Rational and hypergeometric solutions of linear recurrence equations

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How To Do MONTHLY Problems With Your Computer

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Fortunately, on 20 April 1977, all of this kludgery was rendered obsolete when I found a decision procedure for this problem. (A discrete analog to the Risch algorithm for indefinite integration.)

—R. William Gosper, Jr., *Indefinite Hypergeometric Sums in MACSYMA* (1977)

1. INTRODUCTION. The problem of finding simple evaluations of major classes of sums that involve factorials, binomial coefficients, and their q-analogues, has been completely solved. Sums that have the rather general form specified in Section 3 can all be done algorithmically, that is to say, you can do them on your own PC. Your computer evaluates the sum as a simple formula, if that's possible, and gives you a proof that you can check, or gives you a proof that your sum cannot be "done" in simple closed form, if that is the case.

We first briefly describe the algorithms and the theory that have achieved this goal. Second, to illustrate both the scope of the method and the fact that in some interesting cases human intervention still helps, we show how these computer methods would have fared in attacking 27 problems that have appeared over the years in the Problems section of this MONTHLY.

Rational solutions of LREs

Task: Given $p_0, \dots, p_r \in \mathbb{K}[n]$, compute a basis of solutions in $\mathbb{K}(n)$ of

$$\text{(LRE):} \qquad p_r(n)u_{n+r} + \cdots + p_0(n)u_n = 0.$$

▷ Even assuming numerator/denominator bounds, a naive approach by undetermined coefficients would lead to a non-linear polynomial system.

▷ Better strategy [Abramov, 1995]:

1. find a multiple of denominators of all rational solutions $\longrightarrow$ singularity analysis
2. reduce to polynomial solutions for numerators $\longrightarrow$ change of unknowns

Rational solutions of LREs: singularity analysis

Task: Given $p_0, \dots, p_r \in \mathbb{K}[n]$, compute a basis of solutions in $\mathbb{K}(n)$ of

$$\text{(LRE):} \qquad p_r(n)u_{n+r} + \cdots + p_0(n)u_n = 0.$$

Assume $u_n = P(n)/Q(n)$ solution of (LRE) with $\gcd(P, Q) = 1$. Multiplying

$$p_r(n)\frac{P(n+r)}{Q(n+r)} + \cdots + p_0(n)\frac{P(n)}{Q(n)} = 0$$

by

$$M(n) = \operatorname{lcm}(Q(n+1), \dots, Q(n+r))$$

yields

$$Q(n) \mid p_0(n)M(n).$$

▷ Idea in simplest case: Assume Q has no two roots with integer difference. Then $\gcd(Q, M) = 1$. Thus Q divides $p_0(n)$. Similarly, Q divides $p_r(n-r)$.

Task: Given $p_0, \ldots, p_r \in \mathbb{K}[n]$, compute a basis of solutions in $\mathbb{K}(n)$ of

$$\text{(LRE)}: \qquad p_r(n)u_{n+r} + \cdots + p_0(n)u_n = 0.$$

Theorem. Assume $u_n = P(n)/Q(n)$ solution of (LRE) with $\gcd(P,Q) = 1$.
Let α, β be roots in $\overline{\mathbb{K}}$ of Q s.t. $\alpha - \beta = H \in \mathbb{N}$ maximal (H = *dispersion* of Q).
Then $p_0(\alpha) = 0$ and $p_r(\beta - r) = 0$.

Proof: $\alpha + 1, \alpha + 2, \ldots,$ are not roots of Q, i.e., α not pole of $u_{n+1}, \ldots, u_{n+r}$.
So α is not a pole of $p_0(n)u_n$. Since α is a pole of u_n, necessarily $p_0(\alpha) = 0$.

$\beta - 1, \beta - 2, \ldots,$ are not roots of Q, i.e., β not pole of $u_{n-1}, \ldots, u_{n-r}$. So β is not a pole of $p_r(n-r)u_n$. Since β is a pole of u_n, necessarily $p_r(\beta - r) = 0$. $\square$

Corollary. The dispersion H is a root of $\mathrm{Res}_n(p_0(n+h), p_r(n-r)) \in \mathbb{K}[h]$.

