P-recursive sequences and D-finite series

Alin Bostan

informatics mathematics Inria

ENS Lyon

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TOOLS FOR PROOFS

Resultants

The Sylvester matrix of $A = a_m x^m + \cdots + a_0 \in \mathbb{K}[x]$, $(a_m \neq 0)$, and of $B = b_n x^n + \cdots + b_0 \in \mathbb{K}[x]$, $(b_n \neq 0)$, is the square matrix of size $m + n$

$$\mathsf{Syl}(A, B) = \begin{bmatrix} a_m & a_{m-1} & \dots & a_0 & & & \\ & a_m & a_{m-1} & \dots & a_0 & & \\ & & \ddots & \ddots & & \ddots & \\ & & & a_m & a_{m-1} & \dots & a_0 \\ b_n & b_{n-1} & \dots & b_0 & & & \\ & b_n & b_{n-1} & \dots & b_0 & & \\ & & \ddots & \ddots & & \ddots & \\ & & & b_n & b_{n-1} & \dots & b_0 \end{bmatrix}$$

The resultant $\mathsf{Res}(A, B)$ of A and B is the determinant of $\mathsf{Syl}(A, B)$.

▷ Definition extends to polynomials over any commutative ring R.

Key observation

If $A = a_m x^m + \cdots + a_0$ and $B = b_n x^n + \cdots + b_0$, then

$$\begin{bmatrix} a_m & a_{m-1} & \dots & a_0 & & & \\ & \ddots & \ddots & & \ddots & & \\ & & a_m & a_{m-1} & \dots & a_0 & \\ b_n & b_{n-1} & \dots & b_0 & & & \\ & \ddots & \ddots & & \ddots & & \\ & & b_n & b_{n-1} & \dots & b_0 & \end{bmatrix} \times \begin{bmatrix} \alpha^{m+n-1} \\ \vdots \\ \alpha \\ 1 \end{bmatrix} = \begin{bmatrix} \alpha^{n-1} A(\alpha) \\ \vdots \\ A(\alpha) \\ \alpha^{m-1} B(\alpha) \\ \vdots \\ B(\alpha) \end{bmatrix}$$

Corollary: If $A(\alpha) = B(\alpha) = 0$, then $\text{Res}\,(A, B) = 0$.

Example: the discriminant

The discriminant of A is the resultant of A and of its derivative A'.
E.g. for $A = ax^2 + bx + c$,

$$\mathsf{Disc}(A) = \mathsf{Res}\,(A, A') = \det \begin{bmatrix} a & b & c \\ 2a & b & \\ & 2a & b \end{bmatrix} = -a(b^2 - 4ac).$$

E.g. for $A = ax^3 + bx + c$,

$$\mathsf{Disc}(A) = \mathsf{Res}\,(A, A') = \det \begin{bmatrix} a & 0 & b & c & \\ & a & 0 & b & c \\ 3a & 0 & b & & \\ & 3a & 0 & b & \\ & & 3a & 0 & b \end{bmatrix} = a^2(4b^3 + 27ac^2).$$

$\triangleright$ The discriminant vanishes when A and A' have a common root, that is when A has a multiple root.

Main properties

- **Link with gcd** $\text{Res}(A, B) = 0$ if and only if $\gcd(A, B)$ is non-constant.

- **Elimination property**
  There exist $U, V \in \mathbb{K}[x]$ not both zero, with $\deg(U) < n$, $\deg(V) < m$ and such that the following **Bézout identity** holds in $\mathbb{K} \cap (A, B)$:

$$\text{Res}(A, B) = UA + VB.$$

- **Poisson's formula**
  If $A = a(x - \alpha_1) \cdots (x - \alpha_m)$ and $B = b(x - \beta_1) \cdots (x - \beta_n)$, then

$$\text{Res}(A, B) = a^n b^m \prod_{i,j} (\alpha_i - \beta_j) = a^n \prod_{1 \leq i \leq m} B(\alpha_i).$$

