Creative Telescoping
—From 1G to 4G algorithms—

Alin Bostan

Inria informatics mathematics

ENS Lyon

M2, CR06

October 5, 2020

Creative Telescoping

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{2019\cdot 2020}=?$$

Creative Telescoping

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{2019\cdot 2020}=\,?$$

▷ [J. Bernoulli 1682]: Use $\frac{1}{k(k+1)}=\frac{1}{k}-\frac{1}{k+1}$ to *create* a *telescoping sum*

$$\left(\frac{1}{1}-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\cdots+\left(\frac{1}{2019}-\frac{1}{2020}\right)=1-\frac{1}{2020}.$$

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{2019\cdot 2020}=?$$

▷ [J. Bernoulli 1682]: Use $\frac{1}{k(k+1)}=\frac{1}{k}-\frac{1}{k+1}$ to *create* a *telescoping sum*

$$\left(\frac{1}{1}-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\cdots+\left(\frac{1}{2019}-\frac{1}{2020}\right)=1-\frac{1}{2020}.$$

▷ [Knuth 1969] Ex. 1.2.6.63:

> *[50] Develop computer programs for simplifying sums that involve binomial coefficients.*

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{2019\cdot 2020}=?$$

▷ [J. Bernoulli 1682]: Use $\frac{1}{k(k+1)}=\frac{1}{k}-\frac{1}{k+1}$ to *create* a *telescoping sum*

$$\left(\frac{1}{1}-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\cdots+\left(\frac{1}{2019}-\frac{1}{2020}\right)=1-\frac{1}{2020}.$$

▷ [Knuth 1969] Ex. 1.2.6.63:

[50] Develop computer programs for simplifying sums that involve binomial coefficients.

▷ Today: how computer algebra uses Bernoulli's 1682 idea –systematically and algorithmically–, to solve Knuth's 1969 exercise, and more

DIAGONALS

Diagonals of multivariate power series

Definition

If F is a multivariate power series

$$F = \sum_{i_1,\dots,i_n \geq 0} a_{i_1,\dots,i_n} x_1^{i_1} \cdots x_n^{i_n},$$

its diagonal is the univariate power series

$$\mathrm{Diag}(F) \stackrel{def}{=} \sum_i a_{i,\dots,i} t^i.$$

Diagonals of multivariate power series

Definition

If F is a multivariate power series

$$F = \sum_{i_1,\dots,i_n \geq 0} a_{i_1,\dots,i_n} x_1^{i_1} \cdots x_n^{i_n},$$

its diagonal is the univariate power series

$$\mathrm{Diag}(F) \stackrel{def}{=} \sum_i a_{i,\dots,i} t^i.$$

Example: if $n = 1$, then (trivially)

$$\mathrm{Diag}\,(F) = F(t).$$

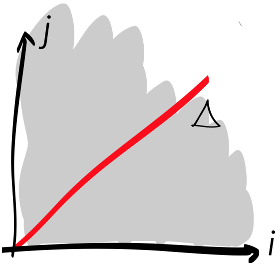
Definition

If F is a multivariate power series

$$F = \sum_{i_1,\dots,i_n \geq 0} a_{i_1,\dots,i_n} x_1^{i_1} \cdots x_n^{i_n},$$

its diagonal is the univariate power series

$$\mathrm{Diag}(F) \stackrel{def}{=} \sum_i a_{i,\dots,i} t^i.$$



Example: if $n = 2$ and $F = \dfrac{1}{1-x-y} = \sum_{i,j\geq 0} \binom{i+j}{i} x^i y^j$, then

$$\mathrm{Diag}\,(F) = \sum_{n\geq 0} \binom{2n}{n} t^n = 1 + 2\,t + 6\,t^2 + 20\,t^3 + 70\,t^4 + \cdots.$$

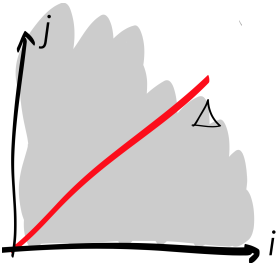
Definition

If F is a multivariate power series

$$F = \sum_{i_1,\dots,i_n \geq 0} a_{i_1,\dots,i_n} x_1^{i_1} \cdots x_n^{i_n},$$

its diagonal is the univariate power series

$$\mathrm{Diag}(F) \stackrel{def}{=} \sum_i a_{i,\dots,i} t^i.$$



Example: if $n = 2$ and $F = \dfrac{1}{1-x-y} = \sum_{i,j \geq 0} \binom{i+j}{i} x^i y^j$, then

$$\mathrm{Diag}\,(F) = \sum_{n \geq 0} \binom{2n}{n} t^n = 1 + 2\,t + 6\,t^2 + 20\,t^3 + 70\,t^4 + \cdots.$$

▷ $\mathrm{Diag}\,(F)$ is not a rational function, even though F is rational.

Pólya's theorem

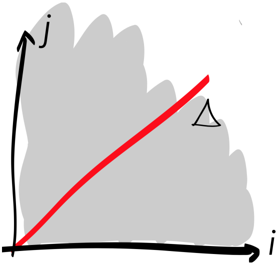
Definition

If F is a multivariate power series

$$F = \sum_{i_1,\dots,i_n \geq 0} a_{i_1,\dots,i_n} x_1^{i_1} \cdots x_n^{i_n},$$

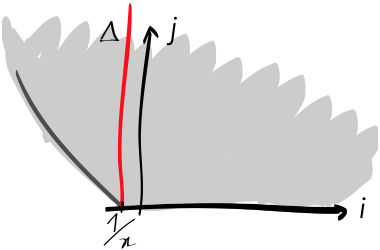
its diagonal is the univariate power series

$$\mathrm{Diag}(F) \stackrel{def}{=} \sum_i a_{i,\dots,i} t^i.$$



Theorem (Pólya, 1922)

Diagonals of bivariate rational functions are algebraic.



