Algorithmic Guessing

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The exercises from last lecture

(1) Show that if $P \in \mathbb{K}[x]$ has degree d, then the sequence $(P(n))_{n\geq 0}$ is C-recursive, and admits $(x-1)^{d+1}$ as a characteristic polynomial.

(2) Let $P = \sum_{i=0}^{2N} p_i x^i \in \mathbb{Z}[x]$ be the polynomial $P(x) = (1+x+x^2)^N$.

1. Show that the parity of all coefficients of P can be determined in $O(\mathsf{M}(N))$ bit ops.
2. Show that P satisfies a linear differential equation of order 1 with polynomial coefficients.
3. Determine a linear recurrence of order 2 satisfied by the sequence $(p_i)_i$.
4. Give an algorithm that computes p_N in $\tilde{O}(N)$ bit ops.

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▷ First solution: there exists a $N_d(x) \in \mathbb{Q}[x]_{\leq d}$ such that

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This can be proved by induction: the case $d = 0$ is obvious, and (1) implies

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▷ Second solution: the operator $\Delta_n = S_n - \mathrm{id} : (u_n)_{n\geq 0} \mapsto (u_{n+1} - u_n)_{n\geq 0}$ decreases by 1 the degree of a polynomial sequence, thus $\Delta_n^{d+1}\left((n^d)_n\right) = 0$.

Solution 2(a)

Let $P = \sum_{i=0}^{2N} p_i x^i \in \mathbb{Z}[x]$ be the polynomial $P(x) = (1 + x + x^2)^N$.

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▷ Conclusion: $\mathsf{C}(N) = O(\mathsf{M}(N))$

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▷ $(i-1)p_{i-1} + ip_i + (i+1)p_{i+1} = 2Np_{i-1} + Np_i$, for all $i \geq 0$

▷ The recurrence satisfied by the sequence (p_i) is

$$p_{i+1} = \frac{1}{i+1}\Big((N-i)p_i + (2N-i+1)p_{i-1}\Big) \quad \text{for} \quad i \geq 0.$$

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▷ The recurrence rewrites in matrix form: $F_i = \frac{1}{i+1} A_i F_{i-1}$, where

$$A_i = \begin{pmatrix} N-i & 2N-i+1 \\ i+1 & 0 \end{pmatrix} \quad \text{and} \quad F_i = \begin{pmatrix} p_{i+1} \\ p_i \end{pmatrix}.$$

▷ By unrolling it, we obtain the equality $F_i = \dfrac{1}{(i+1)!} A(i) \cdots A(1) \begin{pmatrix} N \\ 1 \end{pmatrix}$.

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▷ To compute p_N we determine F_N by binary splitting applied to the integer $f = (N+1)!$, and to the matrix factorial $B = A(N) \cdots A(1)$, followed by the matrix-vector product $v = B \times \begin{pmatrix} N & 1 \end{pmatrix}^T$ and the (exact) division $\frac{1}{f} v$.

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▷ The integer f, and the entries of B and v, have $O(N \log(N))$ bits.

▷ Cost of the whole algorithm: $\tilde{O}(N)$ bit ops.

ALGORITHMIC GUESSING

Guessing: what's the next term of the sequence?

1. 1, 1, 1, 1, 1
2. 1, 1, 2, 3, 5
3. 1, 1, 2, 5, 14
4. 1, 2, 9, 54, 378
5. 1, 2, 16, 192, 2816
6. 1, 3, 30, 420, 6930

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Guessing: what's the next term of the sequence?

1. $1, \; 1, \; 1, \; 1, \; 1$ $\qquad 1/(1-t)$
2. $1, \; 1, \; 2, \; 3, \; 5$ $\qquad 1/(1-t-t^2)$
3. $1, \; 1, \; 2, \; 5, \; 14$ $\qquad (1-\sqrt{1-4t})/(2t)$
4. $1, \; 2, \; 9, \; 54, \; 378$ $\qquad 27\,t^2y^2 + (1-18\,t)\,y + 16\,t = 1$
5. $1, \; 2, \; 16, \; 192, \; 2816$ $\qquad 64\,t^2\,y^3 + 16\,t\,y^2 + (1-72\,t)\,y + 54\,t = 1$
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▷ Automated guessing: algorithmic computation of these equations

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▷ Automated guessing: via Padé, or Hermite-Padé, approximants

PADÉ APPROXIMANTS

—guessing linear recurrences with constant coefficients—

Recall: duality lemma

Duality lemma (link between l.r.s.c.c. and rational functions)
Let $A(x) = \sum_{n \geq 0} a_n x^n \in \mathbb{K}[[x]]$ be the generating function of $(a_n)_{n \geq 0}$.
The following assertions are equivalent:

(i) (a_n) is a l.r.s.c.c., having P as characteristic polynomial of degree d.