Proof: $(n, h) = (\beta, H)$ is solution of system $p_0(n+h) = 0, p_r(n-r) = 0$. $\square$

Task: Given $p_0, \dots, p_r \in \mathbb{K}[n]$, compute a basis of solutions in $\mathbb{K}(n)$ of

$$\text{(LRE):} \qquad p_r(n) u_{n+r} + \cdots + p_0(n) u_n = 0.$$

Theorem. Assume $u_n = P(n)/Q(n)$ solution of (LRE) with $\gcd(P, Q) = 1$. Let H be the dispersion of Q. Then

$$Q(n) \mid \gcd\big(p_0(n) \cdots p_0(n+H),\ p_r(n-r) \cdots p_r(n-r-H)\big)$$

Proof: We've already proved that $Q(n) \mid p_0(n) \operatorname{lcm}(Q(n+1), \dots, Q(n+r))$. Substituting $n \leftarrow n+1$ and repeating yields:

$$\begin{aligned} Q(n) &\mid p_0(n) \operatorname{lcm}(p_0(n+1) \operatorname{lcm}(Q(n+2), \dots, Q(n+r+1)), Q(n+2), \dots, Q(n+r)) \\ &\mid p_0(n) p_0(n+1) \operatorname{lcm}(Q(n+2), \dots, Q(n+r+1)) \\ \cdots &\mid p_0(n) \cdots p_0(n+j) \operatorname{lcm}(Q(n+j+1), \dots, Q(n+j+r)). \end{aligned}$$

As $\gcd(Q(n), Q(n+j)) = 1$ for $j > H$, we get $Q(n) \mid p_0(n) \cdots p_0(n+H)$. $\square$

Rational solutions of LREs: singularity analysis –general case

Task: Given $p_0, \dots, p_r \in \mathbb{K}[n]$, compute a basis of solutions in $\mathbb{K}(n)$ of

$$\text{(LRE):} \qquad p_r(n)u_{n+r} + \cdots + p_0(n)u_n = 0.$$

Theorem. Assume $u_n = P(n)/Q(n)$ solution of (LRE) with $\gcd(P, Q) = 1$. Let H be the dispersion of Q. Then

$$Q(n) \mid \gcd\big(p_0(n) \cdots p_0(n+H),\ p_r(n-r) \cdots p_r(n-r-H)\big)$$

▷ A refinement: Q divides the (generally smaller) polynomial

$$\prod_{i=1}^{m} \prod_{j=0}^{h_i} \gcd(p_0(n-j+h_i), p_r(n-j-r)),$$

where $h_1 > h_2 > \cdots > h_m \geq 0$ are the roots in $\mathbb{N}$ of

$$R(h) = \text{Res}_n\big(p_0(n+h), p_r(n-r)\big).$$

Rational solutions of LREs: Abramov's algorithm

Abramov(LRE)

In: A LRE $p_r(n)u_{n+r} + \cdots + p_0(n)u_n = 0$ with $p_i \in \mathbb{K}[n]$

Out: A multiple of the denominator of all its rational solutions.

1. Compute the polynomial
$$R(h) = \mathrm{Res}_n\big(p_0(n+h), p_r(n-r)\big).$$
2. If R has no roots in $\mathbb{N}$, then return 1; else, let $h_1 > h_2 > \cdots > h_m \geq 0$ be its roots in $\mathbb{N}$. Initialize Q to 1, A to $p_0(n)$, B to $p_r(n-r)$.
3. For $i = 1, \ldots, m$ do
   $g(n) := \gcd\big(A(n+h_i), B(n)\big)$;
   $Q(n) := g(n)g(n-1)\cdots g(n-h_i)Q(n)$;
   $A(n) := A(n)/g(n-h_i)$;
   $B(n) := B(n)/g(n)$.
4. Return Q. // $\prod_{i=1}^{m}\prod_{j=0}^{h_i} \gcd(p_0(n-j+h_i), p_r(n-j-r))$

Hypergeometric solutions of LREs: Petkovšek's algorithm

Task: Given $p_0, \dots, p_r \in \mathbb{K}[n]$, compute a basis of hypergeometric solutions

$$\text{(LRE):} \qquad p_r(n)u_{n+r} + \cdots + p_0(n)u_n = 0.$$

▷ Assume $u_{n+1} = \frac{P(n)}{Q(n)} u_n$, with (u_n) solution of (LRE) with $\gcd(P, Q) = 1$.

