CR06: Modern Algorithms for Symbolic Summation and Integration

Lecture 14: General D-finite functions & their algorithms

Examples of Creative Telescoping

$$\sum_{j,k}(-1)^{j+k}\binom{j+k}{k+l}\binom{r}{j}\binom{n}{k}\binom{s+n-j-k}{m-j}=(-1)^{l}\binom{n+r}{n+l}\binom{s-r}{m-n-l}$$

$$\int_0^{+\infty} xJ_1(ax)I_1(ax)Y_0(x)K_0(x)\,dx=-\frac{\ln(1-a^4)}{2\pi a^2}$$

$$\frac{1}{2\pi i}\oint\frac{(1+2xy+4y^2)\exp\left(\frac{4x^2y^2}{1+4y^2}\right)}{y^{n+1}(1+4y^2)^{\frac{3}{2}}}\,dy=\frac{H_n(x)}{\lfloor n/2\rfloor!}$$

$$\int_{-1}^{1}\frac{e^{-px}T_n(x)}{\sqrt{1-x^2}}\,dx=(-1)^n\pi I_n(p)$$

Today

$$\sum_{k=0}^{n}\frac{q^{k^2}}{(q;q)_k(q;q)_{n-k}}=\sum_{k=-n}^{n}\frac{(-1)^k q^{(5k^2-k)/2}}{(q;q)_{n-k}(q;q)_{n+k}}$$

Aims:

1. Prove them automatically
2. Find the rhs given the lhs

Note: at least one free variable

First: **find a LDE** (or LRE)

Main Example for this Lecture

$$\int_{-1}^{1} \frac{e^{-px} T_n(x)}{\sqrt{1-x^2}}\, dx = (-1)^n \pi\, I_n(p)$$

```
> f:=exp(-p*x)*ChebyshevT(n,x)/sqrt(1-x^2);
```

$$f := \frac{e^{-px}\,\mathrm{ChebyshevT}(n,x)}{\sqrt{1-x^2}}$$

reduction-based creative telescoping

```
> redct(Int(f,x=-1..1),[n::shift,p::diff]);
```

$$[pD_n + pD_p - n, pD_n^2 - 2nD_n - 2D_n - p]$$

Interpretation: the integral $F_n(p)$ satisfies

$$pF_{n+1} + pF_n' - nF_n = 0, \quad pF_{n+2} - 2(n+1)F_{n+1} - pF_n = 0$$

0. In Previous Episodes

Exercise 2 weeks ago

Compute a telescoper for the diagonal of the rational power series

$$\frac{1}{1-x-y} = \sum_{i,j\geq 0} \binom{i+j}{i} x^i y^j$$

3. using the 4G (Griffiths-Dwork reduction-based) creative telescoping algorithm.

1. Diagonal as a residue (p. 6)
2. Homogenize in degree -2 (p. 23,25)
3. Differentiate wrt t
4. Macaulay matrix for $\partial_x Q, \partial_y Q$ in deg 2 (p. 24)
5. Decompose y^2 using A (p. 24)
6. Reduce (p. 25)
7. Find linear dependency (p. 26)

$$\frac{1}{x}\frac{1}{1-x-t/x} = \frac{1}{x-x^2-t}$$

$$f := \frac{1}{xy-x^2-ty^2} =: \frac{1}{Q} = [f]$$

$$f_t = y^2/Q^2$$

$$M_{\phi_2} = \begin{array}{c} \begin{array}{cccc} x\partial_x Q & y\partial_x Q & x\partial_y Q & y\partial_y Q \end{array} \\ \begin{pmatrix} -2 & 0 & 1 & 0 \\ 1 & -2 & -2t & 1 \\ 0 & 1 & 0 & -2t \end{pmatrix} \end{array} \begin{array}{c} x^2 \\ xy \\ y^2 \end{array}$$

A of rank 3

$$y^2 = \left(\frac{2}{1-4t}x + y\right)\partial_x Q + \frac{4}{1-4t}x\partial_y Q$$

$$[f_t] = \left[\frac{2/(1-4t)}{Q}\right]$$

$$[f_t] = 2/(1-4t)[f]$$

Result: $\partial_t - \dfrac{2}{1-4t}$

Last Slide 2 Weeks Ago

This method [multiple binomial sums]

…

5. *deserves a good implementation*, *generalizations…*

Almost the same day

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[Submitted on 18 Oct 2020]

Creative Telescoping on Multiple Sums

Christoph Koutschan, Elaine Wong

We discuss the strategies and difficulties of determining a recurrence which a certain polynomial (in the form of a symbolic multiple sum) satisfies. The polynomial comes from an analysis of integral estimators derived via quasi-Monte

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Creative Telescoping on Multiple Sums

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Abstract. We discuss the strategies and difficulties of determining a recurrence which a certain polynomial (in the form of a symbolic multiple sum) satisfies. The polynomial comes from an analysis of integral estimators derived via quasi-Monte Carlo methods.

Keywords. Symbolic Summation, Creative Telescoping, Holonomic Function, Hypergeometric Series.

1. The Problem

Recently, Wiart and Wong [6] derived a formula for the covariance of an integral estimator for functions satisfying a certain decay condition, based on a quasi-Monte Carlo framework developed by Wiart, Lemieux, and Dong [5]. This formula is written as the following polynomial in x,

$$G_s(x) := \sum_{k=1}^{m+s-1}\left(\sum_{r=1}^{s}\binom{s}{r}\binom{k-1}{r-1}\frac{b-1}{(-b)^r}\sum_{i=0}^{r-1-c_m(k)}(-b)^i\binom{r-1}{i}\right)(bx)^k, \quad (1.1)$$

where $c_m(k) = \max(k-m,0)$. The goal in [6] is to show that (1.1) is not positive for all $b, m, s \in \mathbb{N}$, $b \geq 2$ and $x \in [0,1)$. We approach this problem by employing symbolic computation rather than analysis: using the available computer algebra software for holonomic functions [1, 2, 4], we carry out a guess-and-prove strategy that ultimately leads us to deduce a suitable closed form for (1.1). The result is an expression in terms of regularized beta functions which allows us to show the desired nonpositivity statement. This article serves to highlight the "proving" aspect of the strategy, i.e., the derivation and proof of a third-order recurrence that (1.1) satisfies. At the same time, we discuss some general solutions to address difficulties in summation problems of a similar nature.

