P-recursive sequences and D-finite series

Alin Bostan

ENS Lyon
M2, CR06
September 14, 2020
TOOLS FOR PROOFS

Resultants
**Definition**

The **Sylvester matrix** of $A = a_m x^m + \cdots + a_0 \in \mathbb{K}[x]$, $(a_m \neq 0)$, and of $B = b_n x^n + \cdots + b_0 \in \mathbb{K}[x]$, $(b_n \neq 0)$, is the square matrix of size $m+n$:

\[
Syl(A, B) = \begin{bmatrix}
  a_m & a_{m-1} & \cdots & a_0 \\
  a_m & a_{m-1} & \cdots & a_0 \\
  \vdots & \ddots & \ddots & \vdots \\
  b_n & b_{n-1} & \cdots & b_0 \\
  b_n & b_{n-1} & \cdots & b_0 \\
  \vdots & \ddots & \ddots & \ddots \\
  b_n & b_{n-1} & \cdots & b_0 \\
\end{bmatrix}
\]

The **resultant** $\text{Res}(A, B)$ of $A$ and $B$ is the determinant of $Syl(A, B)$.

- Definition extends to polynomials over any **commutative ring** $\mathbb{R}$. 


Key observation

If $A = a_m x^m + \cdots + a_0$ and $B = b_n x^n + \cdots + b_0$, then

$$
\begin{bmatrix}
  a_m & a_{m-1} & \cdots & a_0 \\
  \vdots & \vdots & \ddots & \vdots \\
  b_n & b_{n-1} & \cdots & b_0 \\
\end{bmatrix}
\times
\begin{bmatrix}
  \alpha^{m+n-1} \\
  \vdots \\
  \alpha \\
  1 \\
\end{bmatrix}
= 
\begin{bmatrix}
  \alpha^{n-1} A(\alpha) \\
  \vdots \\
  A(\alpha) \\
  \alpha^{m-1} B(\alpha) \\
  \vdots \\
  B(\alpha) \\
\end{bmatrix}
$$

Corollary: If $A(\alpha) = B(\alpha) = 0$, then $\text{Res}(A, B) = 0$. 
Example: the discriminant

The **discriminant** of $A$ is the resultant of $A$ and of its derivative $A'$. E.g. for $A = ax^2 + bx + c$,

$$
\text{Disc}(A) = \text{Res}(A, A') = \det \begin{bmatrix}
    a & b & c \\
    2a & b \\
    2a & b
\end{bmatrix} = -a(b^2 - 4ac).
$$

E.g. for $A = ax^3 + bx + c$,

$$
\text{Disc}(A) = \text{Res}(A, A') = \det \begin{bmatrix}
    a & 0 & b & c \\
    a & 0 & b & c \\
    3a & 0 & b \\
    3a & 0 & b
\end{bmatrix} = a^2(4b^3 + 27ac^2).
$$

▶ The discriminant vanishes when $A$ and $A'$ have a common root, that is when $A$ has a multiple root.
Main properties

- **Link with gcd** \( \text{Res} (A, B) = 0 \) if and only if \( \text{gcd}(A, B) \) is non-constant.

- **Elimination property**
  There exist \( U, V \in \mathbb{K}[x] \) not both zero, with \( \deg(U) < n \), \( \deg(V) < m \) and such that the following Bézout identity holds in \( \mathbb{K} \cap (A, B) \):
  \[
  \text{Res} (A, B) = UA + VB.
  \]

- **Poisson’s formula**
  If \( A = a(x - \alpha_1) \cdots (x - \alpha_m) \) and \( B = b(x - \beta_1) \cdots (x - \beta_n) \), then
  \[
  \text{Res} (A, B) = a^n b^m \prod_{i,j} (\alpha_i - \beta_j) = a^n \prod_{1 \leq i \leq m} B(\alpha_i).
  \]

- **Multiplicativity**
  \[
  \text{Res} (A \cdot B, C) = \text{Res} (A, C) \cdot \text{Res} (B, C), \quad \text{Res} (A, B \cdot C) = \text{Res} (A, B) \cdot \text{Res} (A, C).
  \]
Proof of Poisson’s formula

- Direct consequence of the key observation:
  If \( A = (x - \alpha_1) \cdots (x - \alpha_m) \) and \( B = (x - \beta_1) \cdots (x - \beta_n) \) then

\[
\text{Syl}(A, B) \times \begin{bmatrix}
\beta_1^{m+n-1} & \cdots & \beta_n^{m+n-1} & \alpha_1^{m+n-1} & \cdots & \alpha_m^{m+n-1} \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\beta_1 & \cdots & \beta_n & \alpha_1 & \cdots & \alpha_m \\
1 & \cdots & 1 & 1 & \cdots & 1
\end{bmatrix} =
\]

\[
\begin{bmatrix}
\beta_1^{n-1} A(\beta_1) & \cdots & \beta_n^{n-1} A(\beta_n) & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
A(\beta_1) & \cdots & A(\beta_n) & 0 & \cdots & 0 \\
0 & \cdots & 0 & \alpha_1^{m-1} B(\alpha_1) & \cdots & \alpha_m^{m-1} B(\alpha_m) \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & B(\alpha_1) & \cdots & B(\alpha_m)
\end{bmatrix}
\]

- To conclude, take determinants and use Vandermonde’s formula
Let $A = \prod_i (x - \alpha_i)$ and $B = \prod_j (x - \beta_j)$ be polynomials of $\mathbb{K}[x]$. Then

$$\prod_{i,j} (t - (\alpha_i + \beta_j)) = \text{Res}_x(A(x), B(t - x)),$$

$$\prod_{i,j} (t - (\beta_j - \alpha_i)) = \text{Res}_x(A(x), B(t + x)),$$

$$\prod_{i,j} (t - \alpha_i \beta_j) = \text{Res}_x(A(x), x^{\deg B} B(t / x)),$$

$$\prod_i (t - B(\alpha_i)) = \text{Res}_x(A(x), t - B(x)).$$

In particular, the set $\overline{\mathbb{Q}}$ of algebraic numbers is a field.

**Proof:** Poisson’s formula. E.g., first one: $\prod_i B(t - \alpha_i) = \prod_{i,j} (t - \alpha_i - \beta_j)$.

$\triangleright$ The same formulas apply *mutatis mutandis* to algebraic power series.
Application of the application: proving a nice identity

\[ \sqrt[3]{\sqrt[3]{2}} - 1 = \sqrt[3]{\frac{1}{9}} - \sqrt[3]{\frac{2}{9}} + \sqrt[3]{\frac{4}{9}} \]
Application of the application: proving a nice identity

\[ \sqrt[3]{3/2} - 1 = \sqrt[3]{1/9} - \sqrt[3]{2/9} + \sqrt[3]{4/9} \]

If \( a = \sqrt[3]{2} \) and \( b = \sqrt[3]{1/9} \), then RHS is \( bc \) where \( c = 1 - a + a^2 \), LHS is \( (a - 1)^{\frac{1}{3}} \), and resultants provide (equal) annihilating polynomials for RHS and LHS.
Application of the application: proving a nice identity

\[ \sqrt[3]{3/2} - 1 = \sqrt[3]{1/9} - \sqrt[3]{2/9} + \sqrt[3]{4/9} \]

If \( a = \sqrt[3]{2} \) and \( b = \sqrt[3]{1/9} \), then \( \text{RHS} \) is \( bc \) where \( c = 1 - a + a^2 \), \( \text{LHS} \) is \( (a - 1)^{\frac{1}{3}} \), and resultants provide (equal) annihilating polynomials for \( \text{RHS} \) and \( \text{LHS} \):

\[
\begin{align*}
> & \ \text{Pc} := \text{resultant}(a^3 - 2, x - (1 - a + a^2), a); \\
> & \ x^3 - 3x^2 + 9x - 9 \\
> & \ \text{PR} := \text{resultant}(\text{Pc}, \text{numer}(9*(t/x)^3 - 1), x); \\
> & \ 729 \left( t^9 + 3t^6 + 3t^3 - 1 \right) \\
> & \ \text{PL} := \text{expand}((t^3 + 1)^3 - 2); \\
> & \ t^9 + 3t^6 + 3t^3 - 1 
\end{align*}
\]

Why is this a proof?

Hint: The above polynomial has one single real root.

Alin Bostan

P-recursive sequences and D-finite series
Application of the application: proving a nice identity

\[ \sqrt[3]{3/2} - 1 = \sqrt[3]{1/9} - \sqrt[3]{2/9} + \sqrt[3]{4/9} \]

If \( a = \sqrt[3]{2} \) and \( b = \sqrt[3]{1/9} \), then RHS is \( bc \) where \( c = 1 - a + a^2 \), LHS is \((a - 1)^{1/3}\), and resultants provide (equal) annihilating polynomials for RHS and LHS.

\[ \text{Pc := resultant}(a^3 - 2, x - (1 - a + a^2), a); \]
\[ x^3 - 3x^2 + 9x - 9 \]

\[ \text{PR := resultant}(\text{Pc}, \text{numer}(9*(t/x)^3 - 1), x); \]
\[ 729(t^9 + 3t^6 + 3t^3 - 1) \]

\[ \text{PL := expand}((t^{3+1})^3 - 2); \]
\[ t^9 + 3t^6 + 3t^3 - 1 \]

Why is this a proof?
Application of the application: proving a nice identity

\[ \sqrt[3]{3/2} - 1 = \sqrt[3]{1/9} - \sqrt[3]{2/9} + \sqrt[3]{4/9} \]

