Padé-Hermite approximation

16 septembre 2020

Overview

Linearly recurring sequences

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Linearly recurring sequences

Over a field K

$$a_0 = 0, a_1 = 1, a_2 = 1, a_3 = 2, a_4 = 3, a_5 = 5, a_6 =?$$

$$F_{n+2} = F_{n+1} + F_n$$

Linearly recurring sequence:

$$a_i p_0 + a_{i+1} p_1 + \ldots + a_{i+n} p_n = 0,\ \forall i \geq 0$$

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Linearly recurring sequence:

$$a_i p_0 + a_{i+1} p_1 + \ldots + a_{i+n} p_n = 0, \ \forall i \geq 0$$

$a_0 = 2, a_1 = 1$

$$L_{n+2} = L_{n+1} + L_n$$

$$a_i p_0 + a_{i+1} p_1 + \ldots + a_{i+n} p_n = 0, \ \forall i \geq 0$$

- Power series in $\mathsf{K}[[1/x]]$

$p(x)$ is a generating polynomial if and only if

$$\sum_{i \geq 0} \frac{a_i}{x^{i+1}} = \frac{r(x)}{p(x)}, \quad \deg r < \deg p$$

$$a_i p_0 + a_{i+1} p_1 + \ldots + a_{i+n} p_n = 0, \ \forall i \geq 0$$

- Power series in $\mathsf{K}[[x]]$

$p(x)$, $p(0) \neq 0$ is a generating polynomial of degree n if and only if

$$\sum_{i \geq 0} a_i x^i = \frac{t(x)}{\hat{p}(x)}, \quad \deg t < \deg p$$

where $\hat{p}(x) = p(1/x)\, x^n = p_0 x^n + p_1 x^{n-1} + \ldots + p_n$

Matrix sequences

$A \in \mathsf{K}^{m\times m}$

Minimal polynomial of the **matrix**

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Minimal polynomial of a **vector** w.r.t to A, $u \in \mathsf{K}^m$

$$(A^i u) p_0 + (A^{i+1} u) p_1 + \ldots + (A^{i+n} u) p_n = 0, \ \forall i \geq 0$$

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Minimal polynomial of an associated **scalar** sequence, $v \in \mathsf{K}^m$

$$(v^\mathsf{T} A^i u) p_0 + (v^\mathsf{T} A^{i+1} u) p_1 + \ldots + (v^\mathsf{T} A^{i+n} u) p_n = 0, \ \forall i \geq 0$$

Matrix sequences- ctd

$A \in \mathsf{K}[x]^{m \times m}$

Vector generating polynomial, $\vec{p} \in \mathsf{K}[x]^m$

$$A^i \vec{p}_0 + A^{i+1} \vec{p}_1 + \ldots + A^{i+n} \vec{p}_n = 0, \ \forall i \geq 0$$

Matrix sequences- ctd

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$\longrightarrow$ form a $\mathsf{K}[x]$-module

Right (or left) **matrix minimal polynomial**, $P \in \mathsf{K}[x]^{m \times m}$

$$A^i P_0 + A^{i+1} P_1 + \ldots + A^{i+n} P_n = 0, \ \forall i \geq 0$$

$$a_i p_0 + a_{i+1} p_1 + \ldots + a_{i+n} p_n = 0,\ \forall i \geq 0$$

- $p(x) = p_0 + p_1 x + p_2 x^2 + \ldots p_n x^n$ is a **generating (or characteristic) polynomial**
- Principal ideal domain (exercise): minimal polynomial
- Monic polynomial: **the** minimal polynomial

How to compute the minimal polynomial?

$$a_i p_0 + a_{i+1} p_1 + \ldots + a_{i+n} p_n = 0, \ \forall i \geq 0$$

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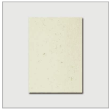
How to compute the minimal polynomial?