- **Multiplicativity**

$$\text{Res}(A \cdot B, C) = \text{Res}(A, C) \cdot \text{Res}(B, C), \quad \text{Res}(A, B \cdot C) = \text{Res}(A, B) \cdot \text{Res}(A, C).$$

Proof of Poisson's formula

▷ Direct consequence of the key observation:
If $A = (x-\alpha_1)\cdots(x-\alpha_m)$ and $B = (x-\beta_1)\cdots(x-\beta_n)$ then

$$\mathsf{Syl}(A,B) \times \begin{bmatrix} \beta_1^{m+n-1} & \dots & \beta_n^{m+n-1} & \alpha_1^{m+n-1} & \dots & \alpha_m^{m+n-1} \\ \vdots & & \vdots & \vdots & & \vdots \\ \beta_1 & \dots & \beta_n & \alpha_1 & \dots & \alpha_m \\ 1 & \dots & 1 & 1 & \dots & 1 \end{bmatrix} =$$

$$= \begin{bmatrix} \beta_1^{n-1}A(\beta_1) & \dots & \beta_n^{n-1}A(\beta_n) & 0 & \dots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ A(\beta_1) & \dots & A(\beta_n) & 0 & \dots & 0 \\ 0 & \dots & 0 & \alpha_1^{m-1}B(\alpha_1) & \dots & \alpha_m^{m-1}B(\alpha_m) \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \dots & 0 & B(\alpha_1) & \dots & B(\alpha_m) \end{bmatrix}$$

▷ To conclude, take determinants and use Vandermonde's formula

Application: computation with algebraic numbers

Let $A = \prod_i (x - \alpha_i)$ and $B = \prod_j (x - \beta_j)$ be polynomials of $\mathbb{K}[x]$. Then

$$\prod_{i,j} (t - (\alpha_i + \beta_j)) = \mathrm{Res}_x(A(x), B(t - x)),$$

$$\prod_{i,j} (t - (\beta_j - \alpha_i)) = \mathrm{Res}_x(A(x), B(t + x)),$$

$$\prod_{i,j} (t - \alpha_i\beta_j) = \mathrm{Res}_x(A(x), x^{\deg B} B(t/x)),$$

$$\prod_{i} (t - B(\alpha_i)) = \mathrm{Res}_x(A(x), t - B(x)).$$

In particular, the set $\overline{\mathbb{Q}}$ of algebraic numbers is a field.

Proof: Poisson's formula. E.g., first one: $\prod_i B(t - \alpha_i) = \prod_{i,j} (t - \alpha_i - \beta_j)$.

▷ The same formulas apply *mutatis mutandis* to algebraic power series.

Application of the application: proving a nice identity

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```
> Pc:=resultant(a^3-2, x-(1-a+a^2), a);
```

$$x^3 - 3\,x^2 + 9\,x - 9$$

```
> PR:=resultant(Pc, numer(9*(t/x)^3-1), x);
```

$$729\,\left(t^9 + 3\,t^6 + 3\,t^3 - 1\right)$$

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▷ Why is this a proof? Hint: The above polynomial has one single real root

Bonus: another beautiful identity of Ramanujan's

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- ▷ In particular our LHS $F(x) = \frac{N(x)}{D(x)}$ is an algebraic number

```
> f:=sin(2*a)/sin(3*a)^2-sin(a)/sin(2*a)^2+sin(3*a)/sin(a)^2:
> expand(convert(f,exp)):
> F:=normal(subs(exp(I*a)=x,%)):
```

$$\frac{2i\left(x^{16}+5x^{14}+12x^{12}+x^{11}+20x^{10}+3x^9+23x^8+3x^7+20x^6+x^5+12x^4+5x^2+1\right)}{x\left(x^2-1\right)\left(x^2+1\right)^2\left(x^4+x^2+1\right)^2}$$

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- Get R in $\mathbb{Q}[t]$ with root $F(x)$, via resultant $\mathrm{Res}_x(x^7+1, t \cdot D(x) - N(x))$