If \( a = \sqrt[3]{2} \) and \( b = \sqrt[3]{1/9} \), then RHS is \( bc \) where \( c = 1 - a + a^2 \), LHS is \((a - 1)^{\frac{1}{3}}\), and resultants provide (equal) annihilating polynomials for RHS and LHS

\[ > \text{Pc:=resultant(a^3-2, x-(1-a+a^2), a);} \]

\[ x^3 - 3 x^2 + 9 x - 9 \]

\[ > \text{PR:=resultant(Pc, numer(9*(t/x)^3-1), x);} \]

\[ 729 \left( t^9 + 3 t^6 + 3 t^3 - 1 \right) \]

\[ > \text{PL:=expand((t^3+1)^3-2);} \]

\[ t^9 + 3 t^6 + 3 t^3 - 1 \]

\( \triangleright \) Why is this a proof? Hint: The above polynomial has one single real root
Bonus: another beautiful identity of Ramanujan’s

\[
\frac{\sin \frac{2\pi}{7}}{\sin^2 \frac{3\pi}{7}} - \frac{\sin \frac{\pi}{7}}{\sin^2 \frac{2\pi}{7}} + \frac{\sin \frac{3\pi}{7}}{\sin^2 \frac{\pi}{7}} = 2\sqrt{7}.
\]
Bonus: another beautiful identity of Ramanujan’s

\[ \frac{\sin \frac{2\pi}{7}}{\sin^2 \frac{3\pi}{7}} - \frac{\sin \frac{\pi}{7}}{\sin^2 \frac{2\pi}{7}} + \frac{\sin \frac{3\pi}{7}}{\sin^2 \frac{\pi}{7}} = 2\sqrt{7}. \]

▶ If \( a = \pi/7 \) and \( x = e^{ia} \), then \( x^7 = -1 \) and \( \sin(ka) = \frac{x^k-x^{-k}}{2i} \)
Bonus: another beautiful identity of Ramanujan’s

\[
\frac{\sin \frac{2\pi}{7}}{\sin^2 \frac{3\pi}{7}} - \frac{\sin \frac{\pi}{7}}{\sin^2 \frac{2\pi}{7}} + \frac{\sin \frac{3\pi}{7}}{\sin^2 \frac{\pi}{7}} = 2\sqrt{7}.
\]

▶ If \( a = \pi/7 \) and \( x = e^{ia} \), then \( x^7 = -1 \) and \( \sin(ka) = \frac{x^k-x^{-k}}{2i} \)

▶ Since \( x \in \mathbb{Q} \), any rational expression in the \( \sin(ka) \) is in \( \mathbb{Q}(x) \), thus in \( \overline{\mathbb{Q}} \)
Bonus: another beautiful identity of Ramanujan’s

\[
\frac{\sin \frac{2\pi}{7}}{\sin^2 \frac{3\pi}{7}} - \frac{\sin \frac{\pi}{7}}{\sin^2 \frac{2\pi}{7}} + \frac{\sin \frac{3\pi}{7}}{\sin^2 \frac{\pi}{7}} = 2\sqrt{7}.
\]

▷ If \( a = \pi/7 \) and \( x = e^{ia} \), then \( x^7 = -1 \) and \( \sin(ka) = \frac{x^k - x^{-k}}{2i} \)

▷ Since \( x \in \mathbb{Q} \), any rational expression in the \( \sin(ka) \) is in \( \mathbb{Q}(x) \), thus in \( \overline{\mathbb{Q}} \)

▷ In particular our LHS \( F(x) = \frac{N(x)}{D(x)} \) is an algebraic number

\[
> f:=\text{sin}(2*\text{a})/\text{sin}(3*\text{a})^2-\text{sin}(\text{a})/\text{sin}(2*\text{a})^2+\text{sin}(3*\text{a})/\text{sin}(\text{a})^2:
> \text{expand(\text{convert}(f,\text{exp}))}:
> F:=\text{normal(\text{subs(\text{exp}(I*\text{a})=x,\%)})}:
\]

\[
\frac{2i \left( x^{16} + 5x^{14} + 12x^{12} + x^{11} + 20x^{10} + 3x^9 + 23x^8 + 3x^7 + 20x^6 + x^5 + 12x^4 + 5x^2 + 1 \right)}{x(x^2 - 1)(x^2 + 1)^2(x^4 + x^2 + 1)^2}
\]
Bonus: another beautiful identity of Ramanujan’s

\[
\frac{\sin \frac{2\pi}{7}}{\sin^2 \frac{3\pi}{7}} - \frac{\sin \frac{\pi}{7}}{\sin^2 \frac{2\pi}{7}} + \frac{\sin \frac{3\pi}{7}}{\sin^2 \frac{\pi}{7}} = 2\sqrt{7}.
\]

▷ If \( a = \frac{\pi}{7} \) and \( x = e^{ia} \), then \( x^7 = -1 \) and \( \sin(ka) = \frac{x^k - x^{-k}}{2i} \)

▷ Since \( x \in \mathbb{Q} \), any rational expression in the \( \sin(ka) \) is in \( \mathbb{Q}(x) \), thus in \( \mathbb{Q} \)

▷ In particular our LHS \( F(x) = \frac{N(x)}{D(x)} \) is an algebraic number

\[
> f:=\text{sin}(2*a)/\text{sin}(3*a)^2-\text{sin}(a)/\text{sin}(2*a)^2+\text{sin}(3*a)/\text{sin}(a)^2:
> \text{expand(\text{convert}(f,\text{exp}))}:
> \text{F:=\text{normal}((\text{subs}(\text{exp}(I*a)=x,%))}):
\]

\[
\frac{2i \left( x^{16} + 5 x^{14} + 12 x^{12} + x^{11} + 20 x^{10} + 3 x^9 + 23 x^8 + 3 x^7 + 20 x^6 + x^5 + 12 x^4 + 5 x^2 + 1 \right)}{ \left( x^2 - 1 \right) \left( x^2 + 1 \right)^2 \left( x^4 + x^2 + 1 \right)^2 } 
\]

▷ Get \( R \) in \( \mathbb{Q}[t] \) with root \( F(x) \), via \( \text{resultant} \ \text{Res}_x(x^7 + 1, t \cdot D(x) - N(x)) \)

\[
> \text{R:=\text{factor}(\text{resultant}(x^7+1,t*\text{denom(F)}-\text{numer(F)},x))};
\]

\[-1274i \left( t^2 - 28 \right)^3 \]
Shanks’ 1974 identities

1. Prove that

\[ \sqrt{11 + 2\sqrt{29}} + \sqrt{16 - 2\sqrt{29} + 2\sqrt{55 - 10\sqrt{29}}} = \sqrt{5} + \sqrt{22 + 2\sqrt{5}} \]
1. Prove that
\[ \sqrt{11 + 2\sqrt{29}} + \sqrt{16 - 2\sqrt{29} + 2\sqrt{55 - 10\sqrt{29}}} = \sqrt{5} + \sqrt{22 + 2\sqrt{5}} \]

2. Prove that for any \( m, n \) in \( \mathbb{N} \)
\[ \sqrt{\sqrt{m + n + \sqrt{n}} + \sqrt{\sqrt{m + n} + m - \sqrt{n} + 2\sqrt{m \left( \sqrt{m + n} - \sqrt{n} \right)}}} = \sqrt{m} + \sqrt{2\sqrt{m + n} + 2\sqrt{m}} \]
1. Prove that
\[ \sqrt{11 + 2\sqrt{29}} + \sqrt{16 - 2\sqrt{29} + 2\sqrt{55 - 10\sqrt{29}}} = \sqrt{5} + \sqrt{22 + 2\sqrt{5}} \]

2. Prove that for any \( m, n \) in \( \mathbb{N} \)
\[ \sqrt{\sqrt{m + n} + \sqrt{n} + \sqrt{m + n + m - \sqrt{n} + 2\sqrt{m \left(\sqrt{m + n} - \sqrt{n}\right)}}} = \sqrt{m} + \sqrt{2\sqrt{m + n} + 2\sqrt{m}} \]

▷ Exercise!
TOOLS FOR PROOFS

D-Finiteness
Definition: A series \( f \in \mathbb{K}[[x]] \) is D-finite (differentially finite), or holonomic, if its derivatives generate a finite-dimensional vector space over \( \mathbb{K}(x) \), i.e.

\[
q_s(x)f^{(s)}(x) + q_{s-1}(x)f^{(s-1)}(x) + \cdots + q_0(x)f(x) = 0.
\]
D-finite Series & P-recursive Sequences

**Definition:** A series \( f \in \mathbb{K}[[x]] \) is **D-finite** (differentially finite), or **holonomic**, if its derivatives generate a finite-dimensional vector space over \( \mathbb{K}(x) \), i.e.

\[
q_s(x)f^{(s)}(x) + q_{s-1}(x)f^{(s-1)}(x) + \cdots + q_0(x)f(x) = 0.
\]

**Definition:** A sequence \( (u_n) \in \mathbb{K}^\mathbb{N} \) is **P-recursive** (polynomially recursive) if its shifts \( (u_n, u_{n+1}, \ldots) \) generate a finite-dimensional vector space over \( \mathbb{K}(n) \), i.e.

\[
p_r(n)u_{n+r} + p_{r-1}(n)u_{n+r-1} + \cdots + p_0(n)u_n = 0, \quad n \geq 0.
\]

Examples: \( \exp, \log, \sin, \cos, \sinh, \cosh, \arccos, \arccosh, \arcsin, \arcsinh, \arctan, \arctanh, \arccot, \arccoth, \arccsc, \arccsch, \text{arcsec}, \text{arcsech} \), \( \text{pFq} \) (includes Bessel \( J, Y, I, \text{and K} \), Airy \( \text{Ai and Bi} \) and polylogarithms), Struve, Weber and Anger functions, the large class of algebraic functions,...
D-finite Series & P-recursive Sequences