▷ Injecting into (LRE) yields:

$$\begin{aligned} p_r(n)P(n+r-1)P(n+r-2)\cdots P(n) + \\ p_{r-1}(n)Q(n+r-1)P(n+r-2)\cdots P(n) + \cdots \\ p_1(n)Q(n+r-1)\cdots Q(n-1)P(n) + \\ p_0(n)Q(n+r-1)\cdots Q(n) = 0. \end{aligned} \tag{1}$$

▷ Easy case: If $\gcd\big(P(n), Q(n+i)\big) = 1$ for $i = 0, \dots, r-1$, then $P(n)$ divides $p_0(n)$ and $Q(n+r-1)$ divides $p_r(n)$.

Task: Given $p_0, \dots, p_r \in \mathbb{K}[n]$, compute a basis of hypergeometric solutions

$$\text{(LRE):} \qquad p_r(n)u_{n+r} + \cdots + p_0(n)u_n = 0.$$

▷ Assume $u_{n+1} = \frac{P(n)}{Q(n)} u_n$, with (u_n) solution of (LRE) with $\gcd(P, Q) = 1$.

▷ Injecting into (LRE) yields:

$$\begin{aligned} p_r(n)P(n+r-1)P(n+r-2)\cdots P(n) &+ \\ p_{r-1}(n)Q(n+r-1)P(n+r-2)\cdots P(n) &+ \cdots \\ p_1(n)Q(n+r-1)\cdots Q(n-1)P(n) &+ \\ p_0(n)Q(n+r-1)\cdots Q(n) &= 0. \end{aligned} \tag{2}$$

▷ General case: use Gosper–Petkovšek decomposition

$$\frac{u_{n+1}}{u_n} = \frac{P(n)}{Q(n)} = Z \frac{A(n)}{B(n)} \frac{C(n+1)}{C(n)},$$

where $Z \in \mathbb{K}$, and A, B, C monic polynomials with (i) $\gcd(A(n), C(n)) = 1$, (ii) $\gcd\big(B(n), C(n+1)\big) = 1$ and (iii) $\gcd\big(A(n), B(n+i)\big) = 1$ for all $i \in \mathbb{N}$.

Task: Given $p_0, \dots, p_r \in \mathbb{K}[n]$, compute a basis of hypergeometric solutions

$$\text{(LRE):} \qquad p_r(n)u_{n+r} + \cdots + p_0(n)u_n = 0.$$

▷ Assume $u_{n+1} = \frac{P(n)}{Q(n)}u_n$, with (u_n) solution of (LRE) with $\gcd(P, Q) = 1$.

▷ Using Gosper–Petkovšek decomposition yields new (LRE):

$$\begin{aligned} Z^r p_r(n)A(n+r-1)\cdots A(n)C(n+r) &+ \\ Z^{r-1}p_{r-1}(n)B(n+r-1)A(n+r-2)\cdots A(n)C(n+r-1) &+ \cdots + \\ Zp_1(n)B(n+r-1)\cdots B(n+1)A(n)C(n+1) &+ \\ p_0(n)B(n+r-1)\cdots B(n)C(n) &= 0. \end{aligned} \tag{3}$$

▷ Properties of Gosper–Petkovšek decomposition imply:

$$A(n) \mid p_0(n), \qquad B(n+r-1) \mid p_r(n).$$

▷ Petkovšek's algorithm (1992): for each pair (A, B) of factors of p_0 and p_r, the lc of (LHS) of (3) gives a polynomial constraint on Z. For each Z, search for polynomial solutions $C(n)$.

Petkovšek(LRE)

In: A LRE $p_r(n)u_{n+r} + \cdots + p_0(n)u_n = 0$ with $p_i \in \mathbb{K}[n]$

Out: The set of its hypergeometric solutions.