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Problem & Result

$$G_s(x) := \sum_{k=1}^{m+s-1}\left(\sum_{r=1}^{s}\binom{s}{r}\binom{k-1}{r-1}\frac{b-1}{(-b)^r}\sum_{i=0}^{r-1-c_m(k)}(-b)^i\binom{r-1}{i}\right)(bx)^k, \quad (1.1)$$

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Several methods, requiring human intervention

Approximate Comp. Time	Strategies Implemented	Result
30 hours	– split sums – sing./comm. corrections – closure properties	fifth-order recurrence in the ideal generated by the guessed recurrence
1.4 hours	– split sums – sing./comm. corrections – closure properties – substitution speedup	
11 minutes	– one triple sum – two gamma insertions – sing. corrections	fourth-order recurrence in the ideal generated by the guessed recurrence
30 seconds	– one triple sum – one gamma insertion – sing. corrections	same third-order recurrence as the guessed recurrence

FIGURE 2. Results and Comparisons

P. Lairez, a few days later:

$$F := \sum_{s\geq 0, m\geq 0} G_s(x) y^m z^s = \sum_{s\geq 0, m\geq 0, k\geq 1, i\geq 0} \sum_{r=1}^{s} \binom{s}{r}\binom{k-1}{r-1}\binom{r-1}{i}\frac{b-1}{(-b)^{r-i}} H_{r-1+m-k-i}(bx)^k y^m z^s$$

Heaviside $H_m = \sum_{k=0}^{m} \delta_k$

```
> sumtores(F, u);
```

$$\frac{xyz(b-1)}{(z-1)(y-1)(bxy-xz+z-1)}, [z, b, y, x]$$

39ms

No integral left

$$= -\frac{xyz(b-1)}{(1-z)(1-y)(1-(1-x)z-bxy)}$$

reveals non-positivity

Exam Next Week

Not defined yet
News in your mail soon

I. General D-finite Functions

Recall: Skew Polynomials

$\mathbb{K}(x)\langle\partial_x\rangle$ with commutation $\partial_x f(x) = f(x)\partial_x + f'(x)$
$\mathbb{K}(n)\langle S_n\rangle$ with $S_n f(n) = f(n+1)S_n$

Rational Weyl algebra & its shift analogue

Main property: A, B in $\mathbb{A}\langle\partial\rangle$, then $\deg AB = \deg A + \deg B$

integral domain

Consequence 1: (non-commutative) Euclidean division
Consequence 2: (non-commutative) Euclidean algorithm
yields **GCRD** and **LCLM.**

Shown in Lect. 5 in the differential case. Extends to shift and other cases.

Ex.: closure by sum

Ore Algebras

$$\mathbb{O} := \mathbb{K}(x_1, \ldots, x_r)\langle \partial_1, \ldots, \partial_r \rangle = \mathbb{K}(x_1, \ldots, x_r)\langle \partial_1 \rangle \cdots \langle \partial_r \rangle$$

with commuting $\partial_i \in \{S_{x_i}, \partial_{x_i}\}$, $i = 1, \ldots, r$.

Restriction to shift/ differential case of a more general definition

Def. $\operatorname{Ann} f := \{P \in \mathbb{O} \mid P \cdot f = 0\}$ annihilating ideal of f.

Def. LM (leading monomial) on next slide.

Main property: A, B in $\mathbb{O}$, then $\mathrm{LM}(AB) = \mathrm{LM}(A)\mathrm{LM}(B)$.

Consequence: (non-commutative) Gröbner bases

Gröbner bases of their annihilating ideal give a data-structure to encode special functions

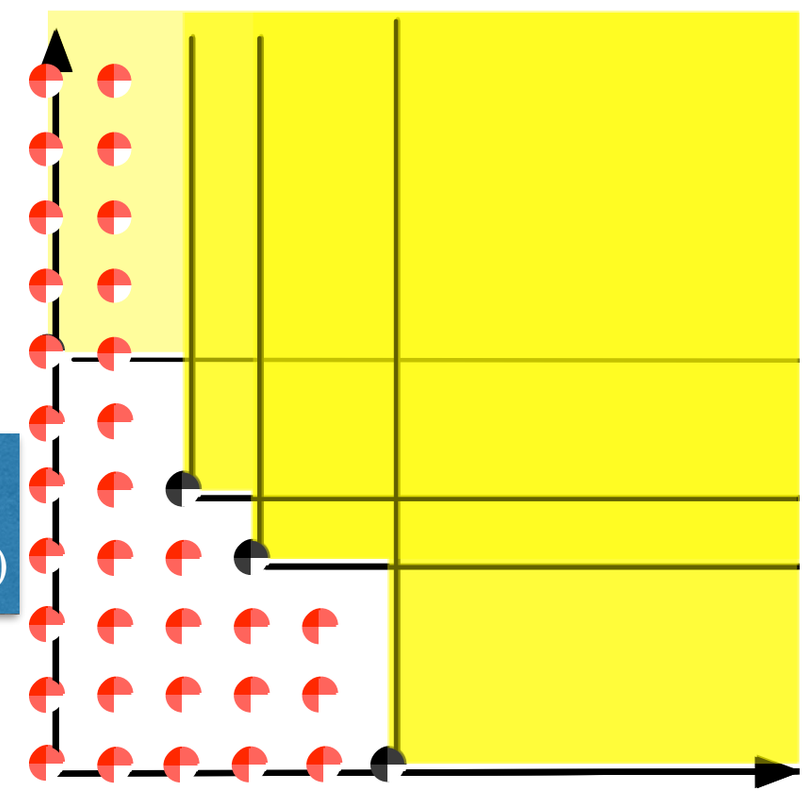
Gröbner Bases

Def. A finite subset G of a (left) ideal $I \subset \mathbb{O}$ is a Gröbner basis of I if for all $P \in I$, there exists $g \in G$ s.t. $\mathrm{LM}(g) \mid \mathrm{LM}(P)$.

