Creative Telescoping
—From 1G to 4G algorithms—

Alin Bostan

informatics mathematics Inria

ENS Lyon

M2, CR06

October 14, 2020

Several generations of Creative Telescoping algorithms

- 1G, *brutal elimination*: [Fasenmyer, 1947], [Lipshitz, 1988], [Zeilberger, 1990], [Takayama, 1990], [Wilf, Zeilberger, 1990], [Chyzak, Salvy, 2000]
- 2G, *linear diff/rec rational solving*: [Zeilberger, 1990], [Zeilberger, 1991], [Almkvist, Zeilberger, 1990], [Chyzak, 2000], [Koutschan, 2010]
- 3G, combines *1G + 2G + linear algebra*: [Apagodu, Zeilberger, 2005], [Koutschan 2010], [Chen, Kauers 2012], [Chen, Kauers, Koutschan 2014]

▷ Advantages:

- 1G–3G: very general algorithms;
- 2G/3G algorithms are able to solve non-trivial problems.

▷ Drawbacks:

- 1G: slow;
- 2G: bad or unknown complexity;
- 1G and 3G: non-minimality of telescopers;
- 1G–3G: all compute (big) certificates.

Several generations of Creative Telescoping algorithms

4G: roots in [Ostrogradsky, 1845], [Hermite, 1872] and [Picard, 1902].

- univariate:
  - rational $\int$: [B., Chen, Chyzak, Li, 2010];
  - hyperexponential $\int$: [B., Chen, Chyzak, Li, Xin, 2013]
  - hypergeometric $\sum$: [Chen, Huang, Kauers, Li, 2015], [Huang, 2016]
  - mixed $\int + \sum$: [B., Dumont, Salvy, 2016]
  - algebraic $\int$: [Chen, Kauers, Koutschan, 2016]
  - D-finite Fuchsian $\int$: [Chen, van Hoeij, Kauers, Koutschan, 2018]
  - D-finite $\int$: [B., Chyzak, Lairez, Salvy, 2018], [van der Hoeven, 2018]
- multiple:
  - rational bivariate $\oiint$: [Chen, Kauers, Singer, 2012]
  - rational: [B., Lairez, Salvy, 2013], [Lairez 2016]
  - binomial sums: [B., Lairez, Salvy, 2017]

▷ Advantages:

- complexity;
- minimality of telescopers;
- does not need to compute certificates;
- fast in practice,

▷ Drawback: not (yet) as general as 1G–3G algorithms.

2G: The Almkvist-Zeilberger Algorithm [1990]

Input: a hyperexponential function $H(t,x)$, i.e., $\frac{\frac{\partial H}{\partial t}}{H}$ and $\frac{\frac{\partial H}{\partial x}}{H}$ are in $\mathbb{Q}(t,x)$

Output:

- a linear differential operator $P(t,\partial_t)$ satisfied by $I(t) = \oint_\gamma H(t,x)\,dx$
- a $G(t,x) \in \mathbb{Q}(t,x)$ such that $P(t,\partial_t) \cdot H(t,x) = \frac{\partial}{\partial x}\Big(G(t,x) \cdot H(t,x)\Big)$.

2G: The Almkvist-Zeilberger Algorithm [1990]

Input: a hyperexponential function $H(t,x)$, i.e., $\frac{\frac{\partial H}{\partial t}}{H}$ and $\frac{\frac{\partial H}{\partial x}}{H}$ are in $\mathbb{Q}(t,x)$
Output:

- a linear differential operator $P(t,\partial_t)$ satisfied by $I(t) = \oint_\gamma H(t,x)\,dx$
- a $G(t,x) \in \mathbb{Q}(t,x)$ such that $P(t,\partial_t) \cdot H(t,x) = \frac{\partial}{\partial x}\Big(G(t,x) \cdot H(t,x)\Big)$.