We are given a sequence

$\exists$ generating polynomial of degree $\leq n$

Hankel matrix

$$H_{n,n+1} = \begin{bmatrix} a_0 & a_1 & \cdots & a_n \\ a_1 & & & \\ & & & \\ a_{n-1} & & & a_{2n-1} \end{bmatrix} \in K^{n \times n+1}$$

Proposition: $H_{n,n+1} \cdot \begin{bmatrix} p_0 \\ \vdots \\ p_n \end{bmatrix} = 0 \iff p$ is a generating poly

Dem: "$\Leftarrow$" definition

$\Rightarrow$ ok for $0 \leq i \leq n-1$

$\forall i$: Using the relation on the left, rows for the infinite matrix can be zeroed.

Certification

$$h(x) = \sum_{i=0}^{2n-1} a_{2n-1-i}\, x^i \qquad \text{(reversed)}$$

from $H_{n,n+1}\, p = 0$ row by row

the terms in $h(x)\, p(x)$ are zero

for powers $2n-1, \dots, n$

$\exists\, r(x)$ $\deg(r) < n$

$$p(x)\, h(x) \equiv r(x) \mod x^{2n}$$

NB: if a polynomial generates the first $2n$ terms then it generates the whole sequence.

$$h(x) = \sum_{i=0}^{2n-1} a_{2n-1-i} \, x^i$$

if $p(x)$ generating of degree k

$$p(x)\, h(x) \equiv r(x) \mod x^{2n}$$

$$\begin{bmatrix} h(x) & -1 \end{bmatrix} \begin{bmatrix} p(x) \\ r(x) \end{bmatrix} \equiv 0 \mod x^{2n}$$

deg $< k$

Padé-Hermite approximation

Minimal approximant basis

$H(x)$ an $m \times 2m$ matrix of power series in $\mathsf{K}[[x]]$

A polynomial matrix $B \in \mathsf{K}[x]^{(2m)\times(2m)}$ is a minimal approximant basis of H at order σ if

- its columns for a basis of the $\mathsf{K}[x]$-module of vectors $v \in \mathsf{K}[x]^{2m}$ such that $Hv = 0 \bmod x^\sigma$
- the basis is minimal

$m = 1, \sigma = 6$

$$\begin{bmatrix} 5 + 3\,x + 2\,x^2 + x^3 + x^4 & -1 \end{bmatrix}$$

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$$\begin{bmatrix} 5+3x+2x^2+x^3+x^4 & -1 \end{bmatrix} \begin{bmatrix} x^2-x-1 & 2x^4-3x^3 \\ -8x-5 & x^4-15x^3 \end{bmatrix} = \begin{bmatrix} x^6 & 2x^8-x^7+x^6 \end{bmatrix}$$

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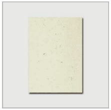

$$M = \begin{bmatrix} m_{ij}(x) \end{bmatrix} \in K(x)^{n \times n}$$

non singular

We look at the polynomial combinations of the columns $\sum M_i(x) p_i(x)$

→ Defines a module over $K(x)$

- free : has a basis
- basis with n polynomial vectors

$$\begin{pmatrix} x^2+1 & x^2+1 \\ x^2+x & x^2 \end{pmatrix} \begin{pmatrix} x \\ 1 \end{pmatrix} = \begin{pmatrix} x^3+x^2+x+1 \\ x^3+2x^2 \end{pmatrix}$$

Minimality of the basis

$$\begin{bmatrix} x^2+1 & x^2+1 \\ x^2+x & x^2 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ x \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} 0 & x^2+1 \\ x & x^2 \end{bmatrix}$$

<u>Proposition:</u> If the leading matrix by colums is non singular then the column degrees are minimal.