1. Initialize S to $\emptyset$; factor $p_0(n)$ and $p_r(n)$.
2. For all monic divisors $A(n)$ of $p_0(n)$ and $B(n)$ of $p_r(n-r+1)$:
   1. extract the leading coefficient $d(Z)$ of
   $$\sum_{i=0}^{r} Z^i p_i(n) B(n+r-1) \cdots B(n+i+1) A(n+i) \cdots A(n) C(n+i)\,; \qquad (4)$$
   2. for each ζ such that $d(\zeta) = 0$
      1. compute the polynomial solutions $C(n)$ of (4);
      2. for each solution C, define a hypergeometric sequence t_n by
      $$t_{n+1}/t_n = \zeta \frac{A(n)}{B(n)} \frac{C(n+1)}{C(n)}$$
      and add it to S;
3. Return S.

Apéry's sequence $1, 5, 73, 1445, 33001, \ldots$ is not hypergeometric

[Apéry 1978]: $A_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2$ satisfies the recurrence

$$(n+1)^3 A_{n+1} + n^3 A_{n-1} = (2\,n+1)\,(17\,n^2 + 17\,n + 5) A_n.$$

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▷ The equation for Z is $Z^2 - 34Z + 1 = 0$, with solutions $Z = 17 \pm 12\sqrt{2}$.

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```
> with(LREtools):
> recA:=(n+1)^3*u(n+1)-(2*n+1)*(17*n^2 +17*n+5)*u(n)+n^3*u(n-1):
> hypergeomsols(recA, u(n), {});
```

[0]

Examples

▷ From the Putnam competition (1990, A-1)

$$2, 3, 6, 14, 40, 152, 784, 5168, \dots$$

```
> with(LREtools):
> recP := u(n) = (n + 4)*u(n-1) - 4*n*u(n-2) + (4*n - 8)*u(n-3):
> polysols(recP, u(n), {});
> ratpolysols(recP, u(n), {});
> hypergeomsols(recP, u(n), {}, output=basis);
```

$$[\ n!,\ 2^n\ ]$$

▷ From a Monthly problem (AMM-10375, 1994)