Definition

If F is a multivariate power series

$$F = \sum_{i_1,\ldots,i_n \geq 0} a_{i_1,\ldots,i_n} x_1^{i_1} \cdots x_n^{i_n},$$

its diagonal is the univariate power series

$$\operatorname{Diag}(F) \overset{def}{=} \sum_i a_{i,\ldots,i} t^i.$$

Theorem (Pólya, 1922)

Diagonals of bivariate rational functions are algebraic.

Proof: Since $F\left(x, \frac{t}{x}\right) = \sum_{i,j} a_{i,j} x^{i-j} t^j$ we have that $\operatorname{Diag}(F) = [x^0] F\left(x, \frac{t}{x}\right)$.

Therefore, by Cauchy's integral theorem,

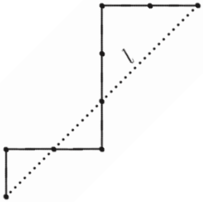
$$\operatorname{Diag}(F) = [x^{-1}] \frac{1}{x} F\left(x, \frac{t}{x}\right) = \frac{1}{2\pi i} \oint_{|x|=\epsilon} F\left(x, \frac{t}{x}\right) \frac{dx}{x}.$$

By the Residue Theorem: last integral is a sum of residues, all algebraic. □

Example: Dyck walks

Let B_n be the number of Dyck bridges, i.e. $\{\mathsf{NE}, \mathsf{SE}\}$-walks of length n in $\mathbb{Z}^2$ starting at $(0,0)$ and ending on the horizontal axis.

Rotating a Dyck bridge



counterclockwise by $\pi/4$

Equivalently, B_n = number of $\{\mathsf{N}, \mathsf{E}\}$-walks in $\mathbb{Z}^2$ from $(0,0)$ to (n,n)

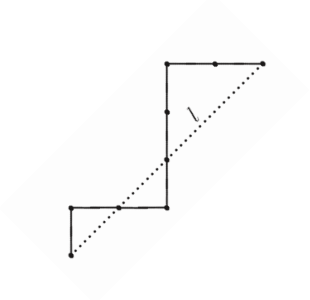
$$\Longrightarrow \qquad B(t) = \sum_{n \geq 0} B_n t^n = \operatorname{Diag}\left(\frac{1}{1-x-y}\right)$$

Then: $$B(t) = \frac{1}{2\pi i}\oint_{|x|=\epsilon} \frac{dx}{x - x^2 - t} = \left.\frac{1}{1-2x}\right|_{x=\frac{1-\sqrt{1-4t}}{2}} = \frac{1}{\sqrt{1-4t}}$$

Example: Dyck walks

Let B_n be the number of Dyck bridges, i.e. $\{\mathsf{NE}, \mathsf{SE}\}$-walks of length n in $\mathbb{Z}^2$ starting at $(0,0)$ and ending on the horizontal axis.

Rotating a Dyck bridge



counterclockwise by $\pi/4$

Equivalently, B_n = number of $\{\mathsf{N}, \mathsf{E}\}$-walks in $\mathbb{Z}^2$ from $(0,0)$ to (n,n) $= \binom{2n}{n}$

$$\Longrightarrow \qquad B(t) = \sum_{n \geq 0} B_n t^n = \operatorname{Diag}\left(\frac{1}{1-x-y}\right)$$

Then: $$B(t) = \frac{1}{2\pi i} \oint_{|x|=\epsilon} \frac{dx}{x - x^2 - t} = \frac{1}{\sqrt{1-4t}} = \sum_{n \geq 0} \binom{2n}{n} t^n$$

Let $A, B \in \mathbb{K}[x]$ be such that $\deg(A) < \deg(B)$, with B squarefree.
In particular, the rational function $F = A/B$ has simple poles only.

Lemma. The residue r_i of F at the pole p_i equals $r_i = \dfrac{A(p_i)}{B'(p_i)}$.

Proof. If $F = \sum_i \dfrac{r_i}{x - p_i}$, then $r_i = (F \cdot (x - p_i))_{|x=p_i} = \dfrac{A(x)}{\prod_{j \neq i}(x - p_j)}(p_i)$

Theorem. The residues r_i of F are roots of the resultant

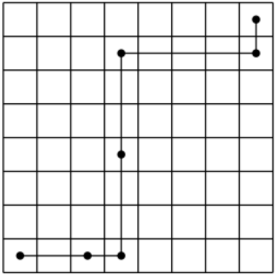
$$R(t) = \mathsf{Res}_{\,x}\big(B(x),\, A(x) - t \cdot B'(x)\big).$$

Proof. By Poisson's formula: $R(t) = \prod_i \Big(A(p_i) - t \cdot B'(p_i)\Big).$ □

▷ Introduced by [Rothstein-Trager 1976] for the (indefinite) integration of rational functions.
▷ Generalized by [Bronstein 1992] to multiple poles.

Example: diagonal Rook paths

Question: A chess Rook can move any number of squares horizontally or vertically in one step. How many paths can a Rook take from the lower-left corner square to the upper-right corner square of an $N \times N$ chessboard? Assume that the Rook moves only right or up at each step.