(ii) $A(x)$ is rational, of the form $A = Q/\mathsf{rev}_d(P)$ for some $Q \in \mathbb{K}[x]_{<d}$, where $\mathsf{rev}_d(P) = P(\frac{1}{x})x^d$.

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Moreover, if P is the minimal polynomial of $(a_n)_{n\geq 0}$, then

$$d = \max\{1 + \deg(Q), \deg(\mathsf{rev}_d(P))\} \quad \text{and} \quad \gcd(Q, \mathsf{rev}_d(P)) = 1.$$

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▷ Computing $\mathsf{MinPol}(a_n)$ is equivalent to solving a Padé approximation pb:

$$\frac{R}{V} \equiv A \bmod x^{2N}, \quad x \nmid V,\ \deg(R) < N,\ \deg(V) \leq N \ \text{ and } \ \gcd(R, V) = 1,$$

where $A = a_0 + a_1 x + a_2 x^2 + \cdots + a_{2N-1} x^{2N-1}$.

Recall: Euclidean-type algorithm for Padé approximation

Pade$(A, 2N)$

In: A in $\mathbb{K}[x]$ with $\deg A < 2N$

Out: (R, V) s.t. $R/V \equiv A \bmod x^{2N}$, $\deg R < N, \deg V \leq N$, or FAIL

1. $R_0 := x^{2N}$; $V_0 := 0$; $R_1 := A$; $V_1 := 1$; $i := 1$.
2. While $\deg R_i \geq N$ do:
    1. $(Q_i, R_{i+1}) := \mathrm{QuotRem}(R_{i-1}, R_i)$ $\qquad \# R_{i-1} = Q_i R_i + R_{i+1}$
    2. $V_{i+1} := V_{i-1} - Q_i V_i$
    3. $i := i + 1$.
3. If $V_i(0) \neq 0$ then return (R_i, V_i); else return FAIL.

▷ Quadratic complexity: $O(N^2)$ operations in $\mathbb{K}$

▷ There exist quasi-linear time algorithms $O(\mathsf{M}(N) \log N)$

In: A bound $N \in \mathbb{N}$ on the degree of the minimal polynomial of $(a_n)_{n\geq 0}$ and the first $2N$ terms $a_0, \ldots, a_{2N-1} \in \mathbb{K}$.

Out: the minimal generating polynomial $(a_n)_{n\geq 0}$.

1. $A = a_0 + a_1 x + \cdots + a_{2N-1} x^{2N-1}$.
2. Compute the solution $(R, V) \in \mathbb{K}[x]^2$ of $\mathsf{Pade}(A, 2N)$ s.t. $V(0) = 1$.
3. $d = \max\{1 + \deg(R), \deg(V)\}$. Return $\mathsf{rev}_d(V) = V(1/x)x^d$.

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Berlekamp-Massey algorithm, a variant

In: A bound $N \in \mathbb{N}$ on the degree of the minimal polynomial of $(a_n)_{n\geq 0}$ and the first $2N$ terms $a_0, \ldots, a_{2N-1} \in \mathbb{K}$.

Out: the minimal generating polynomial $(a_n)_{n\geq 0}$.

1. $R_0 := x^{2N}$; $V_0 := 0$; $R_1 := a_{2n-1} + \cdots + a_0 x^{2N-1}$; $V_1 := 1$; $i := 1$.
2. While $\deg R_i \geq N$, do:
   1. $(Q_i, R_{i+1}) := \mathrm{QuotRem}(R_{i-1}, R_i)$ $\qquad \# R_{i-1} = Q_i R_i + R_{i+1}$
   2. $V_{i+1} := V_{i-1} - Q_i V_i$
   3. $i := i + 1$
3. Return $V_i / \mathrm{lc}(V_i)$.

▷ Quadratic complexity: $O(N^2)$ operations in $\mathbb{K}$

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HERMITE-PADÉ APPROXIMANTS

—guessing equations with polynomial coefficients—

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▷ Very useful concept in number theory (irrationality/transcendence):

- [Hermite, 1873]: e is transcendent.
- [Lindemann, 1882]: π is transcendent; so does e^α for any $\alpha \in \overline{\mathbb{Q}} \setminus \{0\}$.
- [Apéry, 1978; Beukers, 1981]: $\zeta(3) = \sum_{n \geq 1} \frac{1}{n^3}$ is irrational.
- [Rivoal, 2000]: there exist infinite values of k such that $\zeta(2k+1) \notin \mathbb{Q}$.

Sur la généralisation des fractions continues algébriques;

PAR M. H. PADÉ,

Docteur ès Sciences mathématiques,
Professeur au lycée de Lille.