**Definition**: A series \( f \in \mathbb{K}[[x]] \) is D-finite (differentially finite), or holonomic, if its derivatives generate a finite-dimensional vector space over \( \mathbb{K}(x) \), i.e.

\[
q_s(x)f^{(s)}(x) + q_{s-1}(x)f^{(s-1)}(x) + \cdots + q_0(x)f(x) = 0.
\]

**Definition**: A sequence \((u_n) \in \mathbb{K}^\mathbb{N}\) is P-recursive (polynomially recursive) if its shifts \((u_n, u_{n+1}, \ldots)\) generate a finite-dimensional vector space over \( \mathbb{K}(n) \), i.e.

\[
p_r(n)u_{n+r} + p_{r-1}(n)u_{n+r-1} + \cdots + p_0(n)u_n = 0, \quad n \geq 0.
\]

25% of Sloane’s encyclopedia (OEIS) and 60% of Abramowitz & Stegun’s HMF

**Examples**: \( \exp, \log, \sin, \cos, \sinh, \cosh, \arccos, \arccosh, \arcsin, \arccsinh, \arctan, \arctanh, \arccot, \arccoth, \arccsc, \arccsch, \arcsec, \arcsech, pFq \) (includes Bessel \( J, Y, I \) and \( K \), Airy \( Ai \) and \( Bi \) and polylogarithms), Struve, Weber and Anger functions, the large class of algebraic functions, …
Teaser 1: D-finite series and P-recursive seqs can be expanded optimally
\[ \rightarrow O(N) \text{ ops. for first } N \text{ coeffs.} \]
Teaser 1: D-finite series and P-recursive seqs can be expanded optimally
→ \( O(N) \) ops. for first \( N \) coeffs.

Teaser 2: D-finite series and P-recursive seqs are ubiquitous in applications
→ combinatorics, physics, applied maths
Teaser 1: D-finite series and P-recursive seqs can be expanded optimally
   \[ \rightarrow O(N) \text{ ops. for first } N \text{ coeffs.} \]

Teaser 2: D-finite series and P-recursive seqs are ubiquitous in applications
   \[ \rightarrow \text{combinatorics, physics, applied maths} \]

Teaser 3: Equations (recurrence, or differential) form a good data structure
   for their (infinite/transcendental) solutions
   \[ \rightarrow \text{sin}(x) \text{ encoded by } y'' + y = 0, y(0) = 0, y'(0) = 1 \]
Teaser 1: D-finite series and P-recursive seqs can be expanded optimally $\rightarrow O(N)$ ops. for first $N$ coeffs.

Teaser 2: D-finite series and P-recursive seqs are ubiquitous in applications $\rightarrow$ combinatorics, physics, applied maths

Teaser 3: Equations (recurrence, or differential) form a good data structure for their (infinite/transcendental) solutions $\rightarrow \sin(x)$ encoded by $y'' + y = 0, y(0) = 0, y'(0) = 1$

Teaser 4: Their closure properties are effective: analogous of effective closure properties for algebraic numbers via resultants $\rightarrow f + g, f \cdot g$
Teaser 1: D-finite series and P-recursive seqs can be expanded optimally
\[ \longrightarrow O(N) \text{ ops. for first } N \text{ coeffs.} \]

Teaser 2: D-finite series and P-recursive seqs are ubiquitous in applications
\[ \longrightarrow \text{ combinatorics, physics, applied maths} \]

Teaser 3: Equations (recurrence, or differential) form a good data structure
for their (infinite/transcendental) solutions
\[ \longrightarrow \sin(x) \text{ encoded by } y'' + y = 0, y(0) = 0, y'(0) = 1 \]

Teaser 4: Their closure properties are effective: analogous of effective closure
properties for algebraic numbers via resultants
\[ \longrightarrow f + g, f \cdot g \]

Teaser 5: Algorithms on D-finite series and P-recursive seqs allow for the
automatic proof of identities
\[ \longrightarrow \sin(x)^2 + \cos(x)^2 = 1 \]
Classes of power series

- **algebraic**
- **hypergeometric**
- **D-finite**

Alin Bostan

P-recursive sequences and D-finite series
Classes of power series

\[ S(t) = \sum_{n=0}^{\infty} s_n t^n \in \mathbb{Q}[[t]] \text{ is} \]

\( \blacktriangleright \) \text{algebraic if } P(t, S(t)) = 0 \text{ for some } P(x, y) \in \mathbb{Z}[x, y] \setminus \{0\}; \]
Classes of power series

$S(t) = \sum_{n=0}^{\infty} s_n t^n \in \mathbb{Q}[[t]]$ is

- **algebraic** if $P(t, S(t)) = 0$ for some $P(x, y) \in \mathbb{Z}[x, y] \setminus \{0\}$;

- **D-finite** if $c_r(t) S^{(r)}(t) + \cdots + c_0(t) S(t) = 0$ for some $c_i \in \mathbb{Z}[t]$, not all zero;
Classes of power series

\[ S(t) = \sum_{n=0}^{\infty} s_n t^n \in \mathbb{Q}[[t]] \] is

- **algebraic** if \( P(t, S(t)) = 0 \) for some \( P(x, y) \in \mathbb{Z}[x, y] \setminus \{0\} \);

- **D-finite** if \( c_r(t) S^{(r)}(t) + \cdots + c_0(t) S(t) = 0 \) for some \( c_i \in \mathbb{Z}[t] \), not all zero;

- **hypergeometric** if \( \frac{s_{n+1}}{s_n} \in \mathbb{Q}(n) \).
Classes of power series

\[ S(t) = \sum_{n=0}^{\infty} s_n t^n \in \mathbb{Q}[[t]] \text{ is}
\]

- **algebraic** if \( P(t, S(t)) = 0 \) for some \( P(x, y) \in \mathbb{Z}[x, y] \setminus \{0\} \);

- **D-finite** if \( c_r(t) S^{(r)}(t) + \cdots + c_0(t) S(t) = 0 \) for some \( c_i \in \mathbb{Z}[t] \), not all zero;

- **hypergeometric** if \( \frac{s_{n+1}}{s_n} \in \mathbb{Q}(n) \). E.g.,

\[
\frac{1}{1 - \rho t} = \sum_{n \geq 0} \rho^n t^n; \quad (1 + t)^\alpha = \sum_{n \geq 0} \binom{\alpha}{n} t^n, \alpha \in \mathbb{Q}; \quad \ln(1 - t) = - \sum_{n \geq 1} \frac{t^n}{n}.
\]
Classes of power series

- algebraic
- hypergeometric
- differentially finite (holonomic)
- differentially algebraic

Venn diagram showing the relationships between these classes.
Classes of power series

- Algebraic
- Hypergeometric
- Differentially finite (holonomic)
- Differentially algebraic

\[ (1 - t)^\alpha \]
Classes of power series

- algebraic
- hypergeometric
- differentially finite (holonomic)
- differentially algebraic

\[(1-t)\alpha \quad \sqrt{1-t} + \sqrt[3]{1-2t}\]
Classes of power series

- algebraic
- hypergeometric
- differentially finite (holonomic)
- differentially algebraic

The diagram illustrates the relationships between these classes of power series.
Classes of power series

- algebraic
- hypergeometric
- differentially finite (holonomic)
- differentially algebraic

\[
\left(1 - t\right)^\alpha \sqrt{1 - t} + 3 \sqrt{1 - 2t} \ln(1 - t)
\]

\[
\binom{a}{b}{c}{\left| t\right)} = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n} \frac{t^n}{n!}, \text{ where } (a)_n = a(a + 1) \cdots (a + n - 1).
\]
Classes of power series

- Algebraic
- Hypergeometric
- Differentially finite (holonomic)
- Differentially algebraic

\[ \sqrt{1 - t + \sqrt{1 - 2t}} \]

E.g.,

\[ (1 - t)^\alpha = 2F_1\left( -\alpha, 1 \middle| t \right), \quad \ln(1 - t) = -t \cdot 2F_1\left( \frac{1}{2}, 1 \middle| t \right) = -\sum_{n=1}^{\infty} \frac{t^n}{n} \]
Classes of power series

- algebraic
- hypergeometric
- differentially finite (holonomic)
- differentially algebraic

\[ (1-t)^\alpha \sqrt{1-t} + \sqrt[3]{1-2t} \ln(1-t) \]

\[ \binom{a}{b}{c|t} := \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{t^n}{n!}, \quad \text{where} \quad (a)_n = a(a+1) \cdots (a+n-1). \]
Classes of power series

differentially algebraic

differentially finite (holonomic)

algebraic

(1−t)^α

√1−t + 3√1−2t

hypergeometric

ln(1−t)

2F1 \left( \begin{array}{c} a, b \\ c \end{array} \bigg| t \right)

\sqrt{1−t} + \ln(1−t)

\sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{t^n}{n!} \quad \text{where} \quad (a)_n = a(a+1) \cdots (a+n-1).

Alin Bostan

P-recursive sequences and D-finite series
Theorem: A power series \( f \in \mathbb{K}[[x]] \) is D-finite if and only if the sequence \( f_n \) of its coefficients is P-recursive.

Proof (idea): \( x\partial \leftrightarrow n \) and \( x^{-1} \leftrightarrow S_n \) provide a ring isomorphism between

\[
\mathbb{K}[x, x^{-1}, \partial] \quad \text{and} \quad \mathbb{K}[S_n, S_n^{-1}, n].
\]

Snobbish way of saying that the equality \( f = \sum_{n \geq 0} f_n x^n \) implies

\[
[x^n] x f'(x) = n f_n, \quad \text{and} \quad [x^n] x^{-1} f(x) = f_{n+1}.
\]

Both conversions implemented in gfun: \texttt{diffeqtorec} and \texttt{rectodiffeq}

Differential operators of order \( r \) and degree \( d \) give rise to recurrences of order \( \leq d + r \) and coefficients of degree \( \leq r \).
Theorem
(i) D-finite series in $K[[x]]$ form an algebra closed under Hadamard product.
Closure properties

**Theorem**

(i) D-finite series in $\mathbb{K}[[x]]$ form an algebra closed under Hadamard product.
(ii) P-recursive seqs in $\mathbb{K}^\mathbb{N}$ form an algebra closed under Cauchy product.