$1,\ 2,\ 14,\ 106,\ 838,\ 6802,\ 56190,\ 470010,\ \ldots$

Example: diagonal Rook paths

Generating function of the sequence

$$1,\ 2,\ 14,\ 106,\ 838,\ 6802,\ 56190,\ 470010,\ \ldots$$

is

$$\mathsf{Diag}(F) = [x^0]\, F(x, t/x) = \frac{1}{2\pi i} \oint F(x, t/x)\, \frac{dx}{x}, \quad \text{where} \quad F = \frac{1}{1 - \frac{x}{1-x} - \frac{y}{1-y}}.$$

Residue theorem: $\mathsf{Diag}(F)$ is a sum of roots y of the Rothstein-Trager resultant

```
> F:=1/(1-x/(1-x)-y/(1-y)):
> G:=normal(1/x*subs(y=t/x,F)):
> factor(resultant(denom(G),numer(G)-y*diff(denom(G),x),x));
```

$$t^2(1-t)(2y-1)(36ty^2 - 4y^2 + 1 - t)$$

Answer: Generating series of diagonal Rook paths is $\frac{1}{2}\left(1 + \sqrt{\frac{1-t}{1-9t}}\right)$.

Pólya's theorem

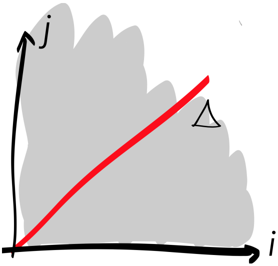
Definition

If F is a formal power series

$$F = \sum_{i_1,\dots,i_n \geq 0} a_{i_1,\dots,i_n} x_1^{i_1} \cdots x_n^{i_n},$$

its diagonal is

$$\mathrm{Diag}(F) \stackrel{def}{=} \sum_i a_{i,\dots,i} t^i.$$



Theorem (Pólya, 1922)

Diagonals of bivariate rational functions are algebraic.[a]

[a] The converse is also true [Furstenberg, 1967]

Pólya's theorem

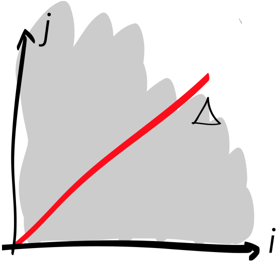
Definition

If F is a formal power series

$$F = \sum_{i_1,\dots,i_n \geq 0} a_{i_1,\dots,i_n} x_1^{i_1} \cdots x_n^{i_n},$$

its diagonal is

$$\mathrm{Diag}(F) \stackrel{def}{=} \sum_i a_{i,\dots,i} t^i.$$



Theorem (Pólya, 1922)

Diagonals of bivariate rational functions are algebraic.

▷ This is false for more than 2 variables. E.g.

$$\mathrm{Diag}\left(\frac{1}{1-x-y-z}\right) = \sum_{n\geq 0} \frac{(3n)!}{n!^3} t^n = {}_2F_1\left(\left.\begin{matrix} \frac{1}{3} & \frac{2}{3} \\ & 1 \end{matrix}\right| 27t\right) \quad \text{is transcendental}$$

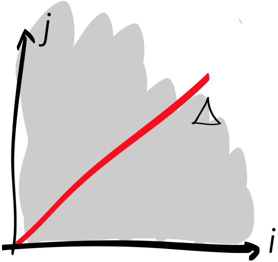
Definition

If F is a formal power series

$$F = \sum_{i_1,\ldots,i_n \geq 0} a_{i_1,\ldots,i_n} x_1^{i_1} \cdots x_n^{i_n},$$

its diagonal is

$$\mathrm{Diag}(F) \stackrel{def}{=} \sum_i a_{i,\ldots,i} t^i.$$



Theorem (Pólya, 1922)

Diagonals of bivariate rational functions are algebraic and thus D-finite.

▷ Algebraic equation has exponential size [B., Dumont, Salvy, 2015]
▷ Differential equation has polynomial size [B., Chen, Chyzak, Li, 2010]

Lipshitz's theorem

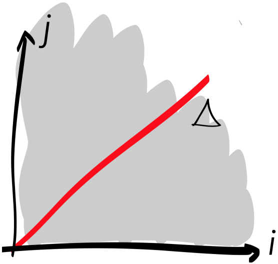
Definition

If F is a formal power series

$$F = \sum_{i_1,\ldots,i_n \geq 0} a_{i_1,\ldots,i_n} x_1^{i_1} \cdots x_n^{i_n},$$

its diagonal is

$$\mathrm{Diag}(F) \stackrel{def}{=} \sum_i a_{i,\ldots,i} t^i.$$



Theorem (Lipshitz, 1988)

Diagonals of multivariate rational functions are D-finite.

Example: Diagonal Rook paths on a 3D chessboard

Question [Erickson 2010]
How many ways can a Rook move from $(0,0,0)$ to (N,N,N), where each step is a positive integer multiple of $(1,0,0), (0,1,0)$, or $(0,0,1)$?

1, 6, 222, 9918, 486924, 25267236, 1359631776, 75059524392, . . .