Algorithm: Write $\mathbb{L} = \mathbb{Q}(t)$. For $r = 0,1,2,\dots$ do

1. compute $a(x) := \frac{\frac{\partial H}{\partial x}}{H} \in \mathbb{L}(x)$ and $b_k(x) := \frac{\frac{\partial^k H}{\partial t^k}}{H} \in \mathbb{L}(x)$ for $k = 0,\dots,r$
2. rational solving: decide whether the inhomogeneous parametrized LDE

$$\frac{\partial G}{\partial x} + a(x) \cdot G = \sum_{k=0}^{r} c_k \cdot b_k(x)$$

   admits a rational solution $G \in \mathbb{L}(x)$, for some $c_0,\dots,c_r \in \mathbb{L}$ not all zero
3. if so, then return $P := \sum_{k=0}^{r} c_k \partial_t^k$ and G; else increase r by 1 and repeat

4G Creative Telescoping: univariate case

Problem: Given $H = P/Q \in \mathbb{K}(t,x)$ compute $\oint_\gamma H(t,x)\mathrm{d}x$

Problem: Given $H = P/Q \in \mathbb{K}(t, x)$ compute $\oint_\gamma H(t, x) \mathrm{d}x$

Hermite reduction: H can be written in reduced form

$$H = \partial_x(g) + \frac{a}{Q^\star},$$

where $Q^\star$ is the squarefree part of Q and $\deg_x(a) < d^\star := \deg_x(Q^\star)$.

Problem: Given $H = P/Q \in \mathbb{K}(t,x)$ compute $\oint_\gamma H(t,x)\mathrm{d}x$

Hermite reduction: H can be written in reduced form

$$H = \partial_x(g) + \frac{a}{Q^\star},$$

where $Q^\star$ is the squarefree part of Q and $\deg_x(a) < d^\star := \deg_x(Q^\star)$.

Algorithm [B., Chen, Chyzak, Li, 2010]

(1) For $i = 0, 1, \ldots, d^\star$ compute Hermite reduction of $\partial_t^i(H)$:

$$\partial_t^i(H) = \partial_x(g_i) + \frac{a_i}{Q^\star}, \quad \deg_x(a_i) < d^\star$$

Problem: Given $H = P/Q \in \mathbb{K}(t,x)$ compute $\oint_\gamma H(t,x)\mathrm{d}x$

Hermite reduction: H can be written in reduced form

$$H = \partial_x(g) + \frac{a}{Q^\star},$$

where $Q^\star$ is the squarefree part of Q and $\deg_x(a) < d^\star := \deg_x(Q^\star)$.

Algorithm [B., Chen, Chyzak, Li, 2010]

(1) For $i = 0, 1, \ldots, d^\star$ compute Hermite reduction of $\partial_t^i(H)$:

$$\partial_t^i(H) = \partial_x(g_i) + \frac{a_i}{Q^\star}, \quad \deg_x(a_i) < d^\star$$

(2) Find the first linear relation over $\mathbb{K}(t)$ of the form $\sum_{k=0}^r c_k a_k = 0$.

4G Creative Telescoping: univariate case

Problem: Given $H = P/Q \in \mathbb{K}(t,x)$ compute $\oint_\gamma H(t,x)\mathrm{d}x$

Hermite reduction: H can be written in reduced form

$$H = \partial_x(g) + \frac{a}{Q^\star},$$

where $Q^\star$ is the squarefree part of Q and $\deg_x(a) < d^\star := \deg_x(Q^\star)$.

Algorithm [B., Chen, Chyzak, Li, 2010]

(1) For $i = 0, 1, \ldots, d^\star$ compute Hermite reduction of $\partial_t^i(H)$:

$$\partial_t^i(H) = \partial_x(g_i) + \frac{a_i}{Q^\star}, \quad \deg_x(a_i) < d^\star$$

(2) Find the first linear relation over $\mathbb{K}(t)$ of the form $\sum_{k=0}^r c_k a_k = 0$.

$\triangleright$ $L = \sum_{k=0}^r c_k \partial_t^k$ is a telescoper (and $\sum_{k=0}^r c_k g_k$ the corresponding certificate).