INTRODUCTION.

M. Hermite s'est, dans un travail récemment paru (¹), occupé de la généralisation des fractions continues algébriques. La question est de déterminer les polynomes $X_1, X_2, \ldots, X_n$, de degrés $\mu_1, \mu_2, \ldots, \mu_n$, qui satisfont à l'équation

$$S_1 X_1 + S_2 X_2 + \ldots + S_n X_n = S\, x^{\mu_1+\mu_2+\ldots+\mu_n+n-1},$$

$S_1, S_2, \ldots, S_n$ étant des séries entières données, et S une série également entière. Ou plutôt, il s'agit d'obtenir un algorithme qui permette le calcul de proche en proche de ces systèmes de n polynomes, et qui

Worked example

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This boils down to finding $\alpha_0, \alpha_1, \beta_0, \beta_1, \gamma_0, \gamma_1$ (not all zero) such that

$$\alpha_0+\alpha_1 x+(\beta_0+\beta_1 x)(1+x+2x^2+5x^3+14x^4)+(\gamma_0+\gamma_1 x)(1+2x+5x^2+14x^3+42x^4)=O(x^5)$$

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▷ Thus the approximant is $(1, -1, x)$, which corresponds to $P = 1 - y + xy^2$ such that $P(x, C(x)) = 0 \bmod x^5$.

Algebraic and differential approximation = guessing

- Hermite-Padé approximants of $n = 2$ power series are related to Padé approximants, i.e. to approximation of series by rational functions
- algebraic approximants = Hermite-Padé approximants for $f_\ell = A^{\ell-1}$, where $A \in \mathbb{K}[[x]]$ seriestoalgeq, listtoalgeq
- differential approximants = Hermite-Padé approximants for $f_\ell = A^{(\ell-1)}$, where $A \in \mathbb{K}[[x]]$ seriestodiffeq, listtodiffeq

```
> listtoalgeq([1,1,2,5,14,42,132,429],y(x));
```

$$1 - y(x) + xy(x)^2$$

```
> listtodiffeq([1,1,2,5,14,42,132,429],y(x))[1];
```

$$\left\{ -2\,y(x) + (2 - 4\,x)\frac{d}{dx}y(x) + x\frac{d^2}{dx^2}y(x), y(0) = 1, D(y)(0) = 1 \right\}$$

Theorem For any vector $\mathbf{F} = (f_1, \ldots, f_n)^T \in \mathbb{K}[[x]]^n$ and for any n-tuple $\mathbf{d} = (d_1, \ldots, d_n) \in \mathbb{N}^n$, there exists a Hermite-Padé approx. of type $\mathbf{d}$ for $\mathbf{F}$.

Proof: The undetermined coefficients of $P_i = \sum_{j=0}^{d_i} p_{i,j} x^j$ satisfy a linear homogeneous system with $\sigma = \sum_i (d_i + 1) - 1$ eqs and $\sigma + 1$ unknowns.

Corollary Computation in $O(\sigma^\omega)$, for $2 \leq \omega \leq 3$ (linear algebra exponent)

▷ There are better algorithms (the linear system is structured, Sylvester-like):

- Derksen's algorithm (Euclidean-like) $O(\sigma^2)$
- Beckermann-Labahn algorithm (DAC) $\tilde{O}(\sigma) = O(\sigma \log^2 \sigma)$
- structured linear algebra algorithms for Toeplitz-like matrices $\tilde{O}(\sigma)$

Theorem [Beckermann, Labahn, 1994] One can compute a Hermite-Padé approximant of type $(d, \dots, d)$ for $\mathbf{F} = (f_1, \dots, f_n)$ in $\tilde{O}(n^\omega d)$ ops. in $\mathbb{K}$.

Ideas:

- Compute a whole matrix of approximants
- Exploit divide-and-conquer

Algorithm:

1. If $\sigma = n(d+1) - 1 \leq$ threshold, call the naive algorithm
2. Else:
   1. recursively compute $\mathbf{P}_1 \in \mathbb{K}[x]^{n \times n}$ s.t. $\mathbf{P}_1 \cdot \mathbf{F} = O(x^{\sigma/2})$, $\deg(\mathbf{P}_1) \approx \frac{d}{2}$
   2. compute "residue" $\mathbf{R}$ such that $\mathbf{P}_1 \cdot \mathbf{F} = x^{\sigma/2} \cdot \left(\mathbf{R} + O(x^{\sigma/2})\right)$
   3. recursively compute $\mathbf{P}_2 \in \mathbb{K}[x]^{n \times n}$ s.t. $\mathbf{P}_2 \cdot \mathbf{R} = O(x^{\sigma/2})$, $\deg(\mathbf{P}_2) \approx \frac{d}{2}$
   4. return $\mathbf{P} := \mathbf{P}_2 \cdot \mathbf{P}_1$

▷ The precise choices of degrees is a delicate issue
▷ Corollary: Gcd, extended gcd, Padé approximants in $\tilde{O}(d)$ ops. in $\mathbb{K}$.