Answer [B., Chyzak, van Hoeij, Pech, 2011]: GF of 3D diagonal Rook paths is

$$G(t) = 1 + 6 \cdot \int_0^t \frac{{}_2F_1\left(\begin{matrix} 1/3 & 2/3 \\ & 2 \end{matrix} \middle| \frac{27x(2-3x)}{(1-4x)^3}\right)}{(1-4x)(1-64x)}\, dx$$

Proof of Lipshitz's theorem on the 3D Rooks example

Problem: Show that $\mathrm{Diag}(F)$ is D-finite, where $F(x,y,z)$ is

$$\left(1-\sum_{n\geq 1} x^n - \sum_{n\geq 1} y^n - \sum_{n\geq 1} z^n\right)^{-1} = \frac{(1-x)(1-y)(1-z)}{1-2(x+y+z)+3(xy+yz+zx)-4xyz}$$

Proof of Lipshitz's theorem on the 3D Rooks example

Problem: Show that $\mathrm{Diag}(F)$ is D-finite, where $F(x,y,z)$ is

$$\left(1-\sum_{n\geq 1} x^n - \sum_{n\geq 1} y^n - \sum_{n\geq 1} z^n\right)^{-1} = \frac{(1-x)(1-y)(1-z)}{1-2(x+y+z)+3(xy+yz+zx)-4xyz}$$

Idea: If one is able to find a nonzero differential operator of the form

$$L(t,\partial_t,\partial_x,\partial_y) = P(t,\partial_t) + (\text{ higher-order terms in } \partial_x \text{ and } \partial_y\ )$$

that annihilates $G = \frac{1}{xy} \cdot F\left(x, \frac{y}{x}, \frac{t}{y}\right)$, then $P(t,\partial_t)$ annihilates $\mathrm{Diag}(F)$.

Proof of Lipshitz's theorem on the 3D Rooks example

Problem: Show that $\operatorname{Diag}(F)$ is D-finite, where $F(x,y,z)$ is

$$\left(1-\sum_{n\geq 1} x^n - \sum_{n\geq 1} y^n - \sum_{n\geq 1} z^n\right)^{-1} = \frac{(1-x)(1-y)(1-z)}{1-2(x+y+z)+3(xy+yz+zx)-4xyz}$$

Idea: If one is able to find a nonzero differential operator of the form

$$L(t,\partial_t,\partial_x,\partial_y) = P(t,\partial_t) + (\text{ higher-order terms in } \partial_x \text{ and } \partial_y\ )$$

that annihilates $G = \frac{1}{xy}\cdot F\left(x,\frac{y}{x},\frac{t}{y}\right)$, then $P(t,\partial_t)$ annihilates $\operatorname{Diag}(F)$.

Proof:

1. $\operatorname{Diag}(F) = [x^0y^0]\,F\left(x,\frac{y}{x},\frac{t}{y}\right)$
2. $0 = L(G) = P(G) + \partial_x(\cdot) + \partial_y(\cdot)$
3. $0 = [x^{-1}y^{-1}]L(G) = [x^{-1}y^{-1}]P(G) = P([x^{-1}y^{-1}]G) = P(\operatorname{Diag}(F))$

Proof of Lipshitz's theorem on the 3D Rooks example

Problem: Show that $\mathrm{Diag}(F)$ is D-finite, where $F(x,y,z)$ is

$$\left(1-\sum_{n\geq 1} x^n - \sum_{n\geq 1} y^n - \sum_{n\geq 1} z^n\right)^{-1} = \frac{(1-x)(1-y)(1-z)}{1-2(x+y+z)+3(xy+yz+zx)-4xyz}$$

Idea: If one is able to find a nonzero differential operator of the form

$$L(t,\partial_t,\partial_x,\partial_y) = P(t,\partial_t) + (\text{ higher-order terms in } \partial_x \text{ and } \partial_y\ )$$

that annihilates $G = \frac{1}{xy} \cdot F\left(x, \frac{y}{x}, \frac{t}{y}\right)$, then $P(t,\partial_t)$ annihilates $\mathrm{Diag}(F)$.

▷ Remaining task: Show that such an L does exist.

Proof of Lipshitz's theorem on the 3D Rooks example

Counting argument: By Leibniz's rule, the $\binom{N+4}{4}$ rational functions

$$t^i \partial_t^j \partial_x^k \partial_y^\ell (G), \quad 0 \leq i + j + k + \ell \leq N$$

are contained in the $\mathbb{Q}$-vector space of dimension $\leq 18(N+1)^3$ spanned by

$$\frac{t^i x^j y^k}{\mathrm{denom}(G)^{N+1}}, \quad 0 \leq i \leq 2N+1,\ 0 \leq j \leq 3N+2,\ 0 \leq k \leq 3N+2.$$

Proof of Lipshitz's theorem on the 3D Rooks example

Counting argument: By Leibniz's rule, the $\binom{N+4}{4}$ rational functions

$$t^i \partial_t^j \partial_x^k \partial_y^\ell (G), \quad 0 \le i + j + k + \ell \le N$$

are contained in the $\mathbb{Q}$-vector space of dimension $\le 18(N+1)^3$ spanned by

$$\frac{t^i x^j y^k}{\mathrm{denom}(G)^{N+1}}, \quad 0 \le i \le 2N+1,\ 0 \le j \le 3N+2,\ 0 \le k \le 3N+2.$$

▷ If N is such that # unknowns $= \binom{N+4}{4} > 18(N+1)^3 =$ # equations, then there exists $L(t, \partial_t, \partial_x, \partial_y)$ of total degree at most N, such that $LG = 0$.

Proof of Lipshitz's theorem on the 3D Rooks example

Counting argument: By Leibniz's rule, the $\binom{N+4}{4}$ rational functions

$$t^i \partial_t^j \partial_x^k \partial_y^\ell (G), \quad 0 \leq i + j + k + \ell \leq N$$

are contained in the $\mathbb{Q}$-vector space of dimension $\leq 18(N+1)^3$ spanned by

$$\frac{t^i x^j y^k}{\operatorname{denom}(G)^{N+1}}, \quad 0 \leq i \leq 2N+1,\ 0 \leq j \leq 3N+2,\ 0 \leq k \leq 3N+2.$$

▷ If N is such that # unknowns $= \binom{N+4}{4} > 18(N+1)^3 =$ # equations, then there exists $P(t, \partial_t)$ of total degree at most N, such that $P(\operatorname{Diag}(F)) = 0$.

