Creative Telescoping
—From 1G to 4G algorithms—

Alin Bostan

ENS Lyon
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Several generations of Creative Telescoping algorithms

- 3G, combines 1G + 2G + linear algebra: [Apagodu, Zeilberger, 2005], [Koutschan 2010], [Chen, Kauers 2012], [Chen, Kauers, Koutschan 2014]

▷ Advantages:
  - 1G–3G: very general algorithms;
  - 2G/3G algorithms are able to solve non-trivial problems.

▷ Drawbacks:
  - 1G: slow;
  - 2G: bad or unknown complexity;
  - 1G and 3G: non-minimality of telescopers;
  - 1G–3G: all compute (big) certificates.
Several generations of Creative Telescoping algorithms

4G: roots in [Ostrogradsky, 1845], [Hermite, 1872] and [Picard, 1902].

- univariate:
  - rational $\int$: [B., Chen, Chyzak, Li, 2010];
  - hyperexponential $\int$: [B., Chen, Chyzak, Li, Xin, 2013]
  - hypergeometric $\sum$: [Chen, Huang, Kauers, Li, 2015], [Huang, 2016]
  - mixed $\int + \sum$: [B., Dumont, Salvy, 2016]
  - algebraic $\int$: [Chen, Kauers, Koutschan, 2016]
  - D-finite Fuchsian $\int$: [Chen, van Hoeij, Kauers, Koutschan, 2018]
  - D-finite $\int$: [B., Chyzak, Lairez, Salvy, 2018], [van der Hoeven, 2018]

- multiple:
  - rational bivariate $\mathcal{f}$: [Chen, Kauers, Singer, 2012]
  - rational: [B., Lairez, Salvy, 2013], [Lairez 2016]
  - binomial sums: [B., Lairez, Salvy, 2017]

▷ Advantages:
  - complexity;
  - minimality of telescopers;
  - does not need to compute certificates;
  - fast in practice,

▷ Drawback: not (yet) as general as 1G–3G algorithms.
Input: a hyperexponential function $H(t,x)$, i.e., $\frac{\partial H}{\partial t}$ and $\frac{\partial H}{\partial x}$ are in $\mathbb{Q}(t,x)$

Output:

- a linear differential operator $P(t, \partial_t)$ satisfied by $I(t) = \int_{\gamma} H(t,x) \, dx$
- a $G(t,x) \in \mathbb{Q}(t,x)$ such that $P(t, \partial_t) \cdot H(t,x) = \frac{\partial}{\partial x} \left( G(t,x) \cdot H(t,x) \right)$. 
2G: The Almkvist-Zeilberger Algorithm [1990]

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Algorithm: Write $\mathcal{L} = Q(t)$. For $r = 0, 1, 2, \ldots$ do

1. compute $a(x) := \frac{\partial H}{\partial x} \in \mathcal{L}(x)$ and $b_k(x) := \frac{\partial^k H}{\partial t^k} \in \mathcal{L}(x)$ for $k = 0, \ldots, r$
2. rational solving: decide whether the inhomogeneous parametrized LDE

\[
\frac{\partial G}{\partial x} + a(x) \cdot G = \sum_{k=0}^{r} c_k \cdot b_k(x)
\]

admits a rational solution $G \in \mathcal{L}(x)$, for some $c_0, \ldots, c_r \in \mathcal{L}$ not all zero
3. if so, then return $P := \sum_{k=0}^{r} c_k \partial_t^k$ and $G$; else increase $r$ by 1 and repeat
Problem: Given $H = P/Q \in \mathbb{K}(t,x)$ compute $\int_{\gamma} H(t,x) \, dx$.
Problem: Given $H = P/Q \in K(t,x)$ compute $\int_H H(t,x) \, dx$

Hermite reduction: $H$ can be written in reduced form

$$H = \partial_x(g) + \frac{a}{Q^*},$$

where $Q^*$ is the squarefree part of $Q$ and $\deg_x(a) < d^* := \deg_x(Q^*)$. 
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Algorithm [B., Chen, Chyzak, Li, 2010]
(1) For $i = 0, 1, \ldots, d^*$ compute Hermite reduction of $\partial_t^i(H)$:

$$\partial_t^i(H) = \partial_x(g_i) + \frac{a_i}{Q^*}, \quad \deg_x(a_i) < d^*$$
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2. Find the first linear relation over $\mathbb{K}(t)$ of the form $\sum_{k=0}^r c_k a_k = 0$. 
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$\blacktriangleright L = \sum_{k=0}^r c_k \partial_t^k$ is a telescoper (and $\sum_{k=0}^r c_k g_k$ the corresponding certificate).
Compute a telescopr for the diagonal of the rational power series

\[
\frac{1}{1 - x - y} = \sum_{i,j \geq 0} \binom{i+j}{i} x^i y^j
\]

in three different ways:

1. using the 2G (Almkvist-Zeilberger) creative telescoping algorithm;
2. using the 4G (Hermite reduction-based) creative telescoping algorithm;
3. using the 4G (Griffiths-Dwork reduction-based) creative telescoping algorithm.
An exercise from a past lecture

Let \((a_n)_{n \geq 0}\) be a sequence with \(a_0 = a_1 = 1\) satisfying the recurrence

\[(n + 3)a_{n+1} = (2n + 3)a_n + 3na_{n-1}, \quad \text{for all } n > 0.\]

Show that \(a_n\) is an integer for all \(n\).
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Show that \(a_n\) is an integer for all \(n\).

Follow the next steps:

1. Compute the first 5 terms of the sequence, \(a_0, \ldots, a_4\);
2. Determine a Hermite-Padé approximant of type \((0, 1, 2)\) for \((1, f, f^2)\), where \(f = \sum_n a_n x^n\);
3. Deduce that \(P(x, f(x)) = 0\) mod \(x^5\) for \(P(x, y) := 1 + (x - 1)y + x^2y^2\);
4. Show that the equation \(P(x, y) = 0\) admits a root \(y = g(x) \in \mathbb{Q}[[x]]\) whose coefficients satisfy the same linear recurrence as \((a_n)_{n \geq 0}\);
5. Deduce that \(a_{n+2} = a_{n+1} + \sum_{k=0}^{n} a_k \cdot a_{n-k}\) for all \(n\), and conclude.
1. Let’s compute the first 5 terms of the sequence:

```maple
> rec:=(n+3)*a(n+1)-(2*n+3)*a(n)-3*n*a(n-1): ini:=a(0)=1,a(1)=1:
> pro:=gfun:-rectoproc({rec,ini}, a(n), list);
> pro(4);

[1, 1, 2, 4, 9]
```
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2. Determine a Hermite-Padé approximant of type (0, 1, 2) for \((1, f, f^2)\):

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\begin{align*}
> & \text{f:=listtoser}(\text{pro}(4), x): \\
> & \text{numapprox[hermite_pade]}([1, f, f^2], x, [0, 1, 2]);
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\[-1, -x + 1, -x^2\]
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3. We guessed $P(x,f(x)) = 0 \mod x^5$ for $P(x,y) = 1 + (x - 1)y + x^2y^2$. 
4. The equation \( P(x, y) = 0 \) admits a root \( y = g(x) = \sum_{n \geq 0} g_n x^n \) in \( \mathbb{Q}[[x]] \):

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g(x) = \frac{1 - x - \sqrt{1 - 2x - 3x^2}}{2x^2} = 1 + x + 2x^2 + 4x^3 + 9x^4 + \cdots
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We show that \((g_n)_{n \geq 0}\) satisfies the same linear recurrence as \((a_n)_{n \geq 0}\). Let

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h(x) := \sqrt{1 - 2x - 3x^2} = 1 - x - 2x^2 \cdot g(x).
\]

Clearly, \(h_0 = 1\), \(h_1 = -1\), \(h_{n+2} = -2g_n\) for \(n \geq 0\), and

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\frac{h'(x)}{h(x)} = \frac{3x + 1}{3x^2 + 2x - 1}
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Hence, $(h_n)_n$ and $(g_n)_n$ satisfy the recurrences

$$(3n - 3) h_n + (2n + 1) h_{n+1} - (n + 2) h_{n+2} = 0,$$

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Conclusion: $a_n = g_n$ is an integer for all $n$. □
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\{ (3n + 3) g_n + (2n + 5) g_{n+1} - (n + 4) g_{n+2} = 0, \ g_0 = 1, \ g_1 = 1 \} 
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\[ ((3n + 3) g_n + (2n + 5) g_{n+1} - (n + 4) g_{n+2} = 0, g_0 = 1, g_1 = 1) \]

Conclusion: $a_n = g_n$ is an integer for all $n$. \qed