A mandatory exercise for 20/10/2020, 11:59pm

Compute a telescoper for the diagonal of the rational power series

$$\frac{1}{1-x-y} = \sum_{i,j\geq 0} \binom{i+j}{i} x^i y^j$$

in three different ways:

1. using the 2G (Almkvist-Zeilberger) creative telescoping algorithm;
2. using the 4G (Hermite reduction-based) creative telescoping algorithm;
3. using the 4G (Griffiths-Dwork reduction-based) creative telescoping algorithm.

An exercise from a past lecture

Let $(a_n)_{n\geq 0}$ be a sequence with $a_0 = a_1 = 1$ satisfying the recurrence

$$(n+3)a_{n+1} = (2n+3)a_n + 3na_{n-1}, \quad \text{for all } n > 0.$$

Show that a_n is an integer for all n.

An exercise from a past lecture

Let $(a_n)_{n\geq 0}$ be a sequence with $a_0 = a_1 = 1$ satisfying the recurrence

$$(n+3)a_{n+1} = (2n+3)a_n + 3na_{n-1}, \quad \text{for all } n > 0.$$

Show that a_n is an integer for all n.

Follow the next steps:

1. Compute the first 5 terms of the sequence, $a_0, \ldots, a_4$;
2. Determine a Hermite-Padé approximant of type $(0,1,2)$ for $(1, f, f^2)$, where $f = \sum_n a_n x^n$;
3. Deduce that $P(x, f(x)) = 0 \bmod x^5$ for $P(x,y) := 1 + (x-1)y + x^2y^2$;
4. Show that the equation $P(x,y) = 0$ admits a root $y = g(x) \in \mathbb{Q}[[x]]$ whose coefficients satisfy the same linear recurrence as $(a_n)_{n\geq 0}$;
5. Deduce that $a_{n+2} = a_{n+1} + \sum_{k=0}^{n} a_k \cdot a_{n-k}$ for all n, and conclude.

Solution

▷ 1. Let's compute the first 5 terms of the sequence:

```
> rec:=(n+3)*a(n+1)-(2*n+3)*a(n)-3*n*a(n-1): ini:=a(0)=1,a(1)=1:
> pro:=gfun:-rectoproc({rec,ini}, a(n), list);
> pro(4);
```

$$[1, 1, 2, 4, 9]$$

Solution

▷ 1. Let's compute the first 5 terms of the sequence:

```
> rec:=(n+3)*a(n+1)-(2*n+3)*a(n)-3*n*a(n-1): ini:=a(0)=1,a(1)=1:
> pro:=gfun:-rectoproc({rec,ini}, a(n), list);
> pro(4);
```

$$[1, 1, 2, 4, 9]$$

▷ 2. Determine a Hermite-Padé approximant of type $(0, 1, 2)$ for $(1, f, f^2)$:

```
> f:=listtoseries(pro(4),x):
> numapprox[hermite_pade]([1, f, f^2], x, [0, 1, 2]);
```

$$[-1, -x+1, -x^2]$$

Solution

▷ 1. Let's compute the first 5 terms of the sequence:

```
> rec:=(n+3)*a(n+1)-(2*n+3)*a(n)-3*n*a(n-1): ini:=a(0)=1,a(1)=1:
> pro:=gfun:-rectoproc({rec,ini}, a(n), list);
> pro(4);
```

$$[1, 1, 2, 4, 9]$$

▷ 2. Determine a Hermite-Padé approximant of type $(0, 1, 2)$ for $(1, f, f^2)$:

```
> f:=listtoseries(pro(4),x):
> numapprox[hermite_pade]([1, f, f^2], x, [0, 1, 2]);
```

$$[-1, -x + 1, -x^2]$$

▷ 3. We guessed $P(x, f(x)) = 0 \bmod x^5$ for $P(x, y) = 1 + (x - 1)y + x^2y^2$.