Counting argument: By Leibniz's rule, the $\binom{N+4}{4}$ rational functions

$$t^i \partial_t^j \partial_x^k \partial_y^\ell (G), \quad 0 \le i+j+k+\ell \le N$$

are contained in the $\mathbb{Q}$-vector space of dimension $\le 18(N+1)^3$ spanned by

$$\frac{t^i x^j y^k}{\operatorname{denom}(G)^{N+1}}, \quad 0 \le i \le 2N+1,\ 0 \le j \le 3N+2,\ 0 \le k \le 3N+2.$$

▷ If N is such that # unknowns $= \binom{N+4}{4} > 18(N+1)^3 =$ # equations, then there exists $P(t, \partial_t)$ of total degree at most N, such that $P(\operatorname{Diag}(F)) = 0$.

▷ $N = 425$ is the smallest integer satisfying $\binom{N+4}{4} > 18(N+1)^3$ (!)

Proof of Lipshitz's theorem on the 3D Rooks example

Counting argument: By Leibniz's rule, the $\binom{N+4}{4}$ rational functions

$$t^i \partial_t^j \partial_x^k \partial_y^\ell (G), \quad 0 \leq i+j+k+\ell \leq N$$

are contained in the $\mathbb{Q}$-vector space of dimension $\leq 18(N+1)^3$ spanned by

$$\frac{t^i x^j y^k}{\operatorname{denom}(G)^{N+1}}, \quad 0 \leq i \leq 2N+1,\ 0 \leq j \leq 3N+2,\ 0 \leq k \leq 3N+2.$$

▷ If N is such that # unknowns $= \binom{N+4}{4} > 18(N+1)^3 =$ # equations, then there exists $P(t, \partial_t)$ of total degree at most N, such that $P(\operatorname{Diag}(F)) = 0$.

▷ $N = 425$ is the smallest integer satisfying $\binom{N+4}{4} > 18(N+1)^3$ (!)

▷ Finding the operator P by Lipshitz' argument would require solving a linear system with 1,391,641,251 unknowns and 1,391,557,968 equations (!)

Proof of Lipshitz's theorem on the 3D Rooks example

Counting argument: By Leibniz's rule, the $\binom{N+4}{4}$ rational functions

$$t^i \partial_t^j \partial_x^k \partial_y^\ell (G), \quad 0 \leq i+j+k+\ell \leq N$$

are contained in the $\mathbb{Q}$-vector space of dimension $\leq 18(N+1)^3$ spanned by

$$\frac{t^i x^j y^k}{\operatorname{denom}(G)^{N+1}}, \quad 0 \leq i \leq 2N+1,\ 0 \leq j \leq 3N+2,\ 0 \leq k \leq 3N+2.$$

▷ If N is such that # unknowns $= \binom{N+4}{4} > 18(N+1)^3 =$ # equations, then there exists $P(t, \partial_t)$ of total degree at most N, such that $P(\operatorname{Diag}(F)) = 0$.

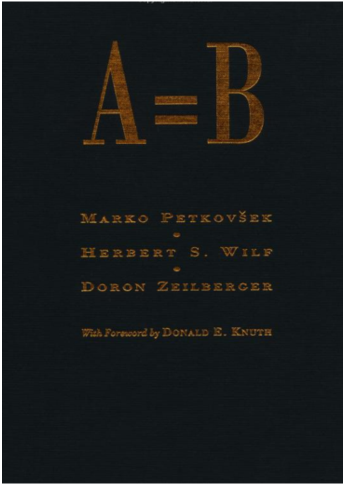
▷ $N = 425$ is the smallest integer satisfying $\binom{N+4}{4} > 18(N+1)^3$ (!)

▷ Finding the operator P by Lipshitz' argument would require solving a linear system with 1,391,641,251 unknowns and 1,391,557,968 equations (!)

▷ A better solution is provided by creative telescoping.

CREATIVE TELESCOPING

General framework in computer algebra –initiated by Zeilberger in the ’90s– for proving identities on multiple integrals and sums with parameters.



Examples I: hypergeometric summation

- $$\sum_{k\in\mathbb{Z}} (-1)^k \binom{a+b}{a+k}\binom{b+c}{b+k}\binom{c+a}{c+k} = \frac{(a+b+c)!}{a!b!c!}$$ [Dixon 1903]

- $A_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2$ satisfies the recurrence [Apéry 1978]:

$$(n+1)^3 A_{n+1} = (34n^3 + 51n^2 + 27n + 5) A_n - n^3 A_{n-1}.$$

(*Neither Cohen nor I had been able to prove this in the intervening two months* [Van der Poorten 1979])

- $$\sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 = \sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k} \sum_{j=0}^{k} \binom{k}{j}^3$$ [Strehl 1992]

Examples I: hypergeometric summation

- $\displaystyle\sum_{k\in\mathbb{Z}} (-1)^k \binom{a+b}{a+k}\binom{b+c}{b+k}\binom{c+a}{c+k} = \frac{(a+b+c)!}{a!b!c!}$ [Dixon 1903]

- $\displaystyle A_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2$ satisfies the recurrence [Apéry 1978]:

$$(n+1)^3 A_{n+1} = (34n^3 + 51n^2 + 27n + 5)A_n - n^3 A_{n-1}.$$

(*The specific problem was mentioned to Don Zagier, who solved it with irritating speed* [Van der Poorten 1979])