```
> recM:=u(n+2) = 2*(2*n+3)^2*u(n+1)-4*(n+1)^2*(2*n+1)*(2*n+3)*u(n):
> hypergeomsols(recM, u(n), {}, output=basis);
```

$$[\ (2n)!\ ]$$

Show that the sequence

$$u_n := \sum_{k=1}^{n} \frac{1}{k!}$$

is not hypergeometric:

1. by using Gosper's algorithm;
2. by using Petkovšek's algorithm (applied to a second-order recurrence equation satisfied by u_n, to be determined).

On the previous homework

Compute a telescoper for the diagonal of the rational power series

$$\frac{1}{1-x-y} = \sum_{i,j \geq 0} \binom{i+j}{i} x^i y^j$$

in three different ways:

1. using the 2G (Almkvist-Zeilberger) creative telescoping algorithm;
2. using the 4G (Hermite reduction-based) creative telescoping algorithm;
3. using the 4G (Griffiths-Dwork reduction-based) creative telescoping algorithm.

Ex 1.

Compute a telescoper for the diagonal of the rational power series

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using the Almkvist-Zeilberger (2G) creative telescoping algorithm;

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▷ We need to compute a linear differential equation for the integral

$$I(t) = \oint H(t,x)\,dx, \quad \text{where } H = \frac{1}{x-x^2-t}$$

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▷ We will use the AZ algorithm to solve the *telescoping equation*

$$L(t,\partial_t)(H) = \partial_x(G \cdot H)$$

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▷ Conclusion: $(1-4t) \cdot H_t - 2 \cdot H = \partial_x \left(\frac{2x-1}{x-x^2-t} \right)$, where $H_t := \partial_t H$

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▷ I.e., we search for $c_0(t) \in \mathbb{C}(t)$ and $G(t,x) \in \mathbb{C}(t,x)$, such that

$$H_t + c_0(t) \cdot H = G_x \cdot H + G \cdot H_x$$

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$$H_t + c_0(t) \cdot H = G_x \cdot H + G \cdot H_x$$

▷ Dividing by H and switching LHS and RHS yields the equivalent equation

$$G_x + G \cdot \frac{H_x}{H} = \frac{H_t}{H} + c_0(t),$$

i.e.

$$G_x + G \cdot \frac{2\,x - 1}{x - x^2 - t} = \frac{1}{x - x^2 - t} + c_0(t)$$

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▷ Possible poles of G at $X_\pm = \frac{1}{2}\cdot\left(1 \pm \sqrt{1-4t}\right)$, with local behavior:

$$G = (x-X_\pm)^v + (\text{h.o.t.}), \quad G_x = v\cdot(x-X_\pm)^{v-1} + (\text{h.o.t.})$$

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▷ Around $x = X_\pm$, LHS of (5) writes

$$\left(v \cdot (x-X_\pm)^{v-1} + \cdots\right) + \left(\frac{1-2x}{x-X_\mp} \cdot (x-X_\pm)^{v-1} + \cdots\right)$$

It has valuation $v-1$, while RHS of (5) has valuation -1; thus $v=0$

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It has valuation $v-1$, while RHS of (5) has valuation -1; thus $v=0$

▷ Look for polynomial $G = g_N(t) \cdot x^N + (\text{l.o.t.})$ with $g_N \neq 0$. Then

$$G_x = N \cdot g_N(t) \cdot x^{N-1} + (\text{l.o.t.})$$

so numer(LHS): $\left(-x^2 \cdot N \cdot g_N(t) \cdot x^{N-1} + \cdots\right) + \left((2x-1) \cdot g_N(t) \cdot x^N + \cdots\right)$
has degree $N+1$ if $N \neq 2$, while numer(RHS) has degree ≤ 2

$$G_x + G \cdot \frac{2\,x-1}{x-x^2-t} = \frac{1}{x-x^2-t} + c_0(t) \qquad (5)$$

▷ Possible poles of G at $X_\pm = \frac{1}{2} \cdot \left(1 \pm \sqrt{1-4t}\right)$, with local behavior:

$$G = (x - X_\pm)^v + (\text{h.o.t.}), \quad G_x = v \cdot (x-X_\pm)^{v-1} + (\text{h.o.t.})$$

▷ Around $x = X_\pm$, LHS of (5) writes

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It has valuation $v-1$, while RHS of (5) has valuation -1; thus $v = 0$

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so numer(LHS): $\left(-x^2 \cdot N \cdot g_N(t) \cdot x^{N-1} + \cdots\right) + \left((2x-1) \cdot g_N(t) \cdot x^N + \cdots\right)$
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▷ Thus $N \leq 2$, and $G = g_2 x^2 + g_1 x + g_0$, so (5) yields a linear system / $\mathbb{C}(t)$

$$c_0 + g_1 + g_2 = 0, \quad 2\,t\,g_2 + c_0 - 2\,g_0 = 0, \quad t \cdot (c_0 - g_1) - 1 - g_0 = 0$$

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▷ Conclusion: $(4t-1)H_t + 2H = \partial_x\left(\frac{1-2x}{x-x^2-t}\right)$.

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Compute a telescoper for the diagonal of the rational power series

$$\frac{1}{1-x-y} = \sum_{i,j\geq 0} \binom{i+j}{i} x^i y^j$$

using the Hermite reduction-based (4G) creative telescoping algorithm.

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▷ Therefore, $(1-4t)H_t - 2H = \partial_x\left(\frac{2x-1}{x-x^2-t}\right).$

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▷ Hermite reduction computation: extended gcd of $g = x - x^2 - t$ and g_x :

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▷ Integrating by parts yields:

$$\frac{1}{g^2} = \frac{1}{2\delta} \cdot \frac{1}{g} + \partial_x \left(\frac{2x-1}{4\delta \cdot g}\right).$$

i.e.,

$$\frac{1}{g^2} = \frac{2}{1-4t} \cdot \frac{1}{g} + \partial_x \left(\frac{2x-1}{(1-4t)g}\right).$$