+ uniqueness

$$\begin{bmatrix} \vdots & \vdots & \vdots & \vdots \\ d_1 & \vdots & | & | \\ \vdots & d_2 & | & d_n \\ \vdots & \vdots & | & \vdots \end{bmatrix}$$

lexicographically

<u>Consequence:</u>

If colums are ordered by increasing degrees

$$d_1, \ldots, d_n$$

$$\left(\begin{bmatrix} M(x) \end{bmatrix} \begin{bmatrix} \vdots \\ * \;\; 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}\right)_i = \text{degree} \geq d_j$$

Algorithm $m = 1$

$$\begin{bmatrix} g(x) & h(x) \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} r_0(x) & t_0(x) \end{bmatrix} = \begin{bmatrix} \rho x^0 + x(\dots) & \tau x^0 + x(\dots) \end{bmatrix}$$

At step i:

$$\begin{bmatrix} g(x) & h(x) \end{bmatrix} \begin{bmatrix} b_i(x) & c_i(x) \end{bmatrix} = \begin{bmatrix} r_i(x) & t_i(x) \end{bmatrix}$$

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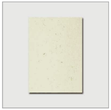

From valuation i to $i+1$ in the residue

$[b_i \;\; c_i]$ gives $[\rho x^i + \dots \quad \tau x^i + \dots]$

Update of the approximant basis

i) if $\rho = \tau = 0$ nothing to do

ii) $\rho \neq 0$ $\tau \neq 0$

$$b_i - \frac{\rho}{\tau} c_i \quad \text{or} \quad c_i - \frac{\tau}{\rho} b_i$$

?

For ensuring minimality

we choose the combination with

no degree increase

→ $\deg b_i \geq \deg c_i$

$\tilde{b}_i \quad b_i - \frac{\rho}{\tau} c_i$

NB: $\deg \tilde{b}_i = \deg b_i$ $\quad$ ($\mathcal{B}_i$ basis minimal)

$$H \cdot [\tilde{b}_i \;\; c_i] = [\tilde{\varphi}\, x^{i+1} + \ldots \quad \tau x^i + \ldots]$$

?

→ $\tilde{c}_i = x c_i$

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iii) $\rho = 0$ or $\tau = 0$ keep one unchanged e.g $\tilde{b}_i = b_i$ and $\tilde{c}_i = x c_i$

Idea of the proof. : by construction in $\mathcal{A}_{i+1}$

Approximant module of order i : $\mathcal{A}_i$

$$\mathcal{A}_{i+1} \subseteq \mathcal{A}_i$$

- $\rho = \tau = 0$ basis unchanged

minimality order i $\Rightarrow$ minimality order $i+1$

- $\rho \neq 0$ or $\tau \neq 0$

a degree must increase since otherwise, by uniqueness of the degrees, a minimal basis of $\mathcal{A}_{i+1}$ would be a basis of $\mathcal{A}_i$, which cannot happen.

- $[b_i \quad c_i]$ $\quad \deg b_i \geq \deg c_i$

if $\tau = 0$ $\quad c_i \in \mathcal{A}_{i+1}$

the minimal degree should be kept $\Rightarrow$ $x b_i$

- $\begin{bmatrix} b_i & c_i \end{bmatrix}$ $\quad$ $\deg b_i \geq \deg c_i$

$\downarrow \tau = 0$

$\tau \neq 0$

augmenting the smallest degree gives an optimal choice

Ex: $(d, d) \longrightarrow (d+1, d)$

$(d+k, d) \longrightarrow (d+k, d+1)$

and not $(d+k+1 \ \ d)$

Cost bound: $O(\sigma^2)$

- For order σ: σ iterations
- operations on polynomials (truncated power series) of degree less than σ
  update of the basis and of the residue

Exercise

$$h(x) = \sum_{i}^{2n-1} a_{2n-i-1} x^i$$

$$p(x)h(x) = r(x) \bmod x^{2n}$$

$$\begin{bmatrix} h(x) & -1 \end{bmatrix} \begin{bmatrix} p(x) & \cdot \\ r(x) & \cdot \end{bmatrix} = \begin{bmatrix} 0 & 0 \end{bmatrix} \bmod x^{2n}$$