- $\displaystyle\sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 = \sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k} \sum_{j=0}^{k} \binom{k}{j}^3$ [Strehl 1992]

Examples II: Integrals and Diagonals

- $\displaystyle \int_0^1 \frac{\cos(zu)}{\sqrt{1-u^2}}\,du = \int_1^{+\infty} \frac{\sin(zu)}{\sqrt{u^2-1}}\,du = \frac{\pi}{2} J_0(z)$

- $\displaystyle \frac{1}{2\pi i}\oint \frac{(1+2xy+4y^2)\exp\left(\frac{4x^2y^2}{1+4y^2}\right)}{y^{n+1}(1+4y^2)^{\frac{3}{2}}}\,dy = \frac{H_n(x)}{\lfloor n/2\rfloor !}$ [Doetsch 1930]

- $\displaystyle \int_0^{+\infty} x J_1(ax) I_1(ax) Y_0(x) K_0(x)\,dx = -\frac{\ln(1-a^4)}{2\pi a^2}$ [Glasser-Montaldi'94]

- $\displaystyle \mathrm{Diag}\,\frac{1}{(1-x-y)(1-z-u)-xyzu} = \sum_{n\geq 0} A_n t^n$ [Straub 2014].

$$I_n := \sum_{k=0}^{n} \binom{n}{k} = 2^n.$$

Principle: **IF** one knows Pascal's triangle:

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1} = 2\binom{n}{k} + \binom{n}{k-1} - \binom{n}{k},$$

then summing over k telescopes and yields

$$I_{n+1} = 2I_n.$$

The initial condition $I_0 = 1$ concludes the proof.

$$F_n = \sum_k u_{n,k} =?$$

IF one knows $P(n, S_n)$ (telescoper) and $R(n, k, S_n, S_k)$ (certificate) such that

$$(P(n, S_n) + \Delta_k R(n, k, S_n, S_k)) \cdot u_{n,k} = 0$$

(where Δ_k is the difference operator, $\Delta_k \cdot v_{n,k} = v_{n,k+1} - v_{n,k}$),
then the sum "telescopes", leading to

$$P(n, S_n) \cdot F_n = 0.$$

Zeilberger's Algorithm [1990]

Input: a hypergeometric term $u_{n,k}$, i.e., $\frac{u_{n+1,k}}{u_{n,k}}$ and $\frac{u_{n,k+1}}{u_{n,k}}$ are in $\mathbb{Q}(n,k)$

Output:

- a linear recurrence, called telescoper, (P) satisfied by $F_n = \sum_k u_{n,k}$
- a certificate (Q), for checking the result: $P(n, S_n) \cdot u_{n,k} = \Delta_k Q \cdot u_{n,k}$.

Zeilberger's Algorithm [1990]

Input: a hypergeometric term $u_{n,k}$, i.e., $\frac{u_{n+1,k}}{u_{n,k}}$ and $\frac{u_{n,k+1}}{u_{n,k}}$ are in $\mathbb{Q}(n,k)$

Output:

- a linear recurrence, called telescoper, (P) satisfied by $F_n = \sum_k u_{n,k}$
- a certificate (Q), for checking the result: $P(n, S_n) \cdot u_{n,k} = \Delta_k Q \cdot u_{n,k}$.

```
> T := binomial(n,k);
> Zpair:=SumTools[Hypergeometric][Zeilberger](T,n,k,Sn):
> tel:=Zpair[1];
```

$$S_n - 2$$

Zeilberger's Algorithm [1990]

Input: a hypergeometric term $u_{n,k}$, i.e., $\frac{u_{n+1,k}}{u_{n,k}}$ and $\frac{u_{n,k+1}}{u_{n,k}}$ are in $\mathbb{Q}(n,k)$

Output:

- a linear recurrence, called telescoper, (P) satisfied by $F_n = \sum_k u_{n,k}$
- a certificate (Q), for checking the result: $P(n, S_n) \cdot u_{n,k} = \Delta_k Q \cdot u_{n,k}$.

```
> T := binomial(n,k);
> Zpair:=SumTools[Hypergeometric][Zeilberger](T,n,k,Sn):
> tel:=Zpair[1];
```

$$S_n - 2$$

▷ This is a proof that $I_n := \sum_{k=0}^{n} \binom{n}{k}$ satisfies $I_{n+1} = 2 \cdot I_n$.

Zeilberger's Algorithm [1990]

Input: a hypergeometric term $u_{n,k}$, i.e., $\frac{u_{n+1,k}}{u_{n,k}}$ and $\frac{u_{n,k+1}}{u_{n,k}}$ are in $\mathbb{Q}(n,k)$

Output:

- a linear recurrence, called telescoper, (P) satisfied by $F_n = \sum_k u_{n,k}$
- a certificate (Q), for checking the result: $P(n, S_n) \cdot u_{n,k} = \Delta_k Q \cdot u_{n,k}$.

```
> T := binomial(n,k);
> Zpair:=SumTools[Hypergeometric][Zeilberger](T,n,k,Sn):
> tel:=Zpair[1];
```

$$S_n - 2$$

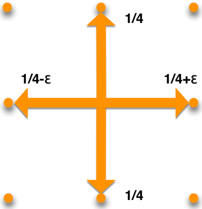
▷ This is a proof that $I_n := \sum_{k=0}^{n} \binom{n}{k}$ satisfies $I_{n+1} = 2 \cdot I_n$.