Solution

▷ 4. The equation $P(x,y)=0$ admits a root $y=g(x)=\sum_{n\geq 0} g_n x^n$ in $\mathbb{Q}[[x]]$:

$$g(x) = \frac{1-x-\sqrt{1-2\,x-3\,x^2}}{2\,x^2} = 1+x+2\,x^2+4\,x^3+9\,x^4+\cdots$$

Solution

▷ 4. The equation $P(x,y)=0$ admits a root $y=g(x)=\sum_{n\geq 0} g_n x^n$ in $\mathbb{Q}[[x]]$:

$$g(x) = \frac{1-x-\sqrt{1-2\,x-3\,x^2}}{2\,x^2} = 1+x+2\,x^2+4\,x^3+9\,x^4+\cdots$$

▷ $(g_n)_{n\geq 0}$ is an integer sequence, as (by coefficient extraction) it satisfies

$$g_{n+2} = g_{n+1} + \sum_{k=0}^{n} g_k \cdot g_{n-k}.$$

Solution

▷ 4. The equation $P(x,y)=0$ admits a root $y=g(x)=\sum_{n\geq 0} g_n x^n$ in $\mathbb{Q}[[x]]$:

$$g(x) = \frac{1-x-\sqrt{1-2\,x-3\,x^2}}{2\,x^2} = 1 + x + 2\,x^2 + 4\,x^3 + 9\,x^4 + \cdots$$

▷ $(g_n)_{n\geq 0}$ is an integer sequence, as (by coefficient extraction) it satisfies

$$g_{n+2} = g_{n+1} + \sum_{k=0}^{n} g_k \cdot g_{n-k}.$$

▷ We show that $(g_n)_{n\geq 0}$ satisfies the same *linear* recurrence as $(a_n)_{n\geq 0}$. Let

$$h(x) := \sqrt{1-2\,x-3\,x^2} = 1 - x - 2\,x^2\,g(x).$$

Clearly, $h_0 = 1$, $h_1 = -1$, $h_{n+2} = -2\,g_n$ for $n \geq 0$, and

$$\frac{h'(x)}{h(x)} = \frac{3\,x+1}{3\,x^2+2\,x-1}$$

Solution

▷ 4. The equation $P(x,y)=0$ admits a root $y=g(x)=\sum_{n\geq 0} g_n x^n$ in $\mathbb{Q}[[x]]$:

$$g(x)=\frac{1-x-\sqrt{1-2\,x-3\,x^2}}{2\,x^2}=1+x+2\,x^2+4\,x^3+9\,x^4+\cdots$$

▷ $(g_n)_{n\geq 0}$ is an integer sequence, as (by coefficient extraction) it satisfies

$$g_{n+2}=g_{n+1}+\sum_{k=0}^{n} g_k\cdot g_{n-k}.$$

▷ We show that $(g_n)_{n\geq 0}$ satisfies the same *linear* recurrence as $(a_n)_{n\geq 0}$. Let

$$h(x):=\sqrt{1-2\,x-3\,x^2}=1-x-2\,x^2\,g(x).$$

Clearly, $h_0=1$, $h_1=-1$, $h_{n+2}=-2\,g_n$ for $n\geq 0$, and

$$\frac{h'(x)}{h(x)}=\frac{3\,x+1}{3\,x^2+2\,x-1}$$

▷ Hence, $(h_n)_n$ and $(g_n)_n$ satisfy the recurrences

$$(3\,n-3)\,h_n+(2\,n+1)\,h_{n+1}-(n+2)\,h_{n+2}=0,$$

$$(3\,n-3)\,g_{n-2}+(2\,n+1)\,g_{n-1}-(n+2)\,g_n=0.$$

Solution

▷ 4. The equation $P(x,y)=0$ admits a root $y=g(x)=\sum_{n\geq 0} g_n x^n$ in $\mathbb{Q}[[x]]$:

$$g(x)=\frac{1-x-\sqrt{1-2\,x-3\,x^2}}{2\,x^2}=1+x+2\,x^2+4\,x^3+9\,x^4+\cdots$$

▷ $(g_n)_{n\geq 0}$ is an integer sequence, as (by coefficient extraction) it satisfies