Ordering in this lecture: degree first, then lexicographic order on the variables.

1. Monomial ordering: order on $\mathbb{N}^r$, compatible with +, 0 minimal.
2. Leading monomial (LM) of a polynomial: the largest one.
3. Gröbner basis of a (left) ideal I: corners of stairs.
4. Quotient mod I: basis below the stairs
5. Reduction of P: Rewrite P mod I on this basis.
6. **D-finiteness**: finite dimension of the quotient.

When $I = \mathrm{Ann}\, f$, $\mathbb{O}/I \simeq \mathrm{Vect}_{\mathbb{K}(\underline{x})}(\{\underline{\partial}^{\underline{\alpha}} f \mid \underline{\alpha} \in \mathbb{N}^r\})$



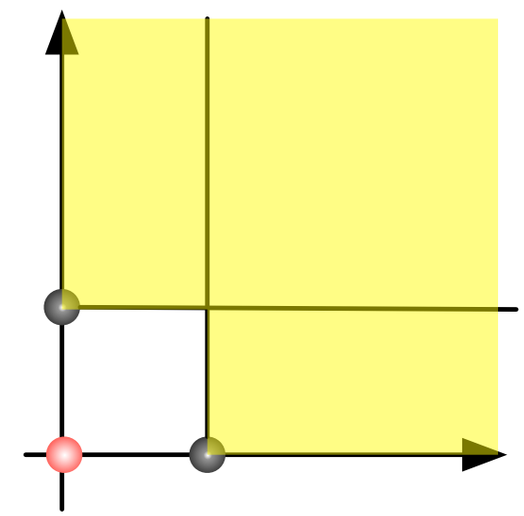
Special case $r = 1$ gives gcd

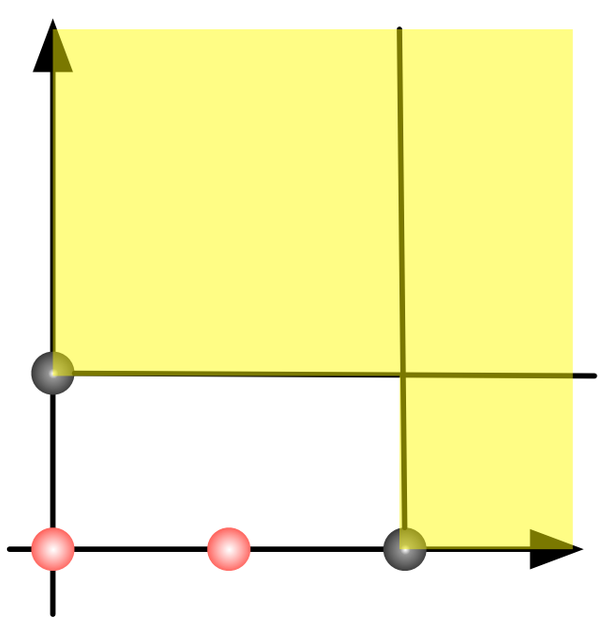
An access to finite-dimensional vector spaces

Examples

Binomial coefficients $\binom{n}{k}$ wrt S_n, S_k

Hypergeometric Sequences





Classical orthogonal polynomials wrt S_n, ∂_x

Bessel $J_\nu(x)$ wrt S_ν, ∂_x

Ex. Chebyshev $T_n(x)$:

$$S_n^2 - 2xS_n + 1,\ (x^2-1)\partial_x - nS_n + nx\,1$$

translates $T_{n+2}(x) - 2xT_{n+1}(x) + T_n(x) = (x^2-1)T_n'(x) - nT_{n+1}(x) + nxT_n(x) = 0$

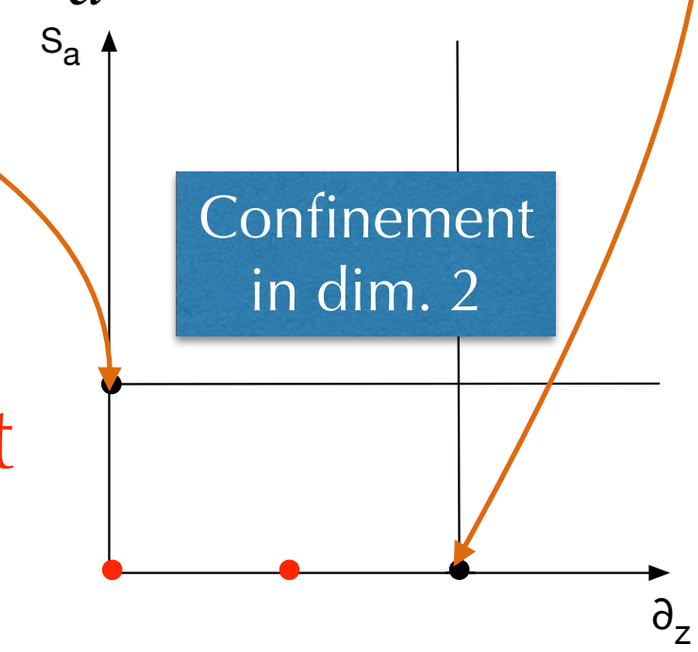
Example: Contiguity of Hypergeometric Series

$$F(a,b;c;z) = \sum_{n=0}^{\infty} \underbrace{\frac{(a)_n (b)_n}{(c)_n n!}}_{u_{a,n}} z^n, \quad (x)_n := x(x+1)\cdots(x+n-1)\,.$$

$$\frac{u_{a,n+1}}{u_{a,n}} = \frac{(a+n)(b+n)}{(c+n)(n+1)} \to z(1-z)F'' + (c-(a+b+1)z)F' - abF = 0,$$

$$\frac{u_{a+1,n}}{u_{a,n}} = \frac{n}{a} + 1 \to S_a F := F(a+1,b;c;z) = \frac{z}{a}F' + F\,.$$

GB: $\{S_a - \frac{z}{a}\partial_z + 1,\ z(1-z)\partial_z^2 + (c-(a+b+1)z)\partial_z - ab\}$



Gauss 1812: contiguity relation.

dim = 2 $\Rightarrow S_a^2 F, S_a F, F$ linearly dependent

(coordinates in $\mathbb{Q}(a,b,c,z)$)

$$(a+1)(z-1)S_a^2 F + ((b-a-1)z + 2 - c + 2a)S_a F + (c-a-1)F = 0.$$

Closure Properties

Prop. f, g D-finite $\Rightarrow f+g, fg$ D-finite too.