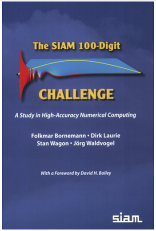
▷ Can check using the certificate:

```
> cert:=Zpair[2];
> iszero:=(subs(n=n+1,T) - 2*T) - (subs(k=k+1,cert) - cert);
> simplify(convert(%,GAMMA));
```

$$0$$

Example: from the SIAM challenges





$$U_{n,k} := \binom{2n}{2k}\binom{2k}{k}\binom{2n-2k}{n-k}\left(\frac{1}{4}+c\right)^k\left(\frac{1}{4}-c\right)^k\frac{1}{4^{2n-2k}},$$

$$p_n = \sum_{k=0}^{n} U_{n,k} \quad = \text{probability of return to } (0,0) \text{ at step } 2n.$$

```
> p:=SumTools[Hypergeometric][Zeilberger](U,n,k,Sn);
```

$$[\left(4\,n^2+16\,n+16\right)Sn^2+\left(-4\,n^2+32\,c^2n^2+96\,c^2n-12\,n+72\,c^2-9\right)Sn$$
$$+128\,c^4n+64\,c^4n^2+48\,c^4,\ \text{...(BIG certificate)...}]$$

Creative Telescoping for Integrals

$$I(t) = \oint_\gamma H(t,x)\,dx = ?$$

IF one knows $P(t,\partial_t)$ (telescoper) and $Q(t,x,\partial_t,\partial_x)$ (certificate) such that

$$\big(P(t,\partial_t) + \partial_x Q(t,x,\partial_t,\partial_x)\big)\cdot H(t,x) = 0,$$

then the integral "telescopes", leading to

$$P(t,\partial_t)\cdot I(t) = 0.$$

The Almkvist-Zeilberger Algorithm [1990]

Input: a hyperexponential function $H(t,x)$, i.e., $\frac{\frac{\partial H}{\partial t}}{H}$ and $\frac{\frac{\partial H}{\partial x}}{H}$ are in $\mathbb{Q}(t,x)$
Output:

- a linear differential operator $P(t,\partial_t)$ satisfied by $I(t) = \oint_\gamma H(t,x)\,dx$
- a $G(t,x) \in \mathbb{Q}(t,x)$ such that $P(t,\partial_t) \cdot H(t,x) = \frac{\partial}{\partial x}\Big(G(t,x) \cdot H(t,x)\Big)$.

The Almkvist-Zeilberger Algorithm [1990]

Input: a hyperexponential function $H(t,x)$, i.e., $\frac{\frac{\partial H}{\partial t}}{H}$ and $\frac{\frac{\partial H}{\partial x}}{H}$ are in $\mathbb{Q}(t,x)$
Output:

- a linear differential operator $P(t,\partial_t)$ satisfied by $I(t) = \oint_\gamma H(t,x)\,dx$
- a $G(t,x) \in \mathbb{Q}(t,x)$ such that $P(t,\partial_t) \cdot H(t,x) = \frac{\partial}{\partial x}\Big(G(t,x) \cdot H(t,x)\Big)$.

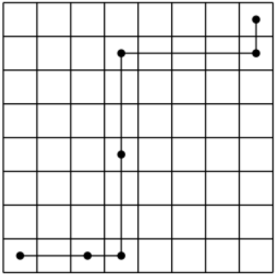
Algorithm: Write $\mathbb{L} = \mathbb{Q}(t)$. For $r = 0, 1, 2, \ldots$ do

1. compute $a(x) := \frac{\frac{\partial H}{\partial x}}{H} \in \mathbb{L}(x)$ and $b_k(x) := \frac{\frac{\partial^k H}{\partial t^k}}{H} \in \mathbb{L}(x)$ for $k = 0, \ldots, r$
2. decide whether the inhomogeneous parametrized LDE

$$\frac{\partial G}{\partial x} + a(x) \cdot G = \sum_{k=0}^{r} c_k \cdot b_k(x)$$

   admits a rational solution $G \in \mathbb{L}(x)$, for some $c_0, \ldots, c_r \in \mathbb{L}$ not all zero
3. if so, then return $P := \sum_{k=0}^{r} c_k \partial_t^k$ and G; else increase r by 1 and repeat

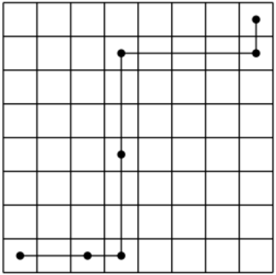
Question: A chess Rook can move any number of squares horizontally or vertically in one step. How many paths can a Rook take from the lower-left corner square to the upper-right corner square of an $N \times N$ chessboard? Assume that the Rook moves only right or up at each step.



$$(r_n)_{n\geq 0}: \quad 1,\ 2,\ 14,\ 106,\ 838,\ 6802,\ 56190,\ 470010,\ \ldots$$

Example: Diagonal Rook paths

Question: A chess Rook can move any number of squares horizontally or vertically in one step. How many paths can a Rook take from the lower-left corner square to the upper-right corner square of an $N \times N$ chessboard? Assume that the Rook moves only right or up at each step.



$$(r_n)_{n\geq 0}: \quad 1,\ 2,\ 14,\ 106,\ 838,\ 6802,\ 56190,\ 470010,\ \ldots$$

Answer: r_N = Nth coefficient in the Taylor expansion of $\displaystyle \frac{1}{2}\left(1+\sqrt{\frac{1-x}{1-9x}}\right)$.