```
> R:=factor(resultant(x^7+1,t*denom(F)-numer(F),x));
```

$$-1274\,i\left(t^2-28\right)^3$$

Shanks' 1974 identities

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$$=\sqrt{m}+\sqrt{2\sqrt{m+n}+2\sqrt{m}}$$

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▷ Exercise!

TOOLS FOR PROOFS

D-Finiteness

D-finite Series & P-recursive Sequences

Definition: A series $f \in \mathbb{K}[[x]]$ is D-finite (differentially finite), or holonomic, if its derivatives generate a finite-dimensional vector space over $\mathbb{K}(x)$, i.e.

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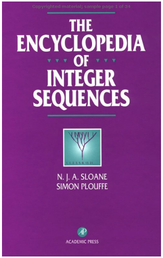
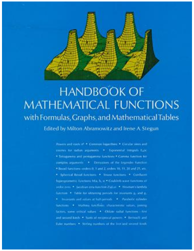
25% of Sloane's encyclopedia (OEIS) and 60% of Abramowitz & Stegun's HMF



Examples: exp, log, sin, cos, sinh, cosh, arccos, arccosh, arcsin, arcsinh, arctan, arctanh, arccot, arccoth, arccsc, arccsch, arcsec, arcsech, ${}_pF_q$ (includes Bessel J, Y, I and K, Airy Ai and Bi and polylogarithms), Struve, Weber and Anger functions, the large class of algebraic functions,...



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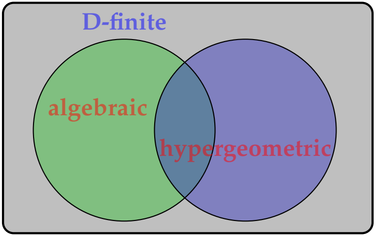
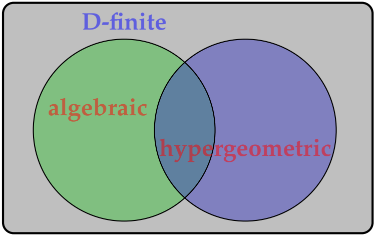
Teaser 5: Algorithms on D-finite series and P-recursive seqs allow for the automatic proof of identities $\longrightarrow$ $\sin(x)^2 + \cos(x)^2 = 1$

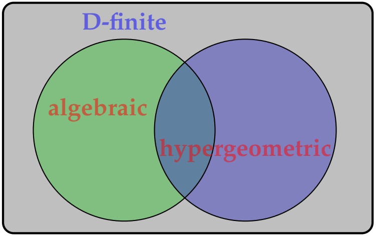
Classes of power series





$$S(t) = \sum_{n=0}^{\infty} s_n t^n \in \mathbb{Q}[[t]] \quad \text{is}$$