Proof as in the 1-variable case:

$$\underline{\partial}^{\underline{\alpha}} f \in \mathrm{Vect}_{\mathbb{K}(\underline{x})}\left(\{\underline{\partial}^{\underline{\alpha}_1} f, \ldots, \underline{\partial}^{\underline{\alpha}_r} f\}\right), \quad \underline{\partial}^{\underline{\alpha}} g \in \mathrm{Vect}_{\mathbb{K}(\underline{x})}\left(\{\underline{\partial}^{\underline{\beta}_1} g, \ldots, \underline{\partial}^{\underline{\beta}_s} g\}\right)$$

so that $\underline{\partial}^{\underline{\alpha}}(f+g) \in \mathrm{Vect}_{\mathbb{K}(\underline{x})}\left(\{\underline{\partial}^{\underline{\alpha}_1} f, \ldots, \underline{\partial}^{\underline{\alpha}_r} f, \underline{\partial}^{\underline{\beta}_1} g, \ldots, \underline{\partial}^{\underline{\beta}_s} g\}\right)$

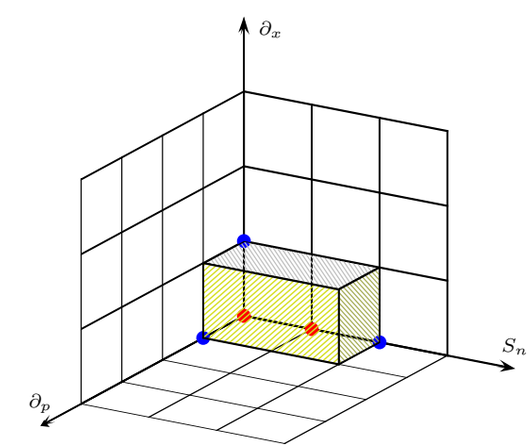
and $\underline{\partial}^{\underline{\alpha}}(fg) \in \mathrm{Vect}_{\mathbb{K}(\underline{x})}\left(\{\underline{\partial}^{\underline{\alpha}_i} f\, \underline{\partial}^{\underline{\beta}_j} g \mid i \leq r, j \leq s\}\right)$

Example

$$f_n(p,x) := \frac{e^{-px}T_n(x)}{\sqrt{1-x^2}}$$

Notation:
$\langle p_1, \ldots, p_s \rangle =$
ideal generated by $p_1, \ldots, p_s$

1. Algebra: $\mathbb{O} := \mathbb{Q}(n,p,x)\langle S_n, \partial_p, \partial_x \rangle$
2. $u := e^{-px}$ annihilated by $\langle S_n - 1, \partial_p + x\,1, \partial_x + p\,1 \rangle$
3. $v := (1-x^2)^{-1/2}$ annihilated by $\langle S_n - 1, \partial_p, (x^2-1)\partial_x + x\,1 \rangle$
4. $w := T_n(x)$ annihilated by $\langle S_n^2 - 2xS_n + 1, \partial_p, (x^2-1)\partial_x - nS_n + nx\,1 \rangle$
5. Reduce successive monomials in increasing order:



$$1 \cdot uvw = uvw$$

$$\partial_p \cdot uvw = -x\,uvw \qquad \to \partial_p + x1 \in \operatorname{Ann} f$$

$$S_n \cdot uvw = uv(S_n \cdot w),$$

$$\partial_x \cdot uvw = \left(-p - \frac{x}{x^2-1} - \frac{nx}{x^2-1}\right) uvw + \frac{n}{x^2-1} uv(S_n \cdot w) \qquad \to \partial_x - \frac{n}{x^2-1} S_n + \left(p + \frac{(n+1)x}{x^2-1}\right) 1 \in \operatorname{Ann} f$$

$$S_n^2 \cdot uvw = 2xuv(S_n \cdot w) - uvw \qquad \to S_n^2 - 2xS_n + 1 \in \operatorname{Ann} f$$

Each reduction brings either a new element in the basis or a new generator in the quotient

FGLM-type Algorithm

Faugère-Gianni-Lazard-Mora (1993)

Algorithm GB

Input: Reduction – a reduction procedure for a D-finite ideal

Output: a Gröbner basis for this ideal

$\mathscr{L} := [1]$ // *list of monomials in* $\partial_1, \ldots, \partial_r$

$G := \{\}$ // *Gröbner basis being computed*

$Q := \{\}$ // *generators of the quotient*

$R := \{\}$ // *leading terms of the basis*

while $\mathscr{L} \neq \emptyset$ **do**

  Remove the first element μ of $\mathscr{L}$

  if μ is not a multiple of an element of R **then**

    $F_\mu :=$ Reduction(μ)

    if $\exists$ a $\mathbb{K}(\underline{x})$-linear combination between F_μ and $\{F_\nu \mid \nu \in Q\}$ **then**

      $(a_\nu)_{\nu \in Q} :=$ coeff. of the relation $F_\mu = \sum_{\nu \in Q} a_\nu F_\nu$