**Proof** by linear algebra:

If $a_r(x)f(r)(x) + \cdots + a_0(x)f(x) = 0$, $b_s(x)g(s)(x) + \cdots + b_0(x)g(x) = 0$, then $f(\ell) \in \text{Vect}_{\mathbb{K}}(x)(f, f', \ldots, f(r-1))$, $g(\ell) \in \text{Vect}_{\mathbb{K}}(x)(g, g', \ldots, g(s-1))$, so that $(f + g)(\ell) \in \text{Vect}_{\mathbb{K}}(x)(f, f', \ldots, f(r-1), g, g', \ldots, g(s-1))$, and $(fg)(\ell) \in \text{Vect}_{\mathbb{K}}(x)(f^i g^j, i < r, j < s)$.

So, $f + g$ satisfies LDE of order $\leq (r + s)$ and $fg$ satisfies LDE of order $\leq rs$.

**Corollary**: D-finite series can be multiplied mod $x^N$ in linear time $O(N)$.

⊿

Implemented in gfun: `diffeq+diffeq`, `diffeq*rec`, `hadamardproduct`, `rec+rec`, `rec*rec`, `cauchyproduct`
Closure properties

Theorem
(i) D-finite series in $\mathbb{K}[[x]]$ form an algebra closed under Hadamard product.
(ii) P-recursive seqs in $\mathbb{K}^\mathbb{N}$ form an algebra closed under Cauchy product.

Proof by linear algebra:

implemented in gfun:

- `diffeq+diffeq`
- `diffeq*rec`
- `rec+rec`
- `rec*rec`
- `cauchyproduct`
Closure properties

Theorem
(i) D-finite series in $\mathbb{K}[[x]]$ form an algebra closed under Hadamard product.
(ii) P-recursive seqs in $\mathbb{K}^\mathbb{N}$ form an algebra closed under Cauchy product.

Proof by linear algebra: If

$$a_r(x)f^{(r)}(x) + \cdots + a_0(x)f(x) = 0, \quad b_s(x)g^{(s)}(x) + \cdots + b_0(x)g(x) = 0,$$

then
Closure properties

**Theorem**

(i) D-finite series in $\mathbb{K}[[x]]$ form an algebra closed under Hadamard product.
(ii) P-recursive seqs in $\mathbb{K}^\mathbb{N}$ form an algebra closed under Cauchy product.

**Proof** by linear algebra: If

\[ a_r(x)f^{(r)}(x) + \cdots + a_0(x)f(x) = 0, \quad b_s(x)g^{(s)}(x) + \cdots + b_0(x)g(x) = 0, \]

then

\[ f^{(\ell)} \in \text{Vect}_{\mathbb{K}(x)} \left( f, f', \ldots, f^{(r-1)} \right), \quad g^{(\ell)} \in \text{Vect}_{\mathbb{K}(x)} \left( g, g', \ldots, g^{(s-1)} \right), \]
Closure properties

**Theorem**

(i) D-finite series in $\mathbb{K}[[x]]$ form an algebra closed under Hadamard product.

(ii) P-recursive seqs in $\mathbb{K}^\mathbb{N}$ form an algebra closed under Cauchy product.

**Proof** by linear algebra: If

$$a_r(x)f^{(r)}(x) + \cdots + a_0(x)f(x) = 0, \quad b_s(x)g^{(s)}(x) + \cdots + b_0(x)g(x) = 0,$$

then

$$f^{(\ell)} \in \text{Vect}_{\mathbb{K}(x)}\left(f, f', \ldots, f^{(r-1)}\right), \quad g^{(\ell)} \in \text{Vect}_{\mathbb{K}(x)}\left(g, g', \ldots, g^{(s-1)}\right),$$

so that

$$(f + g)^{(\ell)} \in \text{Vect}_{\mathbb{K}(x)}\left(f, f', \ldots, f^{(r-1)}, g, g', \ldots, g^{(s-1)}\right),$$
Closure properties

Theorem
(i) D-finite series in $\mathbb{K}[[x]]$ form an algebra closed under Hadamard product.
(ii) P-recursive seqs in $\mathbb{K}^\mathbb{N}$ form an algebra closed under Cauchy product.

Proof by linear algebra: If
\[ a_r(x)f^{(r)}(x) + \cdots + a_0(x)f(x) = 0, \quad b_s(x)g^{(s)}(x) + \cdots + b_0(x)g(x) = 0, \]
then
\[ f^{(\ell)} \in \text{Vect}_{\mathbb{K}(x)}(f, f', \ldots, f^{(r-1)}), \quad g^{(\ell)} \in \text{Vect}_{\mathbb{K}(x)}(g, g', \ldots, g^{(s-1)}), \]
so that \( (f + g)^{(\ell)} \in \text{Vect}_{\mathbb{K}(x)}(f, f', \ldots, f^{(r-1)}, g, g', \ldots, g^{(s-1)}) \),
and \( (fg)^{(\ell)} \in \text{Vect}_{\mathbb{K}(x)}(f^{(i)}g^{(j)}), \quad i < r, \; j < s \).
Closure properties

Theorem
(i) D-finite series in $\mathbb{K}[[x]]$ form an algebra closed under Hadamard product.
(ii) P-recursive seqs in $\mathbb{K}^\mathbb{N}$ form an algebra closed under Cauchy product.

Proof by linear algebra: If
\[ a_r(x)f^{(r)}(x) + \cdots + a_0(x)f(x) = 0, \quad b_s(x)g^{(s)}(x) + \cdots + b_0(x)g(x) = 0, \]
then
\[ f^{(\ell)} \in \text{Vect}_{\mathbb{K}(x)} \left( f, f', \ldots, f^{(r-1)} \right), \quad g^{(\ell)} \in \text{Vect}_{\mathbb{K}(x)} \left( g, g', \ldots, g^{(s-1)} \right), \]
so that \( (f + g)^{(\ell)} \in \text{Vect}_{\mathbb{K}(x)} \left( f, f', \ldots, f^{(r-1)}, g, g', \ldots, g^{(s-1)} \right), \)
and \( (fg)^{(\ell)} \in \text{Vect}_{\mathbb{K}(x)} \left( f^{(i)}g^{(j)}, \quad i < r, \ j < s \right). \)

So, $f + g$ satisfies LDE of order $\leq (r + s)$ and $fg$ satisfies LDE of order $\leq (rs)$.  

Corollary: D-finite series can be multiplied mod $x^N$ in linear time $O(N)$.  

⊿  
Implemented in $gfun$: $\text{diffeq}+\text{diffeq}$, $\text{diffeq}*\text{diffeq}$, $\text{hadamardproduct}$, $\text{rec}+\text{rec}$, $\text{rec}*\text{rec}$, $\text{cauchyproduct}$.
Closure properties

Theorem
(i) D-finite series in $\mathbb{K}[[x]]$ form an algebra closed under Hadamard product.
(ii) P-recursive seqs in $\mathbb{K}^\mathbb{N}$ form an algebra closed under Cauchy product.

Proof by linear algebra: If
\[
a_r(x)f^{(r)}(x) + \cdots + a_0(x)f(x) = 0, \quad b_s(x)g^{(s)}(x) + \cdots + b_0(x)g(x) = 0,
\]
then
\[
f^{(\ell)} \in \text{Vect}_{\mathbb{K}(x)} \left( f, f', \ldots, f^{(r-1)} \right), \quad g^{(\ell)} \in \text{Vect}_{\mathbb{K}(x)} \left( g, g', \ldots, g^{(s-1)} \right),
\]
so that \((f + g)^{(\ell)} \in \text{Vect}_{\mathbb{K}(x)} \left( f, f', \ldots, f^{(r-1)}, g, g', \ldots, g^{(s-1)} \right),
and \((fg)^{(\ell)} \in \text{Vect}_{\mathbb{K}(x)} \left( f^{(i)}g^{(j)}, \quad i < r, j < s \right).

So, $f + g$ satisfies LDE of order $\leq (r + s)$ and $fg$ satisfies LDE of order $\leq (rs)$.

Corollary: D-finite series can be multiplied mod $x^N$ in linear time $O(N)$. 

Implemented in gfun: diffeq+diffeq, diffeq*diffq, hadamardproduct, rec+rec, rec*rec, cauchyproduct.
Closure properties

Theorem
(i) D-finite series in $\mathbb{K}[[x]]$ form an algebra closed under Hadamard product.
(ii) P-recursive seqs in $\mathbb{K}^\mathbb{N}$ form an algebra closed under Cauchy product.

Proof by linear algebra: If
$a_r(x)f^{(r)}(x) + \cdots + a_0(x)f(x) = 0, \quad b_s(x)g^{(s)}(x) + \cdots + b_0(x)g(x) = 0$, then

\[ f^{(\ell)} \in \text{Vect}_{\mathbb{K}(x)} \left( f, f', \ldots, f^{(r-1)} \right), \quad g^{(\ell)} \in \text{Vect}_{\mathbb{K}(x)} \left( g, g', \ldots, g^{(s-1)} \right), \]

so that \( (f + g)^{(\ell)} \in \text{Vect}_{\mathbb{K}(x)} \left( f, f', \ldots, f^{(r-1)}, g, g', \ldots, g^{(s-1)} \right) \),

and \( (fg)^{(\ell)} \in \text{Vect}_{\mathbb{K}(x)} \left( f^{(i)}g^{(j)}, \quad i < r, \; j < s \right). \)

So, \( f + g \) satisfies LDE of order \( \leq (r + s) \) and \( fg \) satisfies LDE of order \( \leq (rs) \).