▷ *algebraic* if $P\big(t, S(t)\big) = 0$ for some $P(x,y) \in \mathbb{Z}[x,y] \setminus \{0\}$;



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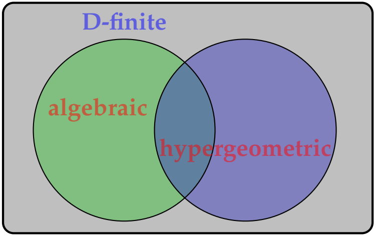

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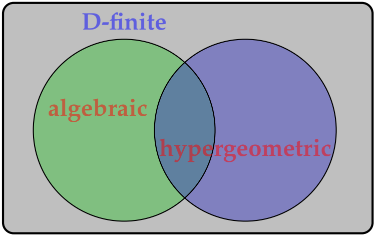

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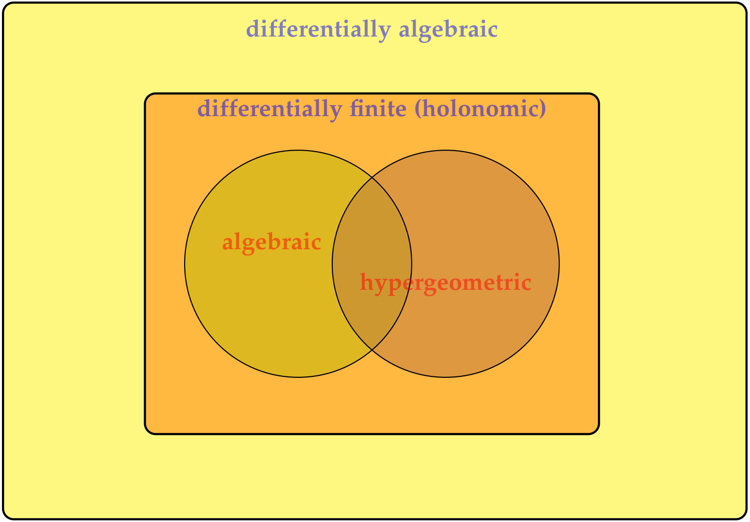
$$\frac{1}{1 - \rho t} = \sum_{n \geq 0} \rho^n t^n; \quad (1+t)^\alpha = \sum_{n \geq 0} \binom{\alpha}{n} t^n, \alpha \in \mathbb{Q}; \quad \ln(1-t) = -\sum_{n \geq 1} \frac{t^n}{n}.$$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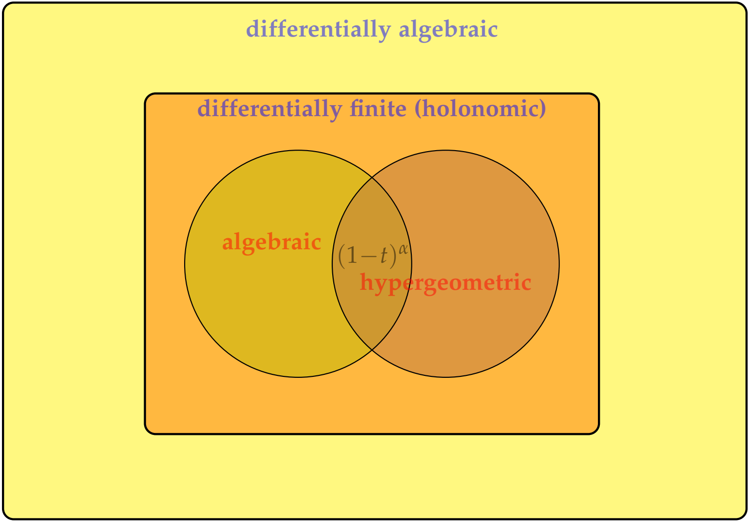
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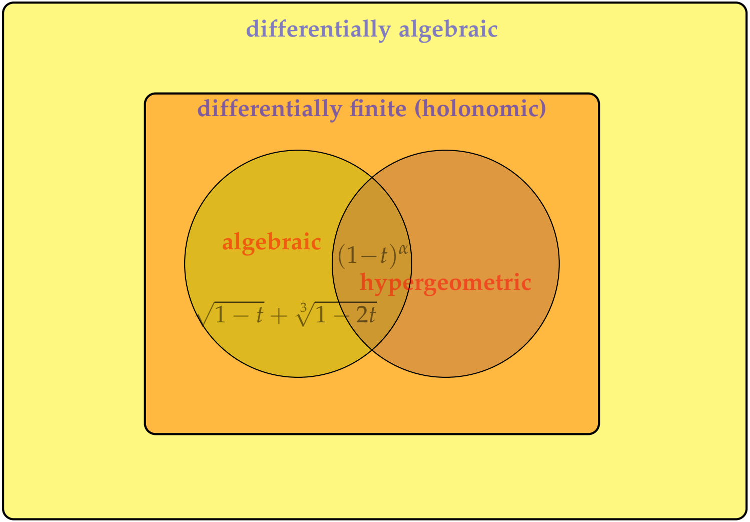

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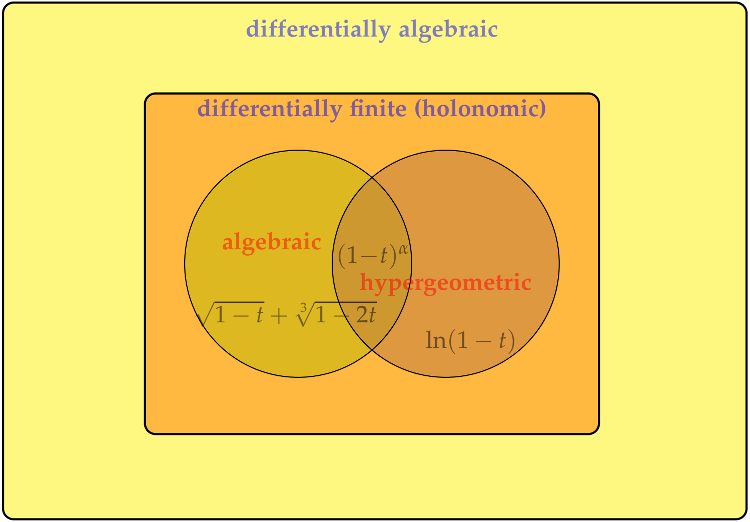

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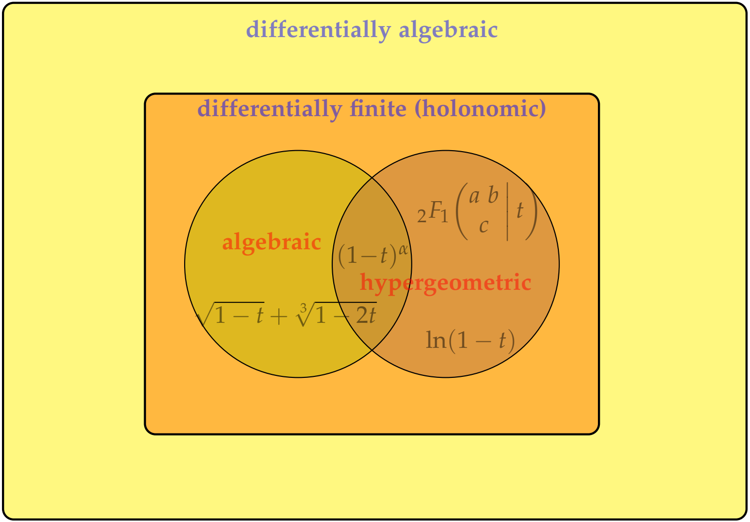
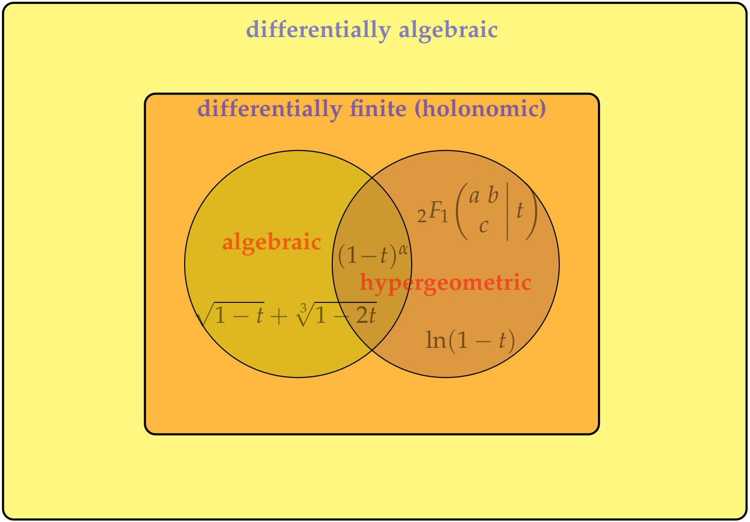
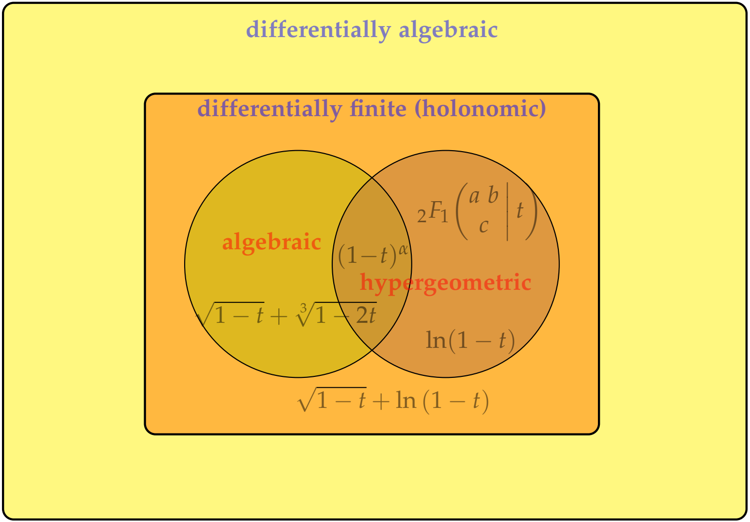


$${}_2F_1\left(\begin{matrix} a\ b \\ c \end{matrix} \middle| t\right) := \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{t^n}{n!}, \quad \text{where} \quad (a)_n = a(a+1)\cdots(a+n-1).$$



E.g., $(1-t)^\alpha = {}_2F_1\left(\begin{matrix} -\alpha\ 1 \\ 1 \end{matrix} \,\middle|\, t\right)$, $\ln(1-t) = -t \cdot {}_2F_1\left(\begin{matrix} 1\ 1 \\ 2 \end{matrix} \,\middle|\, t\right) = -\sum_{n=1}^{\infty} \frac{t^n}{n}$