      Add $\mu - \sum_{\nu \in Q} a_\nu \nu$ to G; add μ to R

    else

      Add μ to Q

      for $1 \leq i \leq r$ **do** Append the monomial $\partial_i \mu$ to $\mathscr{L}$

return G

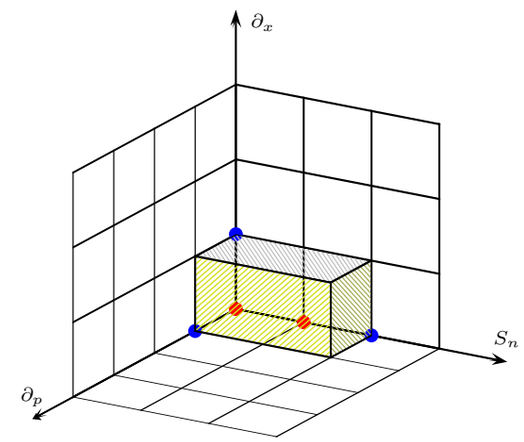
Linear algebra over polynomial matrices

Termination: clear if ideal D-finite

Original application: change of ordering

Example

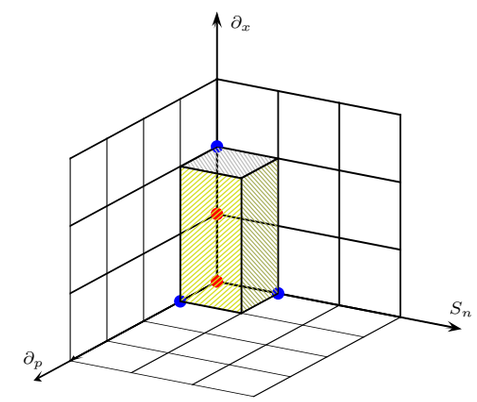
from $(S_n, \partial_x, \partial_p)$ *to* $(\partial_x, S_n, \partial_p)$



Reduce using:

$$\partial_p + x1, \quad S_n^2 - 2xS_n + 1, \quad (x^2-1)\partial_x - nS_n + \big(p(x^2-1) + (n+1)x\big)1$$

Successive monomials:



$1 \to 1$	$1 \in \mathbb{O}/I$
$\partial_p \to -x1$	$\partial_p + x1 \in G$
$\partial_x \to \dfrac{n}{x^2-1}S_n - \dfrac{p(x^2-1)+(n+1)x}{x^2-1}1$	$\partial_x \in \mathbb{O}/I$
$S_n \to S_n$	$-nS_n + (x^2-1)\partial_x + (p(x^2-1)+(n+1)x)1 \in G$
$\partial_x^2 \to \text{big} \times S_n + \text{BIG} \times 1$	$(x^2-1)\partial_x^2 + (2px^2-2p+3x)\partial_x + (p^2x^2-n^2-p^2+3px+1)1 \in G$

Result:

$$\partial_p + x1, \quad -nS_n + (x^2-1)\partial_x + (p(x^2-1)+(n+1)x)$$

$$(x^2-1)\partial_x^2 + (2px^2-2p+3x)\partial_x + (p^2x^2-n^2-p^2+3px+1)1$$

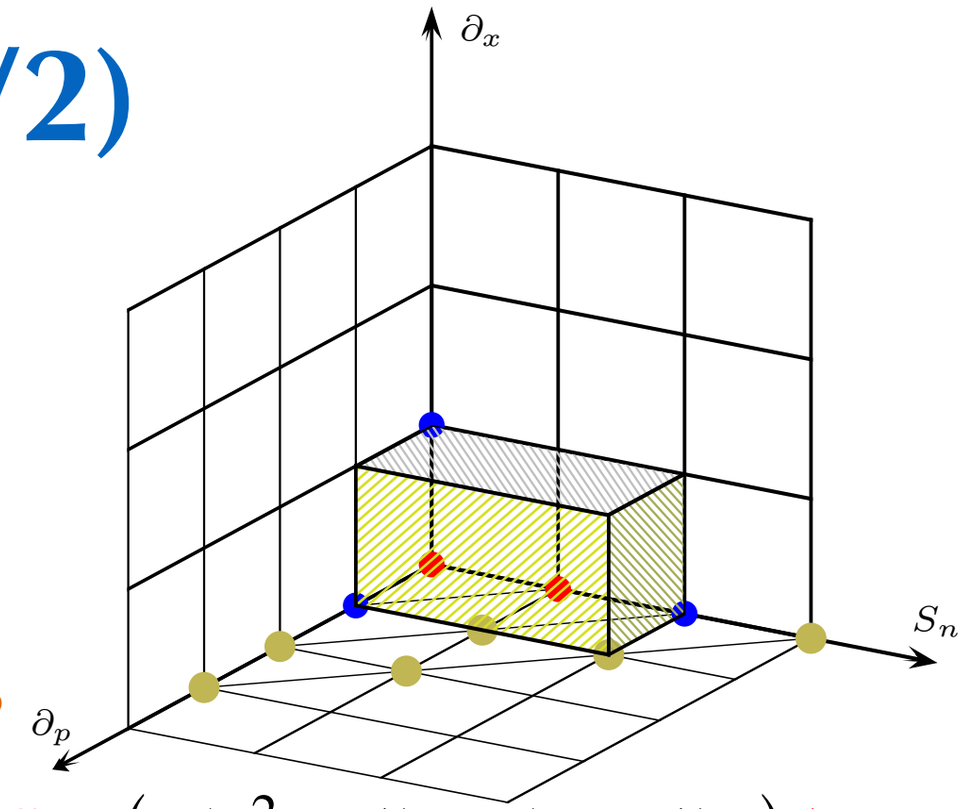
II. Chyzak's Algorithm for Creative Telescoping

A Generalization of Zeilberger's Algorithm

Example (1/2)

$$\int_{-1}^{1} \frac{e^{-px}T_n(x)}{\sqrt{1-x^2}}\,dx = (-1)^n \pi I_n(p)$$

f



Ann f generated by the Gröbner basis

$$\partial_p + x1, \quad S_n^2 - 2xS_n + 1, \quad (x^2-1)\partial_x - nS_n + \big(p(x^2-1) + (n+1)x\big)1$$