Corollary: D-finite series can be multiplied mod $x^N$ in linear time $O(N)$.

Implemented in gfun: difeq+diffeq, diffeq*diffeq, hadamardproduct, rec+rec, rec*rec, cauchyproduct
Closure properties

**Theorem**

(i) D-finite series in $\mathbb{K}[[x]]$ form a $\mathbb{K}$-algebra closed by Hadamard product.

(ii) P-recursive seqs in $\mathbb{K}^\mathbb{N}$ form an algebra closed under Cauchy product.

▸ Previous proof is effective: it contains algorithms.
Closure properties

Theorem
(i) D-finite series in $\mathbb{K}[[x]]$ form a $\mathbb{K}$-algebra closed by Hadamard product.
(ii) P-recursive seqs in $\mathbb{K}^\mathbb{N}$ form an algebra closed under Cauchy product.

▷ Previous proof is effective: it contains algorithms.

▷ To find explicit differential (resp., recurrence) equations for $\odot \in \{+, \times, \star\}$:

compute derivatives (or, shifts) of $h = f \odot g$, and express them on a system of generators by using the input equations compute enough such derivatives, or shifts (at most: order of output) form a (rational function) matrix whose rows contain coordinates of successive derivatives (resp., shifts) compute a non-trivial element in its kernel, and output the corresponding equation
Theorem
(i) D-finite series in $\mathbb{K}[[x]]$ form a $\mathbb{K}$-algebra closed by Hadamard product.
(ii) P-recursive seqs in $\mathbb{K}^\mathbb{N}$ form an algebra closed under Cauchy product.

Previous proof is effective: it contains algorithms.

To find explicit differential (resp., recurrence) equations for $\odot \in \{+ , \times , \star\}$:
- compute derivatives (or, shifts) of $h := f \odot g$, and express them on a system of generators by using the input equations.
Closure properties

Theorem
(i) D-finite series in $\mathbb{K}[[x]]$ form a $\mathbb{K}$-algebra closed by Hadamard product.
(ii) P-recursive seqs in $\mathbb{K}^\mathbb{N}$ form an algebra closed under Cauchy product.

▷ Previous proof is effective: it contains algorithms.

▷ To find explicit differential (resp., recurrence) equations for $\odot \in \{+, \times, \star\}$:
  - compute derivatives (or, shifts) of $h := f \odot g$, and express them on a system of generators by using the input equations
  - compute enough such derivatives, or shifts (at most: order of output)
Theorem
(i) D-finite series in $\mathbb{K}[[x]]$ form a $\mathbb{K}$-algebra closed by Hadamard product.
(ii) P-recursive seqs in $\mathbb{K}^\mathbb{N}$ form an algebra closed under Cauchy product.

Previous proof is effective: it contains algorithms.

To find explicit differential (resp., recurrence) equations for $\odot \in \{+, \times, \star\}$:
- compute derivatives (or, shifts) of $h := f \odot g$, and express them on a system of generators by using the input equations
- compute enough such derivatives, or shifts (at most: order of output)
- form a (rational function) matrix whose rows contain coordinates of successive derivatives (resp., shifts)
Theorem
(i) D-finite series in $\mathbb{K}[[x]]$ form a $\mathbb{K}$-algebra closed by Hadamard product.
(ii) P-recursive seqs in $\mathbb{K}^\mathbb{N}$ form an algebra closed under Cauchy product.

- Previous proof is effective: it contains algorithms.

- To find explicit differential (resp., recurrence) equations for $\odot \in \{+ , \times , \star\}$:
  - compute derivatives (or, shifts) of $h := f \odot g$, and express them on a system of generators by using the input equations
  - compute enough such derivatives, or shifts (at most: order of output)
  - form a (rational function) matrix whose rows contain coordinates of successive derivatives (resp., shifts)
  - compute a non-trivial element in its kernel, and output the corresponding equation
Application: Proof of Identities (I)

\[ > \text{series} (\sin(x)^2 + \cos(x)^2, x, 4); \]

\[ 1 + O(x^4) \]

This proves \( \sin^2(x) + \cos^2(x) = 1 \).

Why?

1. \( \sin \) and \( \cos \) satisfy a 2nd order LDE:
   \[ y'' + y = 0; \]
2. Their squares (and their sum) satisfy a 3rd order LDE;
3. The constant \( -1 \) satisfies a 1st order LDE:
   \[ y' = 0; \]
4. \( \Rightarrow \) \( \sin^2(x) + \cos^2(x) - 1 \) satisfies a LDE of order at most 4;
5. Since it is not singular at 0, Cauchy’s theorem concludes.

\( \Delta \)

Cassini’s identity (same idea):

\[ F_{2n} - F_{n+1}F_{n-1} = (-1)^{n+1} \]

for \( n \) to 8 do

\( \text{fibonacci}(n)^2 - \text{fibonacci}(n+1) \times \text{fibonacci}(n-1) + (-1)^n \)

od;

0, 0, 0, 0, 0, 0, 0, 0
Application: Proof of Identities (I)

```maple
> series(sin(x)^2+cos(x)^2,x,4);

1 + O(x^4)
```

This proves \( \sin(x)^2 + \cos(x)^2 = 1 \). Why?
This proves \( \sin(x)^2 + \cos(x)^2 = 1 \). Why?

(1) \( \sin \) and \( \cos \) satisfy a 2nd order LDE: \( y'' + y = 0 \);
This proves $\sin(x)^2 + \cos(x)^2 = 1$. Why?

(1) $\sin$ and $\cos$ satisfy a 2nd order LDE: $y'' + y = 0$;
(2) their squares (and their sum) satisfy a 3rd order LDE;
> \text{series}(\sin(x)^2+\cos(x)^2,x,4);

\[1 + O(x^4)\]

This proves \(\sin(x)^2 + \cos(x)^2 = 1\). Why?

(1) \text{sin} and \text{cos} satisfy a 2nd order LDE: \(y'' + y = 0\);
(2) their squares (and their sum) satisfy a 3rd order LDE;
(3) the constant \(-1\) satisfies a 1st order LDE: \(y' = 0\);
This proves \( \sin(x)^2 + \cos(x)^2 = 1 \). Why?

(1) \( \sin \) and \( \cos \) satisfy a 2nd order LDE: \( y'' + y = 0 \);
(2) their squares (and their sum) satisfy a 3rd order LDE;
(3) the constant \(-1\) satisfies a 1st order LDE: \( y' = 0 \);
(4) \( \implies \sin^2 + \cos^2 - 1 \) satisfies a LDE of order at most 4;
Application: Proof of Identities (I)

> series(sin(x)^2+cos(x)^2,x,4);

$$1 + O(x^4)$$

This proves \(\sin(x)^2 + \cos(x)^2 = 1\). Why?

1. \textit{sin} and \textit{cos} satisfy a 2nd order LDE: \(y'' + y = 0\);
2. their squares (and their \textit{sum}) satisfy a 3rd order LDE;
3. the constant \(-1\) satisfies a 1st order LDE: \(y' = 0\);
4. \(\implies \sin^2 + \cos^2 - 1\) satisfies a LDE of order at most 4;
5. Since it is not singular at 0, Cauchy’s theorem concludes.
> series(sin(x)^2+cos(x)^2,x,4);

\[ 1 + O(x^4) \]

This proves \( \sin(x)^2 + \cos(x)^2 = 1 \). Why?

1. \( \sin \) and \( \cos \) satisfy a 2nd order LDE: \( y'' + y = 0 \);
2. their squares (and their sum) satisfy a 3rd order LDE;
3. the constant \( -1 \) satisfies a 1st order LDE: \( y' = 0 \);
4. \( \implies \sin^2 + \cos^2 - 1 \) satisfies a LDE of order at most 4;
5. Since it is not singular at 0, Cauchy's theorem concludes.

Cassini's identity (same idea): \( F_n^2 - F_{n+1}F_{n-1} = (-1)^{n+1} \)

for n to 8 do
  fibonacci(n)^2-fibonacci(n+1)*fibonacci(n-1)+(-1)^n 
od;

0, 0, 0, 0, 0, 0, 0, 0
Let’s compute an explicit differential equation satisfied by $\sin^2 + \cos^2 - 1$
Let’s compute an explicit differential equation satisfied by $\sin^2 + \cos^2 - 1$

1. sin and cos satisfy a 2nd order LDE: $y'' + y = 0$;
Let’s compute an explicit differential equation satisfied by $\sin^2 + \cos^2 - 1$

1. $\sin$ and $\cos$ satisfy a 2nd order LDE: $y'' + y = 0$;
2. their squares (and their sum) satisfy a 3rd order LDE:
Let’s compute an explicit differential equation satisfied by \( \sin^2 + \cos^2 - 1 \)

(1) \( \sin \) and \( \cos \) satisfy a 2nd order LDE: \( y'' + y = 0 \);

(2) their squares (and their sum) satisfy a 3rd order LDE:

\[ z = y^2, \quad z' = 2yy', \quad z'' = 2yy'' + 2y'^2 = 2y'^2 - 2y^2, \quad z''' = 4y'y'' - 4yy' = -8yy' \]
Let’s compute an explicit differential equation satisfied by $\sin^2 + \cos^2 - 1$

(1) $\sin$ and $\cos$ satisfy a 2nd order LDE: $y'' + y = 0$;

(2) their squares (and their sum) satisfy a 3rd order LDE:

1. $z = y^2, z' = 2yy', z'' = 2yy'' + 2y'^2 = 2y'^2 - 2y^2, z''' = 4y'y'' - 4yy' = -8yy'$