$$g_{n+2}=g_{n+1}+\sum_{k=0}^{n} g_k\cdot g_{n-k}.$$

▷ We show that $(g_n)_{n\geq 0}$ satisfies the same *linear* recurrence as $(a_n)_{n\geq 0}$. Let

$$h(x):=\sqrt{1-2\,x-3\,x^2}=1-x-2\,x^2\,g(x).$$

Clearly, $h_0=1$, $h_1=-1$, $h_{n+2}=-2\,g_n$ for $n\geq 0$, and

$$\frac{h'(x)}{h(x)}=\frac{3\,x+1}{3\,x^2+2\,x-1}$$

▷ Hence, $(h_n)_n$ and $(g_n)_n$ satisfy the recurrences

$$(3\,n-3)\,h_n+(2\,n+1)\,h_{n+1}-(n+2)\,h_{n+2}=0,$$

$$(3\,n-3)\,g_{n-2}+(2\,n+1)\,g_{n-1}-(n+2)\,g_n=0.$$

▷ Conclusion: $a_n=g_n$ is an integer for all n. □

Solution

▷ 4. The equation $P(x, y) = 0$ admits a root $y = g(x) = \sum_{n\geq 0} g_n x^n$ in $\mathbb{Q}[[x]]$:

$$g(x) = \frac{1 - x - \sqrt{1 - 2\,x - 3\,x^2}}{2\,x^2} = 1 + x + 2\,x^2 + 4\,x^3 + 9\,x^4 + \cdots$$

▷ $(g_n)_{n\geq 0}$ is an integer sequence, as (by coefficient extraction) it satisfies

$$g_{n+2} = g_{n+1} + \sum_{k=0}^{n} g_k \cdot g_{n-k}.$$

Solution

▷ 4. The equation $P(x,y)=0$ admits a root $y=g(x)=\sum_{n\geq 0} g_n x^n$ in $\mathbb{Q}[[x]]$:

$$g(x) = \frac{1-x-\sqrt{1-2\,x-3\,x^2}}{2\,x^2} = 1+x+2\,x^2+4\,x^3+9\,x^4+\cdots$$

▷ $(g_n)_{n\geq 0}$ is an integer sequence, as (by coefficient extraction) it satisfies

$$g_{n+2} = g_{n+1} + \sum_{k=0}^{n} g_k \cdot g_{n-k}.$$

▷ We show that $(g_n)_{n\geq 0}$ satisfies the same *linear* recurrence as $(a_n)_{n\geq 0}$.

```
> pol:=x^2*y^2 + (x-1)*y + 1;
> deq:=gfun:-algeqtodiffeq(pol,y(x)):
> recg:=gfun:-diffeqtorec(deq,y(x),g(n));
```

$$\{(3\,n+3)\,g_n + (2\,n+5)\,g_{n+1} - (n+4)\,g_{n+2} = 0,\ g_0 = 1,\ g_1 = 1\}$$

Solution

▷ 4. The equation $P(x,y)=0$ admits a root $y=g(x)=\sum_{n\geq 0} g_n x^n$ in $\mathbb{Q}[[x]]$:

$$g(x) = \frac{1-x-\sqrt{1-2x-3x^2}}{2x^2} = 1+x+2x^2+4x^3+9x^4+\cdots$$

▷ $(g_n)_{n\geq 0}$ is an integer sequence, as (by coefficient extraction) it satisfies

$$g_{n+2} = g_{n+1} + \sum_{k=0}^{n} g_k \cdot g_{n-k}.$$

▷ We show that $(g_n)_{n\geq 0}$ satisfies the same *linear* recurrence as $(a_n)_{n\geq 0}$.

```
> pol:=x^2*y^2 + (x-1)*y + 1;
> deq:=gfun:-algeqtodiffeq(pol,y(x)):
> recg:=gfun:-diffeqtorec(deq,y(x),g(n));
```

$$\{(3n+3)\,g_n + (2n+5)\,g_{n+1} - (n+4)\,g_{n+2} = 0,\ g_0 = 1,\ g_1 = 1\}$$

▷ Conclusion: $a_n = g_n$ is an integer for all n. $\square$