Undetermined coefficients

certificate

telescoper $\displaystyle\sum_{(k,m)} t_{k,m}(n,p)\partial_p^k S_n^m - \partial_x\big(c_0(n,p,x) + c_1(n,p,x)S_n\big) \bmod \operatorname{Ann} f$

Reduces to

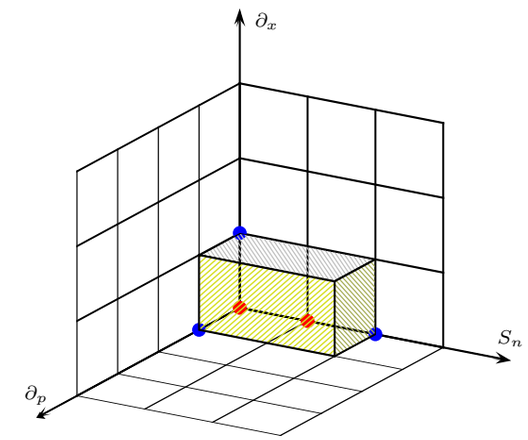
$$\begin{cases} \dfrac{\partial c_0}{\partial x} - \dfrac{p(x^2-1)+(n+1)x}{x^2-1}c_0 - \dfrac{n+1}{x^2-1}c_1 = \sum_{(k,m)} t_{k,m}u_{k,m}^{(0)}, \\[2ex] \dfrac{\partial c_1}{\partial x} + \dfrac{n}{x^2-1}c_0 + \dfrac{nx-p(x^2-1)}{x^2-1}c_1 = \sum_{(k,m)} t_{k,m}u_{k,m}^{(1)}. \end{cases}$$

use the GB and extract coeffs of 1 and S_n

Look for $t_{k,m}$ s.t. a rational solution exists

Increase support until a soln is found

Example (2/2)



Telescoper in $1, S_n, \partial_p$? Starting point: reductions

$$1 \mapsto 1, \quad S_n \mapsto S_n, \quad \partial_p \mapsto -x$$

System:
$$\begin{cases} \dfrac{\partial c_0}{\partial x} - \dfrac{p(x^2-1)+(n+1)x}{x^2-1} c_0 - \dfrac{n+1}{x^2-1} c_1 = t_{0,0} - x t_{1,0}, \\ \dfrac{\partial c_1}{\partial x} + \dfrac{n}{x^2-1} c_0 + \dfrac{nx - p(x^2-1)}{x^2-1} c_1 = t_{0,1}\,. \end{cases}$$

A 1st order system of dim n can always be reduced to an equation of order $\leq n$

Next page

1. Uncoupling leads to:

$$(x^2-1)c_0'' + (x+2p-2px^2)c_0' + (p^2(x^2-1) - px - (n+1)^2)c_0 =$$
$$(px^3 - (n+3)x^2 - px + 1)t_{1,0} - (px^2 - (n+2)x - p)t_{0,0} + (n+1)t_{0,1}$$

2. Indicial equation at ± 1 : $\alpha(\alpha - 1/2) = 0$
3. Bound on the degree: 1
4. Linear system in the coeffs of c_0 and $t_{0,0}, t_{1,0}, t_{0,1}$ gives

$$p\partial_p + pS_n - n \in \operatorname{Ann} f$$

Uncoupling

$$Y'(x) = A(x)Y(x) + B(x), \quad Y = \begin{pmatrix} y_1 \\ \vdots \\ y_s \end{pmatrix} \text{ unknown}$$

Using $Y' = AY + B$ to rewrite $y_2', \ldots, y_n'$ in terms of $y_1, \ldots, y_{n'}$ reduce $y_1', y_1'', \ldots$ until a linear dependency is found.

A analogous method for $Y_{n+1} = A(n)Y_n + B(n)$

Chyzak's Algorithm (2000)

Algorithm CreativeTelescoping

Input: a Gröbner basis G of a D-finite ideal $I \subset \mathbb{O} = \mathbb{K}(x_1, \ldots, x_r)\langle \partial_1, \ldots, \partial_r \rangle$

a set M of monomials in $\partial_1, \ldots, \partial_{r-1}$

Output: a telescoper $\sum_{m \in M} t_m m \in I$ with $t_m \in \mathbb{K}(x_1, \ldots, x_{r-1})$ if one exists

// $Q := (Q_1, \ldots, Q_n)$ *is a basis of* $\mathbb{O}/I$ *(obtained from* G*)*

1. Compute a matrix A with A_{ij} the coefficient of Q_i in the reduction of $\partial_r Q_j$
2. Compute a matrix B with B_{im} the coefficient of Q_i in the reduction of $m \in M$
3. Setup the system $\partial_r(C) + AC = BT$ (differential case)

   or $A\partial_r(C) - C = BT$ (shift case)
4. Find its rational solutions // *e.g., by uncoupling and parameterized Liouville/Abramov*
5. Return it if it is nonzero, FAIL otherwise

Can be coupled with the technique of Algorithm GB to return a basis of telescopers if their ideal is D-finite