2. systems of generators: $\{y^2, yy', y'^2\}$
Application: Proof of Identities (II)

Let’s compute an explicit differential equation satisfied by \( \sin^2 + \cos^2 - 1 \)

1. \( \sin \) and \( \cos \) satisfy a 2\text{nd} order LDE: \( y'' + y = 0 \);
2. their squares (and their sum) satisfy a 3\text{rd} order LDE:
   \[ z = y^2, \quad z' = 2yy', \quad z'' = 2yy'' + 2y'^2 = 2y'^2 - 2y^2, \quad z''' = 4y'y'' - 4yy' = -8yy' \]
3. systems of generators: \( \{y^2, yy', y'^2\} \)
4. matrix

\[
M = \begin{bmatrix}
1 & 0 & -2 & 0 \\
0 & 2 & 0 & -8 \\
0 & 0 & 2 & 0 \\
\end{bmatrix}
\]
Let's compute an explicit differential equation satisfied by $\sin^2 + \cos^2 - 1$

1. $\sin$ and $\cos$ satisfy a 2nd order LDE: $y'' + y = 0$;
2. their squares (and their sum) satisfy a 3rd order LDE:
   - $z = y^2$, $z' = 2yy'$, $z'' = 2yy'' + 2y'^2 = 2y'^2 - 2y^2$, $z''' = 4y'y'' - 4yy' = -8yy'$
3. systems of generators: $\{y^2, yy', y'^2\}$
4. matrix
   
   $$M = \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 2 & 0 & -8 \\ 0 & 0 & 2 & 0 \end{bmatrix}$$
5. kernel $\ker(M) = \{v \mid Mv = 0\}$ generated by $v = [0 \ 4 \ 0 \ 1]^T$
Let’s compute an explicit differential equation satisfied by $\sin^2 + \cos^2 - 1$

1. $\sin$ and $\cos$ satisfy a 2nd order LDE: $y'' + y = 0$;
2. their squares (and their sum) satisfy a 3rd order LDE:
   - $z = y^2$, $z' = 2yy'$, $z'' = 2yy'' + 2y'^2 = 2y'^2 - 2y^2$, $z''' = 4y'y'' - 4yy' = -8yy'$
   - systems of generators: $\{y^2, yy', y'^2\}$
   - matrix
     \[
     M = \begin{bmatrix}
     1 & 0 & -2 & 0 \\
     0 & 2 & 0 & -8 \\
     0 & 0 & 2 & 0 \\
     \end{bmatrix}
     \]
   - kernel $\ker(M) = \{v \mid Mv = 0\}$ generated by $v = [0 \ 4 \ 0 \ 1]^T$
3. output equation satisfied by $z = \sin(x)^2 + \cos(x)^2$ is $z''' + 4z' = 0$
Let’s compute an explicit differential equation satisfied by $\sin^2 + \cos^2 - 1$

(1) $\sin$ and $\cos$ satisfy a 2nd order LDE: $y'' + y = 0$;

(2) their squares (and their sum) satisfy a 3rd order LDE:

1. $z = y^2, z' = 2yy', z'' = 2yy'' + 2y'^2 = 2y'^2 - 2y^2, z''' = 4y'y'' - 4yy' = -8yy'$
2. systems of generators: $\{y^2, yy', y'^2\}$
3. matrix

$$M = \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 2 & 0 & -8 \\ 0 & 0 & 2 & 0 \end{bmatrix}$$

4. kernel $\ker(M) = \{v \mid Mv = 0\}$ generated by $v = [0, 4, 0, 1]^T$
5. output equation satisfied by $z = \sin(x)^2 + \cos(x)^2$ is $z''' + 4z' = 0$

(3) the constant $c = -1$ satisfies a 1st order LDE: $c' = 0$;
Let’s compute an explicit differential equation satisfied by $\sin^2 + \cos^2 - 1$

1. $\sin$ and $\cos$ satisfy a 2nd order LDE: $y'' + y = 0$;
2. their squares (and their sum) satisfy a 3rd order LDE:
   - $z = y^2$, $z' = 2yy'$, $z'' = 2yy'' + 2y'^2 = 2y'^2 - 2y^2$, $z''' = 4y'y'' - 4yy' = -8yy'$
   - systems of generators: $\{y^2, yy', y'^2\}$
3. matrix
   $$M = \begin{bmatrix}
   1 & 0 & -2 & 0 \\
   0 & 2 & 0 & -8 \\
   0 & 0 & 2 & 0
   \end{bmatrix}$$
4. kernel $\ker(M) = \{v \mid Mv = 0\}$ generated by $v = [0 \ 4 \ 0 \ 1]^T$
5. output equation satisfied by $z = \sin(x)^2 + \cos(x)^2$ is $z''' + 4z' = 0$

3. the constant $c = -1$ satisfies a 1st order LDE: $c' = 0$;
4. $\implies \sin^2 + \cos^2 - 1 = z + c$ satisfies a LDE of order at most 4;
Let’s compute an explicit differential equation satisfied by $\sin^2 + \cos^2 - 1$

1. $\sin$ and $\cos$ satisfy a 2nd order LDE: $y'' + y = 0$;
2. their squares (and their sum) satisfy a 3rd order LDE:
   - $z = y^2, z' = 2yy', z'' = 2yy'' + 2y'^2 = 2y'^2 - 2y^2, z''' = 4y'y'' - 4yy' = -8yy'$
   - systems of generators: $\{y^2, yy', y'^2\}$
   - matrix

\[
M = \begin{bmatrix}
1 & 0 & -2 & 0 \\
0 & 2 & 0 & -8 \\
0 & 0 & 2 & 0
\end{bmatrix}
\]

3. kernel $\ker(M) = \{v \mid Mv = 0\}$ generated by $v = [0 \ 4 \ 0 \ 1]^T$
4. output equation satisfied by $z = \sin(x)^2 + \cos(x)^2$ is $z''' + 4z' = 0$

5. the constant $c = -1$ satisfies a 1st order LDE: $c' = 0$;
6. $\Rightarrow \sin^2 + \cos^2 - 1 = z + c$ satisfies a LDE of order at most 4;
   - $u = z + c, u' = z' + c' = z', u'' = z'', u''' = z''' = -4z', u'''' = -4z''$
Application: Proof of Identities (II)

Let’s compute an explicit differential equation satisfied by $\sin^2 + \cos^2 - 1$

1. $\sin$ and $\cos$ satisfy a 2nd order LDE: $y'' + y = 0$;
2. their squares (and their sum) satisfy a 3rd order LDE:
   1. $z = y^2, z' = 2yy', z'' = 2yy'' + 2y'^2 = 2y'^2 - 2y^2, z''' = 4y'y'' - 4yy' = -8yy'$
   2. systems of generators: $\{y^2, yy', y'^2\}$
   3. matrix
   $$M = \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 2 & 0 & -8 \\ 0 & 0 & 2 & 0 \end{bmatrix}$$
4. kernel $\ker(M) = \{v \mid Mv = 0\}$ generated by $v = [0 \ 4 \ 0 \ 1]^T$
5. output equation satisfied by $z = \sin(x)^2 + \cos(x)^2$ is $z''' + 4z' = 0$

3. the constant $c = -1$ satisfies a 1st order LDE: $c' = 0$;
4. $\Rightarrow \sin^2 + \cos^2 - 1 = z + c$ satisfies a LDE of order at most 4;
   1. $u = z + c, u' = z' + c' = z', u'' = z'', u''' = z''' = -4z', u'''' = -4z''$
   2. systems of generators: $\{z, z', z'', c\}$
Let's compute an explicit differential equation satisfied by $\sin^2 + \cos^2 - 1$

1. $\sin$ and $\cos$ satisfy a 2nd order LDE: $y'' + y = 0$;
2. their squares (and their sum) satisfy a 3rd order LDE:
   - $z = y^2$, $z' = 2yy'$, $z'' = 2yy'' + 2y'^2 = 2y'^2 - 2y^2$, $z''' = 4y'y'' - 4yy' = -8yy'$
   - systems of generators: $\{y^2, yy', y'^2\}$
   - matrix
     $$M = \begin{bmatrix}
        1 & 0 & -2 & 0 \\
        0 & 2 & 0 & -8 \\
        0 & 0 & 2 & 0 \\
        -2 & 0 & 0 & 0
     \end{bmatrix}$$
   - kernel $\ker(M) = \{v \mid Mv = 0\}$ generated by $v = [0 \ 4 \ 0 \ 1]^T$
   - output equation satisfied by $z = \sin(x)^2 + \cos(x)^2$ is $z'''' + 4z' = 0$

3. the constant $c = -1$ satisfies a 1st order LDE: $c' = 0$;
4. $\implies \sin^2 + \cos^2 - 1 = z + c$ satisfies a LDE of order at most 4;
   - $u = z + c$, $u' = z' + c' = z'$, $u'' = z''$, $u''' = z''' = -4z'$, $u'''' = -4z''$
   - systems of generators: $\{z, z', z'', c\}$
   - matrix
     $$\begin{bmatrix}
        1 & 0 & 0 & 0 & 0 \\
        0 & 1 & 0 & -4 & 0 \\
        0 & 0 & 1 & 0 & -4 \\
        0 & 0 & 0 & 1 & 0 \\
        0 & 0 & 0 & 0 & 1
     \end{bmatrix}$$
Let’s compute an explicit differential equation satisfied by $\sin^2 + \cos^2 - 1$

