\begin{cases} \Phi_1 = x_1\,F + x_2\,G, \\ \Phi_2 = (-x_1^2\,\partial_2 + x_1\,x_2\,\partial_1 - x_2)\,G - x_1^3\,\partial_3\,H, \end{cases}$$

which both satisfy the system:

$$\begin{cases} \partial_3\,\Phi_i = 0, \\ (-x_1\,\partial_2 + x_2\,\partial_1)\,\Phi_i = 0, \end{cases} \qquad i = 1, 2.$$

We recover the torsion element Φ_1 exhibited in [38] as well as the parametrization of the torsion-free left $B_3(\mathbb{Q})$-module $M'/t(M')$ given in [3].

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