Almkvist-Zeilberger: $r = 2, n = 1$, differential

Zeilberger: $r = 2, n = 1$, shift

III. Reduction-Based Creative Telescoping for Integrals

A certificate-free approach

Example

$$\int_{-1}^{1} \frac{e^{-px}T_n(x)}{\sqrt{1-x^2}}\,dx = (-1)^n \pi I_n(p)$$

Recall: creative telescoping= diff. under ∫ + integration by parts

certificate

Ann f generated by

$$\partial_p + x1, \quad -nS_n + (x^2-1)\partial_x + (p(x^2-1)+(n+1)x)$$

$$(x^2-1)\partial_x^2 + (2px^2-2p+3x)\partial_x + (p^2x^2-n^2-p^2+3px+1)1$$

Modulo derivatives in x

$$\partial_p f \equiv -xf$$

$$nS_n f \equiv (p(x^2-1)+(n+1)x)f - 2xf + \cancel{\partial_x((x^2-1)f)}$$

$$(p^2x^2-n^2-p^2+3px+1)f - (4xp+3)f + 2f \equiv 0$$

Combinations of f, xf only

$$\partial_p^2 f \equiv x^2 f$$

Conclusion: the integral $F_n(p)$ satisfies

$$F_n' + F_{n+1} = \tfrac{n}{p}F_n \qquad p^2F_n'' + pF_n' = (n^2+p^2)F_n$$

Integration by Parts

∂ derivative: $u\partial v + (\partial u)v = \partial(uv)$

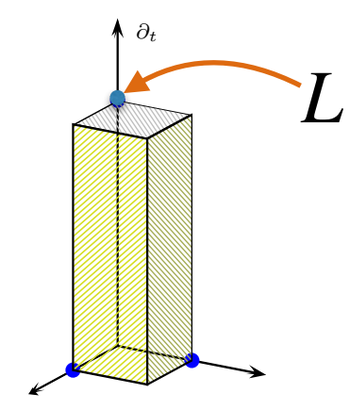
By induction, $u\partial^r f = (-\partial u)\partial^{r-1} f + \partial(\ldots) = \cdots = ((-\partial)^r u)f + \partial(\ldots)$

Def. Adjoint of $M = c_r\partial^r + \cdots + c_0$: $M^\star = c_0 + \cdots + (-\partial)^r c_r$

Lagrange's identity: $uM(f) - M^\star(u)f = \partial(P_M(f,u))$

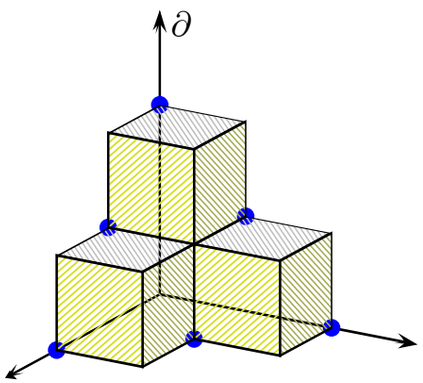
Cor. $g = M(f) \Rightarrow \exists q, g = M^\star(1)f + \partial(q)$ rational fcn

Prop. If $L(f) = 0$ and $(1, \partial, \ldots, \partial^{s-1})$ is a basis of $\mathbb{O}/\mathrm{Ann}\, f$, then $\mathbb{K}(\underline{x}) \cap \partial(\mathbb{O}/\mathrm{Ann}\, f) = L^\star(\mathbb{K}(\underline{x}))$.



Computation reduced to rational fcns x f and working modulo $\mathrm{Im}\, L^\star$

Cyclic Vectors are Easy to Find



Def. An element v of a vector space V is cyclic for a linear $A : V \to V$ if $(v, Av, A^2v, \ldots)$ span V.

Here: $V = \mathbb{O}/I$ with I D-finite, $A = \partial$.

Las Vegas algorithm

Algorithm Cyclic Vector:

Input: a Gröbner basis G of a D-finite ideal

$$I \subset \mathbb{K}(\underline{x})\langle \partial_1, \ldots, \partial_r \rangle,\ \partial \text{ — one of the } \partial_i$$

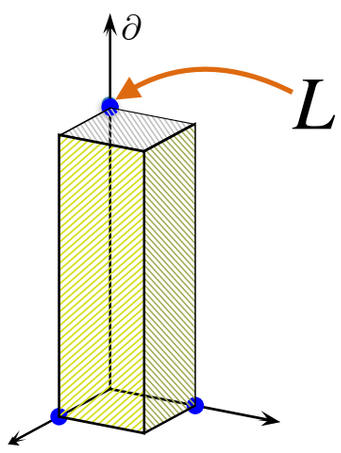
Output: a cyclic vector for ∂ in $\mathbb{K}(\underline{x})\langle \underline{\partial} \rangle / I$ of dim n

repeat:

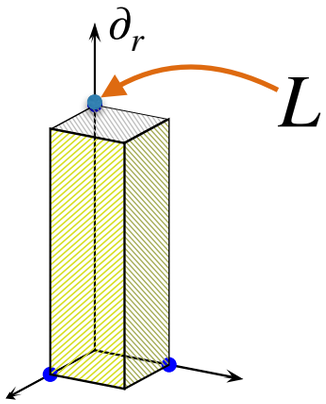
Choose a random $v \in \mathbb{K}(\underline{x})\langle \underline{\partial} \rangle / I$

Reduce $v, \partial v, \ldots, \partial^{n-1} v$ using B

If they are linearly independent, **return** v



Algorithm



∂_r a derivative

Algorithm Reduction-Based Creative Telescoping

Input: B a Gröbner basis of $\operatorname{Ann} f$, with $B \cap \mathbb{K}(\underline{x})\langle \partial_r \rangle = \{L(\partial_r)\}$
and the other elements of total degree 1 in $\partial_1, \ldots, \partial_{r-1}$
$P \in \mathbb{K}(\underline{x})\langle \partial_1, \ldots, \partial_r \rangle$

Output: a Gröbner basis of telescopers for Pf

$\mathscr{L} := [1]$ // *list of monomials in* $\partial_1, \ldots, \partial_{r-1}$
$G := \{\}$ // *Gröbner basis being computed*
$Q := \{\}$ // *generators of the quotient*
$R := \{\}$ // *leading terms of the basis*
while $\mathscr{L} \neq \varnothing$ **do**
  Remove the first element μ of $\mathscr{L}$
  if μ is not a multiple of an element of R **then**
    Reduce $\mu P \to M(\partial_r)$; $F_\mu :=$ GeneralizedHermiteReduction$(M^\star(1), L^\star)$
    if $\exists$ a $\mathbb{K}(\underline{x})$-linear combination between F_μ and $\{F_\nu \mid \nu \in Q\}$ **then**
      $(a_\nu)_{\nu \in Q} :=$ coeff. of the relation $F_\mu = \sum_{\nu \in Q} a_\nu F_\nu$
      Add $\mu - \sum_{\nu \in Q} a_\nu \nu$ to G; add μ to R
    else
      Add μ to Q
      for $1 \leq i < r$ **do** Append the monomial $\partial_i \mu$ to $\mathscr{L}$
return G

Almost Identical to Algorithm GB

using the basis B

next pages

Linear algebra over polynomial matrices

Termination: if ideal of telescopers D-finite

Stopping prematurely gives correct telescopers

Generalized Hermite Reduction

Input: $f \in \mathbb{K}(x)$, $M(x, \partial_x)$ a linear differential operator

Output: $[f] \in \mathbb{K}(x)$ s.t. $f = [f] + M(g)$ for some $g \in \mathbb{K}(x)$

$f \mapsto [f]$ is $\mathbb{K}$-linear and $[f] = 0 \Leftrightarrow f \in \operatorname{Im} M$.