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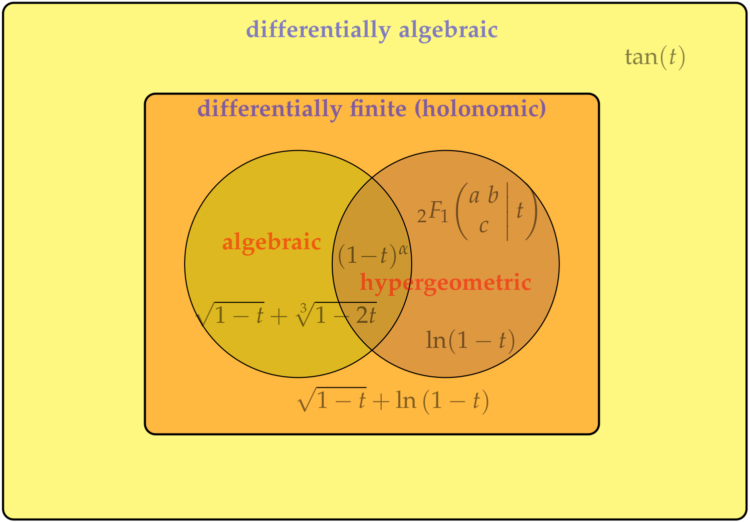

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Link D-finite $\leftrightarrow$ P-recursive

Theorem: A power series $f \in \mathbb{K}[[x]]$ is D-finite if and only if the sequence f_n of its coefficients is P-recursive

Proof (idea): $x\partial \leftrightarrow n$ and $x^{-1} \leftrightarrow S_n$ provide a ring isomorphism between

$$\mathbb{K}[x, x^{-1}, \partial] \quad \text{and} \quad \mathbb{K}[S_n, S_n^{-1}, n].$$

Snobbish way of saying that the equality $f = \sum_{n \geq 0} f_n x^n$ implies

$$[x^n]\, x f'(x) = n f_n, \quad \text{and} \quad [x^n]\, x^{-1} f(x) = f_{n+1}.$$

▷ Both conversions implemented in gfun: diffeqtorec and rectodiffeq

▷ Differential operators of order r and degree d give rise to recurrences of order $\leq d + r$ and coefficients of degree $\leq r$

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▷ Implemented in gfun: diffeq+diffeq, diffeq*diffeq, hadamardproduct, rec+rec, rec*rec, cauchyproduct

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- form a (rational function) matrix whose rows contain coordinates of successive derivatives (resp., shifts)
- compute a non-trivial element in its kernel, and output the corresponding equation

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▷ Cassini's identity (same idea): $F_n^2 - F_{n+1}F_{n-1} = (-1)^{n+1}$