1. $\sin$ and $\cos$ satisfy a 2nd order LDE: $y'' + y = 0$;
2. their squares (and their sum) satisfy a 3rd order LDE:
   - $z = y^2$, $z' = 2yy'$, $z'' = 2yy'' + 2y'^2 = 2y'^2 - 2y^2$, $z''' = 4y'y'' - 4yy' = -8yy'$
   - systems of generators: $\{y^2, yy', y'^2\}$
   - matrix
     \[
     M = \begin{bmatrix}
     1 & 0 & -2 & 0 \\
     0 & 2 & 0 & -8 \\
     0 & 0 & 2 & 0 \\
     \end{bmatrix}
     \]
   - kernel $\ker(M) = \{v \mid Mv = 0\}$ generated by $v = [0 \ 4 \ 0 \ 1]^T$
   - output equation satisfied by $z = \sin(x)^2 + \cos(x)^2$ is $z''' + 4z' = 0$
3. the constant $c = -1$ satisfies a 1st order LDE: $c' = 0$;
4. $\implies \sin^2 + \cos^2 - 1 = z + c$ satisfies a LDE of order at most 4;
   - $u = z + c$, $u' = z' + c' = z'$, $u'' = z''$, $u''' = z''' = -4z'$, $u'''' = -4z''$
   - systems of generators: $\{z, z', z'', c\}$
   - matrix
     \[
     \begin{bmatrix}
     1 & 0 & 0 & 0 & 0 \\
     0 & 1 & 0 & -4 & 0 \\
     0 & 0 & 1 & 0 & -4 \\
     1 & 0 & 0 & 0 & 0 \\
     \end{bmatrix}
     \]
   - kernel generated by $[0 \ 4 \ 0 \ 1 \ 0]^T$
Let’s compute an explicit differential equation satisfied by \( \sin^2 + \cos^2 - 1 \)

1. \( \sin \) and \( \cos \) satisfy a 2nd order LDE: \( y'' + y = 0 \);
2. their squares (and their sum) satisfy a 3rd order LDE:
   \[
   z = y^2, \quad z' = 2yy', \quad z'' = 2yy'' + 2y'^2 = 2y'^2 - 2y^2, \quad z''' = 4y'y'' - 4yy' = -8yy'
   \]
   systems of generators: \( \{y^2, yy', y'^2\} \)
3. matrix
   \[
   M = \begin{bmatrix}
   1 & 0 & -2 & 0 \\
   0 & 2 & 0 & -8 \\
   0 & 0 & 2 & 0 \\
   \end{bmatrix}
   \]
4. kernel \( \text{ker}(M) = \{v \mid Mv = 0\} \) generated by \( v = [0 \ 4 \ 0 \ 1]^T \)
5. output equation satisfied by \( z = \sin(x)^2 + \cos(x)^2 \) is \( z'''' + 4z' = 0 \)

(3) the constant \( c = -1 \) satisfies a 1st order LDE: \( c' = 0 \);

(4) \( \implies \sin^2 + \cos^2 - 1 = z + c \) satisfies a LDE of order at most 4;
   \[
   u = z + c, \quad u' = z' + c' = z', \quad u'' = z'', \quad u''' = z''' = -4z', \quad u'''' = -4z''
   \]
   systems of generators: \( \{z, z', z'', c\} \)
3. matrix
   \[
   \begin{bmatrix}
   1 & 0 & 0 & 0 & 0 \\
   0 & 1 & 0 & -4 & 0 \\
   0 & 0 & 1 & 0 & -4 \\
   1 & 0 & 0 & 0 & 0 \\
   \end{bmatrix}
   \]
4. kernel generated by \( [0 \ 4 \ 0 \ 1 \ 0]^T \)
5. output equation satisfied by \( u = \sin(x)^2 + \cos(x)^2 - 1 \) is \( u'''' + 4u' = 0 \)
Let’s compute an explicit differential equation satisfied by \( \sin^2 + \cos^2 - 1 \)

1. \( \sin \) and \( \cos \) satisfy a 2nd order LDE: \( y'' + y = 0 \);
2. their squares (and their sum) satisfy a 3rd order LDE:
   - \( z = y^2, z' = 2yy', z'' = 2yy'' + 2y'^2 = 2y'^2 - 2y'^2, z''' = 4y'y'' - 4yy' = -8yy' \)
   - systems of generators: \( \{y^2, yy', y'^2\} \)
   - matrix
     \[
     M = \begin{bmatrix}
     1 & 0 & -2 & 0 \\
     0 & 2 & 0 & -8 \\
     0 & 0 & 2 & 0
     \end{bmatrix}
     \]
   - kernel \( \ker(M) = \{v \mid Mv = 0\} \) generated by \( v = [0 \ 4 \ 0 \ 1]^T \)
   - output equation satisfied by \( z = \sin(x)^2 + \cos(x)^2 \) is \( z''' + 4z' = 0 \)
3. the constant \( c = -1 \) satisfies a 1st order LDE: \( c' = 0 \);
4. \( \Rightarrow \) \( \sin^2 + \cos^2 - 1 = z + c \) satisfies a LDE of order at most 4;
   - \( u = z + c, u' = z' + c' = z', u'' = z'', u''' = z''' = -4z', u'''' = -4z'' \)
   - systems of generators: \( \{z, z', z'', c\} \)
   - matrix
     \[
     \begin{array}{cccc}
     1 & 0 & 0 & 0 \\
     0 & 1 & 0 & -4 \\
     0 & 0 & 1 & 0 \\
     1 & 0 & 0 & 0
     \end{array}
     \]
   - kernel generated by \( [0 \ 4 \ 0 \ 1]^T \)
   - output equation satisfied by \( u = \sin(x)^2 + \cos(x)^2 - 1 \) is \( u'''' + 4u' = 0 \)
5. Since \( u(0) = u'(0) = u''(0) = 0 \), Cauchy’s theorem concludes \( u(x) \equiv 0 \).
Algebraic series are D-finite

Theorem [Abel 1827, Cockle 1860, Harley 1862] Algebraic series are D-finite, i.e., they satisfy linear differential equations with polynomial coefficients.
Algebraic series are D-finite

Theorem [Abel 1827, Cockle 1860, Harley 1862] Algebraic series are D-finite, thus, their coefficients satisfy linear recurrences with polynomial coefficients.
Algebraic series are D-finite

Theorem [Abel 1827, Cockle 1860, Harley 1862] Algebraic series are D-finite.

Proof: Let $f(t) \in \mathbb{Q}[[t]]$ such that $P(t, f(t)) = 0$, with $P \in \mathbb{Q}[t, y]$ irreducible.
Algebraic series are D-finite

Theorem [Abel 1827, Cockle 1860, Harley 1862] Algebraic series are D-finite

Proof: Let \( f(t) \in \mathbb{Q}[[t]] \) such that \( P(t, f(t)) = 0 \), with \( P \in \mathbb{Q}[t, y] \) irreducible.

Differentiate w.r.t. \( t \):

\[
P_t(t, f(t)) + f'(t)P_y(t, f(t)) = 0 \quad \implies \quad f' = -\frac{P_t}{P_y}(t, f(t)).
\]
Algebraic series are D-finite

Theorem [Abel 1827, Cockle 1860, Harley 1862] Algebraic series are D-finite

Proof: Let \( f(t) \in \mathbb{Q}[[t]] \) such that \( P(t, f(t)) = 0 \), with \( P \in \mathbb{Q}[t, y] \) irreducible.

Differentiate w.r.t. \( t \):

\[
P_t(t, f(t)) + f'(t)P_y(t, f(t)) = 0 \quad \implies \quad f' = -\frac{P_t}{P_y}(t, f(t)).
\]

Extended gcd: \( \gcd(P, P_y) = 1 \quad \implies \quad UP + VP_y = 1, \text{ for } U, V \in \mathbb{Q}(t)[y] \)

\[
\implies f' = -\left( P_t V \mod P \right)(t, f) \in \text{Vect}_{\mathbb{Q}(t)} \left( 1, f, f^2, \ldots, f^{\deg_y(P) - 1} \right).
\]
Algebraic series are D-finite

Theorem [Abel 1827, Cockle 1860, Harley 1862] Algebraic series are D-finite

Proof: Let \( f(t) \in \mathbb{Q}[[t]] \) such that \( P(t, f(t)) = 0 \), with \( P \in \mathbb{Q}[t,y] \) irreducible.

Differentiate w.r.t. \( t \):

\[
P_t(t, f(t)) + f'(t)P_y(t, f(t)) = 0 \implies f' = -\frac{P_t}{P_y}(t, f(t)).
\]

Extended gcd: \( \gcd(P, P_y) = 1 \implies UP + VP_y = 1 \), for \( U, V \in \mathbb{Q}(t)[y] \)

\[
\implies f' = -\left(P_t V \mod P\right)(t, f) \in \text{Vect}_{\mathbb{Q}(t)} \left(1, f, f^2, \ldots, f^{\deg_y(P)-1}\right).
\]

By induction, \( f^{(\ell)} \in \text{Vect}_{\mathbb{Q}(t)} \left(1, f, f^2, \ldots, f^{\deg_y(P)-1}\right) \), for all \( \ell \).

\[\square\]
Algebraic series are D-finite

Theorem [Abel 1827, Cockle 1860, Harley 1862] Algebraic series are D-finite

Proof: Let $f(t) \in \mathbb{Q}[[t]]$ such that $P(t, f(t)) = 0$, with $P \in \mathbb{Q}[t, y]$ irreducible.