Canonical form

Hermite: special case when $M = \partial_x$

Algorithm similar to Liouville for rational solutions:
local analysis at singularities; cleanup on polynomials.

Example

$$\int_{-1}^{1} \frac{e^{-px} T_n(x)}{\sqrt{1-x^2}}\, dx = (-1)^n \pi\, I_n(p)$$

Ann f generated by G : $\partial_p + x1, \quad -nS_n + (x^2-1)\partial_x + (p(x^2-1)+(n+1)x),$

$$L(\partial_x) = (x^2-1)\partial_x^2 + (2p(x^2-1)+3x)\partial_x + (p^2(x^2-1) - n^2 + 3px + 1)1\,.$$

$$L^\star = (p^2(x^2-1) - n^2 + 3px + 1)\partial_x^2 + (-2px^2 + 4p^2x - 3x + 8p)\partial_x + (x^2 - 4px + 2p^2 - 4)$$

$q(x)$

Reduction step: $nS_n \to_G M = (x^2-1)\partial_x + (p(x^2-1)+(n+1)x)$

$$\to M^\star(1) = p(x^2-1) + (n-1)x$$

Local analysis:

$$L^\star((x-\alpha)^{-k}) \sim \begin{cases} C(x-\alpha)^{-k-2}, & q(\alpha) \neq 0 \\ C(x-\alpha)^{-k-1}, & \text{otherwise} \end{cases} \quad \text{and} \quad L^\star(x^k) \sim Cx^{k+2}\,.$$

$\to$ all polynomials reduce to $\mathbb{K} + \mathbb{K}x$.

Generalized Hermite Reduction

$$M = c_m(x)\partial_x^m + \cdots + c_0(x)$$

Local analysis: for $\alpha \in \overline{\mathbb{K}}, s \in \mathbb{Z}$,

indicial polynomial at α

$$M((x-\alpha)^s) = \mathrm{ind}_\alpha(s)(x-\alpha)^{s+\sigma_\alpha}(1+o(1)), \quad x \to \alpha\,.$$

$$M(x^s) = \mathrm{ind}_\infty(s)x^{s+\sigma_\infty}(1+o(1)), \quad x \to \infty\,.$$

Weak reduction of $f = \dfrac{c_k}{(x-\alpha)^k} + \cdots + \dfrac{c_1}{x-\alpha}$

from a partial fraction decomposition

incremental Gaussian reduction

$$H_\alpha(f) = \begin{cases} H_\alpha\left(f - \dfrac{c_k}{\mathrm{ind}_\alpha(-k-\sigma_\alpha)} M((x-\alpha)^{-k-\sigma_\alpha})\right) & \text{if } \mathrm{ind}_\alpha(-k-\sigma_\alpha) \neq 0, \\ \dfrac{c_k}{(x-\alpha)^k} + H_\alpha\left(f - \dfrac{c_k}{(x-\alpha)^k}\right) & \text{otherwise.} \end{cases}$$

Exercise: $H_\infty(f) = ?$

Full reduction: also use those that have been skipped, ie,

$$H_\alpha(M((x-\alpha)^{-k})), \quad c_m(\alpha) = 0 \text{ and } \mathrm{ind}_\alpha(-k) = 0 \text{ or } 0 < k \leq \sigma_\alpha\,.$$

Exercise: analogous set at ∞

Main Example for this Lecture

$$\int_{-1}^{1} \frac{e^{-px} T_n(x)}{\sqrt{1-x^2}}\, dx = (-1)^n \pi\, I_n(p)$$

```
> f:=exp(-p*x)*ChebyshevT(n,x)/sqrt(1-x^2);
```

$$f := \frac{e^{-px}\,\mathrm{ChebyshevT}(n,x)}{\sqrt{1-x^2}}$$

```
> redct(Int(f,x=-1..1),[n::shift,p::diff]);
```

$$[pD_n + pD_p - n, pD_n^2 - 2nD_n - 2D_n - p]$$

Interpretation: the integral $F_n(p)$ satisfies

$$pF_{n+1} + pF_n' - nF_n = 0, \quad pF_{n+2} - 2(n+1)F_{n+1} - pF_n = 0$$

Conclusion: Panorama of Summation & Integration Algorithms in this Course

With certificates:

Rely on finding rational solutions of parameterized linear differential/recurrence equations

dim $\mathbb{O}/I$ =1

Integration of hyperexponential functions — Almkvist-Zeilberger 1990

Summation of hypergeometric sequences — Zeilberger 1992

finite

General D-finite functions/sequences — Chyzak 2000

Reduction-based, certificate free:

Rely on solving (large) linear systems with polynomial coefficients

single $\int$

Hermite for bivariate rational functions — Bostan-Chen-Chyzak-Li 2010

Generalized Hermite for gl D-finite $\int$ — Bostan-Chyzak-Lairez-S. 2018

multiple $\int$, $\sum$ restricted fcns

Griffiths-Dwork for multivariate rational fcns — Bostan-Lairez-S. 2013

Multiple binomial sums via generating fcns — Bostan-Lairez-S. 2017

Algorithms waiting to be designed: gl D-finite $\sum$, multiple D-finite $\sum$ and $\int$,…

The End