Generating function of the sequence

$$1,\ 2,\ 14,\ 106,\ 838,\ 6802,\ 56190,\ 470010,\ \ldots$$

is

$$\mathsf{Diag}(F) = [x^0]\, F(x,t/x) = \frac{1}{2\pi i} \oint F(x,t/x)\, \frac{dx}{x}, \quad \text{where} \quad F = \frac{1}{1 - \frac{x}{1-x} - \frac{y}{1-y}}.$$

Creative telescoping computes a differential equation satisfied by $\mathsf{Diag}(F)$:

```
> F:=1/(1-x/(1-x)-y/(1-y)):
> G:=normal(1/x*subs(y=t/x,F)):
> Zeilberger(G, t, x, Dt)[1];
```

$$(9t^2 - 10t + 1)\partial_t^2 + (18t - 14)\partial_t$$

Conclusion: Generating series of diagonal Rook paths is $\displaystyle \frac{1}{2}\left(1 + \sqrt{\frac{1-t}{1-9t}}\right).$

Creative Telescoping for multiple rational integrals

Problem:

$$\left.\begin{array}{rl} \mathbf{x} = x_1, \ldots, x_n & \text{— integration variables} \\ t & \text{— parameter} \\ H(t, \mathbf{x}) & \text{— rational function} \\ \gamma & \text{— } n\text{-cycle in } \mathbb{C}^n \end{array}\right\} \oint_\gamma H(t, \mathbf{x}) \mathrm{d}\mathbf{x}$$

Principle of creative telescoping

$$\underbrace{\sum_{k=0}^{r} c_k(t) \frac{\partial^k H}{\partial t^k} = \overbrace{\sum_{i=1}^{n} \frac{\partial A_i}{\partial x_i}}^{\text{certificate}}}_{\text{telescopic relation}} \implies \overbrace{\left(\sum_{k=0}^{r} c_k(t) \partial_t^k \right)}^{\text{telescoper}} \cdot \oint_\gamma H \mathrm{d}\mathbf{x} = 0$$

Task:

1. find the $c_k(t)$ which satisfy a telescopic relation,
2. ideally, without computing the certificate (A_i).

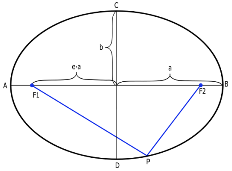
Example: Perimeter of an ellipse

Example [Euler, 1733]: Perimeter of an ellipse of eccentricity e, semi-major axis 1

Example: Perimeter of an ellipse

Example [Euler, 1733]: Perimeter of an ellipse of eccentricity e, semi-major axis 1

$$p(e) = 4\int_0^1 \sqrt{\frac{1-e^2u^2}{1-u^2}}\mathrm{d}u = 4 \oint\!\!\oint \frac{\mathrm{d}u\mathrm{d}v}{1-\frac{1-e^2u^2}{(1-u^2)v^2}}$$

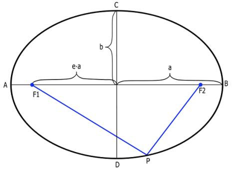


Principle: Find algorithmically

Example: Perimeter of an ellipse

Example [Euler, 1733]: Perimeter of an ellipse of eccentricity e, semi-major axis 1

$$p(e) = 4\int_0^1 \sqrt{\frac{1-e^2u^2}{1-u^2}}\,\mathrm{d}u = 4\oint\!\!\oint \frac{\mathrm{d}u\mathrm{d}v}{1-\frac{1-e^2u^2}{(1-u^2)v^2}}$$



Principle: Find algorithmically

$$\left((e-e^3)\partial_e^2 + (1-e^2)\partial_e + e\right)\cdot\left(\frac{1}{1-\frac{1-e^2u^2}{(1-u^2)v^2}}\right) =$$

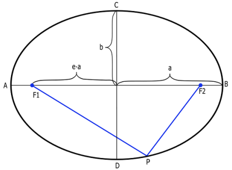
$$\partial_u\left(-\frac{e(-1-u+u^2+u^3)v^2(-3+2u+v^2+u^2(-2+3e^2-v^2))}{(-1+v^2+u^2(e^2-v^2))^2}\right)$$

$$+\,\partial_v\left(\frac{2e(-1+e^2)u(1+u^3)v^3}{(-1+v^2+u^2(e^2-v^2))^2}\right)$$

▷ Conclusion: $p(e) = \frac{\pi}{2}\cdot {}_2F_1\left(\begin{matrix}-\frac{1}{2} & \frac{1}{2}\\ & 1\end{matrix}\,\middle|\, e^2\right) = 2\pi - \frac{\pi}{2}e^2 - \frac{3\pi}{32}e^4 - \cdots.$

Example [Euler, 1733]: Perimeter of an ellipse of eccentricity e, semi-major axis 1

$$p(e) = 4\int_0^1 \sqrt{\frac{1-e^2u^2}{1-u^2}}\,\mathrm{d}u = 4\oint\!\!\oint \frac{\mathrm{d}u\mathrm{d}v}{1-\frac{1-e^2u^2}{(1-u^2)v^2}}$$



Principle: Find algorithmically

$$\left((e-e^3)\partial_e^2 + (1-e^2)\partial_e + e\right)\cdot\left(\frac{1}{1-\frac{1-e^2u^2}{(1-u^2)v^2}}\right) =$$

$$\partial_u\left(-\frac{e\left(-1-u+u^2+u^3\right)v^2\left(-3+2u+v^2+u^2\left(-2+3e^2-v^2\right)\right)}{\left(-1+v^2+u^2(e^2-v^2)\right)^2}\right)$$

$$+\,\partial_v\left(\frac{2e\left(-1+e^2\right)u\left(1+u^3\right)v^3}{\left(-1+v^2+u^2(e^2-v^2)\right)^2}\right)$$

▷ Drawback: Size(certificate) $\gg$ Size(telescoper).