```
for n to 8 do
   fibonacci(n)^2-fibonacci(n+1)*fibonacci(n-1)+(-1)^n
od;
```

$$0, 0, 0, 0, 0, 0, 0, 0$$

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(5) Since $u(0) = u'(0) = u''(0) = 0$, Cauchy's theorem concludes $u(x) \equiv 0$.

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Theorem [Abel 1827, Cockle 1860, Harley 1862] Algebraic series are D-finite, i.e., they satisfy linear differential equations with polynomial coefficients.

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Let $F(z) = \frac{1}{3} + \frac{4\sqrt{2}}{9}z - \frac{2}{9}z^2 + \cdots$ be the unique solution in $\mathbb{C}[[z]]$ of

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▷ Here is a short proof based on D-finiteness.

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Step 0. Find a polynomial equation for F:

```
> P:= resultant(F-G^2, resultant((F+1)-H^2,
                                  (2*F-1)*H+2*G^3-sqrt(6)*z,H),G);
```

$$-96\,F^3z^2 + 36\,z^4 + 36\,Fz^2 + 9\,F^2 - 12\,z^2 - 6\,F + 1$$

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Step 1. Find a linear differential equation for F:

```
> deqF:=algeqtodiffeq(P,F(z));
```

$$\left(18\,z^4 - 3\,z^2\right)\frac{d^2}{dz^2}F(z) + \left(18\,z^3 - 6\,z\right)\frac{d}{dz}F(z) + \left(6 - 8\,z^2\right)F(z) = 2$$

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Step 2. Find a linear recurrence for (the coefficients of) F:

```
> recF:=diffeqtorec(deqF, F(z), u(n))[1];
```

$$2\,(3\,n-2)\,(3\,n+2)\,u\,(n) = 3\,(n+4)\,(n+1)\,u\,(n+2)$$

Application

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Step 3. Check that the result satisfies the same recurrence, and the same initial conditions:

```
> res :=  n-> (-1)^(n+1)/n! * (3*sqrt(2))^(-n) * 2^(2*n+1)/(n+2) *
                                            GAMMA(3*n/2-1)/GAMMA(n/2):
> simplify(coeff(recF, u(n))*res(n) + coeff(recF, u(n+2))*res(n+2));
> res(1), res(2);
```

$$0$$

$$\frac{4}{9}\sqrt{2},\ -\frac{2}{9}$$

An exercise for next week

Prove the identity

$$\arcsin(x)^2 = \sum_{k\geq 0} \frac{k!}{\left(\frac{1}{2}\right)\cdots\left(k+\frac{1}{2}\right)} \frac{x^{2k+2}}{2k+2},$$

by performing the following steps:

1. Show that $y = \arcsin(x)$ can be represented by the differential equation $(1-x^2)y'' - xy' = 0$ and the initial conditions $y(0) = 0$, $y'(0) = 1$.
2. Compute a linear differential equation satisfied by $z(x) = y(x)^2$.
3. Deduce a linear recurrence relation satisfied by the coefficients of $z(x)$.
4. Conclude.