PROOF OF THE PROPOSITION ON P. 26

This is the proof that could not be given during the lecture, apparently because the server BBB was overloaded.

The proposition is the following.

Proposition 1. *If* $L(f) = 0$ *and* $(1, \partial, \ldots, \partial^{s-1})$ *is a basis of* $\mathbb{O}/\operatorname{Ann} f$*, then* $\mathbb{K}(\underline{x}) \cap \partial(\mathbb{O}/\operatorname{Ann} f) = L^\star(\mathbb{K}(\underline{x}))$*.*

Proof. First, using $M = L$ in Lagrange's identity with $L(f) = 0$ shows that for any $u \in \mathbb{K}(\underline{x})$, $L^\star(u)$, which is a rational function, is also the derivative of an element of $\mathbb{O}/\operatorname{Ann} f$. This proves the $\supset$ part of the identity.

Let $r \in \mathbb{K}(\underline{x}) \cap \partial(\mathbb{O}/\operatorname{Ann} f)$. There exists $Q \in \mathbb{O}/\operatorname{Ann} f$ such that $r = \partial Q$. Since $(1, \partial, \ldots, \partial^{s-1})$ is a basis of $\mathbb{O}/\operatorname{Ann} f$, Q can be written as a polynomial $Q(\partial)$ of degree at most $s-1$ in ∂. Since $\partial Q - r \in \operatorname{Ann} f$, it is a multiple of L, by minimality of L (which follows from the linear independence of $(1, \partial, \ldots, \partial^{s-1})$ in $\mathbb{O}/\operatorname{Ann} f$). Degree considerations then imply that there exists a rational function $q \in \mathbb{K}(\underline{x})$ such that

$$\partial Q - r = qL.$$

Taking adjoints on both sides then imply

$$-Q^\star \partial - r = L^\star q.$$

Applying these operators to 1 yields

$$-r = L^\star q$$

and thus $r = L^\star(-q) \in L^\star(\mathbb{K}(\underline{x}))$ showing the $\subset$ part of the identity. $\square$