Differentiate w.r.t. $t$:

$$P_t(t, f(t)) + f'(t)P_y(t, f(t)) = 0 \quad \implies \quad f' = -\frac{P_t}{P_y}(t, f(t)).$$

Extended gcd: $\gcd(P, P_y) = 1 \quad \implies \quad UP + VP_y = 1$, for $U, V \in \mathbb{Q}(t)[y]$

$$\implies \quad f' = -\left(P_tV \mod P\right)(t, f) \in \text{Vect}_{\mathbb{Q}(t)}\left(1, f, f^2, \ldots, f^{\deg_y(P)-1}\right).$$

By induction, $f^{(\ell)} \in \text{Vect}_{\mathbb{Q}(t)}\left(1, f, f^2, \ldots, f^{\deg_y(P)-1}\right)$, for all $\ell$. \hfill \Box

> Implemented, e.g., in maple’s package gfun `algeqtodiffeq, diffeqtorec`
Theorem [Abel 1827, Cockle 1860, Harley 1862] Algebraic series are D-finite

Proof: Let \( f(t) \in \mathbb{Q}[[t]] \) such that \( P(t, f(t)) = 0 \), with \( P \in \mathbb{Q}[t, y] \) irreducible. Differentiate w.r.t. \( t \):

\[
P_t(t, f(t)) + f'(t)P_y(t, f(t)) = 0 \quad \implies \quad f' = -\frac{P_t}{P_y}(t, f(t)).
\]

Extended gcd: \( \gcd(P, P_y) = 1 \implies UP + VP_y = 1, \) for \( U, V \in \mathbb{Q}(t)[y] \)

\[
\implies f' = -\left(P_t V \mod P\right)(t, f) \in \text{Vect}_{\mathbb{Q}(t)} \left(1, f, f^2, \ldots, f^{\deg_y(P)-1}\right).
\]

By induction, \( f^{(\ell)} \in \text{Vect}_{\mathbb{Q}(t)} \left(1, f, f^2, \ldots, f^{\deg_y(P)-1}\right), \) for all \( \ell \). \( \square \)

- Implemented, e.g., in maple’s package \textit{gfun} algeqtodiffeq, diffeqtorec
- Generalization: \( g \) D-finite, \( f \) algebraic \( \rightarrow \) g \( \circ \) f D-finite algebraicsubs
Theorem 12 in [Le Gall & Riera, 2018] amounts to showing the following:

Let $F(z) = \frac{1}{3} + \frac{4\sqrt{2}}{9}z - \frac{2}{9}z^2 + \cdots$ be the unique solution in $C[[z]]$ of

$$(2F(z) - 1)\sqrt{F(z)} + 1 + 2F(z)^{3/2} = \sqrt{6}z.$$

Then, for $n \geq 1$, the $n$th coefficient $u(n)$ of $F = \sum_{n \geq 0} u(n)z^n$ is equal to:

$$\frac{(-1)^{n+1}}{n!} (3\sqrt{2})^{-n} 2^{2n+1} \frac{1}{n+2} \frac{\Gamma\left(\frac{3n}{2} - 1\right)}{\Gamma\left(\frac{n}{2}\right)}.$$
Theorem 12 in [Le Gall & Riera, 2018] amounts to showing the following: Let $F(z) = \frac{1}{3} + \frac{4\sqrt{2}}{9}z - \frac{2}{9}z^2 + \cdots$ be the unique solution in $\mathbb{C}[[z]]$ of

$$(2F(z) - 1)\sqrt{F(z)} + 1 + 2F(z)^{3/2} = \sqrt{6}z.$$ 

Then, for $n \geq 1$, the $n$th coefficient $u(n)$ of $F = \sum_{n \geq 0} u(n)z^n$ is equal to:

$$\frac{(-1)^{n+1}}{n!} (3\sqrt{2})^{-n} 2^{2n+1} \frac{1}{n+2} \frac{\Gamma\left(\frac{3n}{2} - 1\right)}{\Gamma\left(\frac{n}{2}\right)}.$$ 

Original proof is tricky and uses several non-trivial mathematical facts.
Theorem 12 in [Le Gall & Riera, 2018] amounts to showing the following:
Let \( F(z) = \frac{1}{3} + \frac{4\sqrt{2}}{9}z - \frac{2}{9}z^2 + \cdots \) be the unique solution in \( \mathbb{C}[[z]] \) of
\[
(2 F(z) - 1) \sqrt{F(z)} + 1 + 2F(z)^{3/2} = \sqrt{6}z.
\]
Then, for \( n \geq 1 \), the \( n \)th coefficient \( u(n) \) of \( F = \sum_{n \geq 0} u(n)z^n \) is equal to:
\[
\frac{(-1)^{n+1}}{n!} (3\sqrt{2})^{-n} 2^{2n+1} \frac{1}{n+2} \frac{\Gamma\left(\frac{3n}{2} - 1\right)}{\Gamma\left(\frac{n}{2} + 1\right)}.
\]

▷ Original proof is tricky and uses several non-trivial mathematical facts.

▷ Here is a short proof based on D-finiteness.
Let $F(z) = \frac{1}{3} + \frac{4\sqrt{2}}{9} z - \frac{2}{9} z^2 + \cdots$ be the unique solution in $\mathbb{R}[[z]]$ of

$$(2F(z) - 1)\sqrt{F(z)} + 1 + 2F(z)^{3/2} = \sqrt{6} z.$$ 

Then, for $n \geq 1$, the $n$th coefficient $u(n)$ of $F = \sum_{n \geq 0} u(n)z^n$ is equal to:

$$\frac{(-1)^{n+1}}{n!}(3\sqrt{2})^{-n}2^{2n+1} \frac{1}{n+2} \frac{\Gamma(\frac{3n}{2} - 1)}{\Gamma(\frac{n}{2})}.$$ 

**Step 0.** Find a polynomial equation for $F$:

$$P := \text{resultant}(F-G^2, \text{resultant}((F+1)-H^2, (2*F-1)*H+2*G^3-sqrt(6)*z,H),G);$$

$$-96 F^3 z^2 + 36 z^4 + 36 F z^2 + 9 F^2 - 12 z^2 - 6 F + 1$$
Let $F(z) = \frac{1}{3} + \frac{4\sqrt{2}}{9}z - \frac{2}{9}z^2 + \cdots$ be the unique solution in $\mathbb{R}[[z]]$ of

$$(2 F(z) - 1)^{\frac{1}{2}} F(z) + 1 + 2F(z)^{3/2} = \sqrt{6} z.$$ 

Then, for $n \geq 1$, the $n$th coefficient $u(n)$ of $F = \sum_{n \geq 0} u(n)z^n$ is equal to:

$$\frac{(-1)^{n+1}}{n!} (3\sqrt{2})^{-n} 2^{2n+1} \frac{1}{n+2} \frac{\Gamma\left(\frac{3n}{2}-1\right)}{\Gamma\left(\frac{n}{2}\right)}.$$ 

**Step 1.** Find a linear differential equation for $F$:

> deqF:=algeqtodiffeq(P,F(z));

$$\left(18 z^4 - 3 z^2\right) \frac{d^2}{dz^2} F(z) + \left(18 z^3 - 6 z\right) \frac{d}{dz} F(z) + \left(6 - 8 z^2\right) F(z) = 2$$
Let $F(z) = \frac{1}{3} + \frac{4\sqrt{2}}{9}z - \frac{2}{9}z^2 + \cdots$ be the unique solution in $\mathbb{R}[[z]]$ of

$$(2F(z) - 1)\sqrt{F(z)} + 1 + 2F(z)^{3/2} = \sqrt{6}z.$$ 

Then, for $n \geq 1$, the $n$th coefficient $u(n)$ of $F = \sum_{n \geq 0} u(n)z^n$ is equal to:

$$\frac{(-1)^{n+1}}{n!} (3\sqrt{2})^{-n} 2^{2n+1} \frac{1}{n+2} \frac{\Gamma\left(\frac{3n}{2} - 1\right)}{\Gamma\left(\frac{n}{2}\right)}.$$ 

**Step 2.** Find a linear recurrence for (the coefficients of) $F$:

```
> recF := diffeqtorec(deqF, F(z), u(n))[1];
```

$$2 (3n - 2)(3n + 2) u(n) = 3 (n + 4)(n + 1) u(n + 2)$$
Let $F(z) = \frac{1}{3} + \frac{4\sqrt{2}}{9}z - \frac{2}{9}z^2 + \cdots$ be the unique solution in $\mathbb{R}[[z]]$ of

$$(2F(z) - 1)\sqrt{F(z)} + 1 + 2F(z)^{3/2} = \sqrt{6}z.$$ 

Then, for $n \geq 1$, the $n$th coefficient $u(n)$ of $F = \sum_{n}\geq 0 u(n)z^n$ is equal to:

$$\frac{(-1)^{n+1}}{n!} \left(3\sqrt{2}\right)^{-n} 2^{2n+1} \frac{1}{n+2} \frac{\Gamma\left(\frac{3n}{2} - 1\right)}{\Gamma\left(\frac{n}{2}\right)}.$$

Step 3. Check that the result satisfies the same recurrence, and the same initial conditions:

```plaintext
> res := n-> (-1)^(n+1)/n! * (3*sqrt(2))^(n) * 2^(2*n+1)/(n+2) * GAMMA(3*n/2-1)/GAMMA(n/2):
> simplify(coeff(recF, u(n))*res(n) + coeff(recF, u(n+2))*res(n+2));

0

\[
\begin{align*}
\frac{4}{9} \sqrt{2}, & \quad -\frac{2}{9}
\end{align*}
\]
An exercise for next week

Prove the identity

\[
\arcsin(x)^2 = \sum_{k \geq 0} \frac{k!}{\left(\frac{1}{2}\right) \cdots \left(k + \frac{1}{2}\right)} \frac{x^{2k+2}}{2k + 2},
\]

by performing the following steps:

1. Show that \( y = \arcsin(x) \) can be represented by the differential equation \((1 - x^2)y'' - xy' = 0\) and the initial conditions \( y(0) = 0, \ y'(0) = 1 \).
2. Compute a linear differential equation satisfied by \( z(x) = y(x)^2 \).
3. Deduce a linear recurrence relation satisfied by the coefficients of \( z(x) \).