Application: certified algebraic guessing

Theorem. Suppose $A \in \mathbb{K}[[x]]$ is an algebraic series, and that it is a root of a (unknown) polynomial in $\mathbb{K}[x,y]$ of degree at most d in x and at most n in y. Let $\mathbf{Q} = (Q_0, Q_1, \ldots, Q_n)$ be a Hermite-Padé approximant of type $(d, \ldots, d)$ for $\mathbf{F} = (1, A, \ldots, A^n)$. If $\mathbf{Q} \cdot \mathbf{F} = O(x^{2dn+1})$, then $\mathbf{Q} \cdot \mathbf{F} = 0$.

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Remark: If $n = 1$, this simply says that if $A \in \mathbb{K}(x)_{\leq d}$ and if $Q_0(x) + Q_1(x)A = O(x^{2d+1})$ with $\deg(Q_i) \leq d$, then $Q_0(x) + Q_1(x)A = 0$.

Indeed, if $A = P_0/P_1$ with $\deg(P_i) \leq d$, then $Q_0 P_1 + Q_1 P_0 = O(x^{2d+1})$ and $\deg(Q_0 P_1 + Q_1 P_0) \leq 2d$ implies $Q_0 P_1 + Q_1 P_0 = 0$.

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Proof:

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- Thus $R = 0$, that is $\gcd(P, Q) \neq 1$, and thus $P \mid Q$, and A is a root of Q.

Application: algebraicity of a hypergeometric series

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5. Conclude that $f_n = r_n$ for all n, thus $f(t) = r(t)$ is algebraic.

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```
> f5:=sum(binomial(5*n,n)*t^n, n=0..infinity):
> simplify(f5) assuming t>0 and t<1/100;
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$$ {}_4F_3\left(\left[\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}\right]; \left[\frac{1}{4}, \frac{1}{2}, \frac{3}{4}\right]; \frac{3125\,t}{256}\right) $$

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> P5:=subs(y(t) = y, seriestoalgeq(series(f5,t,20), y(t))[1]);
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$$ 1+15\,y+80\,y^2+160\,y^3+(3125\,t-256)\,y^5 $$

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```
> subs({t=0, y=1}, P5), subs({t=0, y=1}, diff(P5,y));
```

$$ 0, \quad -625 $$

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> deq5:=algeqtodiffeq(P5, y(t))[1];
```

$$120\,y(t) + (15000\,t - 24)\,\frac{d}{dt}y(t) + \left(45000\,t^2 - 816\,t\right)\frac{d^2}{dt^2}y(t) + \left(25000\,t^3 - 1152\,t^2\right)\frac{d^3}{dt^3}y(t) + \left(3125\,t^4 - 256\,t^3\right)\frac{d^4}{dt^4}y(t) = 0$$

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> rec5:=map(factor, diffeqtorec(deq5, y(t), r(n)));
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$$5\,(5\,n+1)\,(5\,n+2)\,(5\,n+3)\,(5\,n+4)\,r(n) - 8\,(4\,n+1)\,(2\,n+1)\,(4\,n+3)\,(n+1)\,r(n+1) = 0$$

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```
> f:=n -> binomial(5*n,n):
> simplify(convert(subs({r(n)=f(n), r(n+1)=f(n+1)}, rec5), GAMMA));
```

$$0$$

An exercise for next time (18/11/2019)

Let (a_n) be a sequence with $a_0 = a_1 = 1$ satisfying the recurrence

$$(n+3)a_{n+1} = (2n+3)a_n + 3na_{n-1}, \quad \text{for all } \ n > 0.$$

Show that all a_n is an integer for all n.

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Show that all a_n is an integer for all n.

Follow the next steps:

1. Compute the first 5 terms of the sequence, $a_0, \dots, a_4$;
2. Determine a Hermite-Padé approximant of type $(0, 1, 2)$ for $(1, f, f^2)$, where $f = \sum_n a_n x^n$;
3. Deduce that $P(x, f(x)) = 0 \bmod x^5$ for $P(x, y) := 1 + (x-1)y + x^2y^2$;
4. Show that the equation $P(x, y) = 0$ admits a root $y = g(x) \in \mathbb{Q}[[x]]$ whose coefficients satisfy the same linear recurrence as (a_n);
5. Deduce that $a_{n+2} = a_{n+1} + \sum_{k=0}^{n} a_k \cdot a_{n-k}$ for all n, and conclude.