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A short proof of a non-vanishing result by Conca, Krattenthaler and Watanabe

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In their paper *Regular sequences of symmetric polynomials* [CKW09], Aldo Conca, Christian Krattenthaler and Junzo Watanabe needed to prove, as an intermediate result, the fact that for any $h \geq 1$, the rational number

$$\sum_{b=0}^{\lfloor h/3 \rfloor} \frac{(-1)^{h-b}}{h-b} \binom{h-b}{2\,b} \left(\frac{2}{3}\right)^b$$

is non-zero, except for $h = 3$. The proof in [CKW09, Appendix, pp. 190–199] performs a (quite intricate) 3-adic analysis. In this note, we propose a shorter and elementary proof, based on the following observation.

Theorem 1. *For any $h \geq 1$, consider the polynomials*

$$a_h := \sum_{b=0}^{\lfloor h/3 \rfloor} \frac{(-1)^{h-b}}{h-b} \binom{h-b}{2b} U^b \ \in \mathbb{Q}[U]$$

and $s_h := h \cdot a_h$. Then, the sequence $(s_h)_{h \geq 1}$ satisfies the linear recurrence

$$s_{n+3} + 2\,s_{n+2} + s_{n+1} = U \cdot s_n, \quad \textit{for all } n \geq 1. \tag{1}$$

PROOF. Using $h/(h-b) = 1 + b/(h-b)$ and $2b \cdot \binom{h-b}{2b} = (h-b) \cdot \binom{h-b-1}{2b-1}$ yields the additive decomposition $s_h = p_h + q_h$, where

$$p_h := \sum_{b=0}^{\lfloor h/3 \rfloor} (-1)^{h-b} \binom{h-b}{2b} U^b \quad \text{and} \quad q_h := \frac{1}{2} \cdot \sum_{b=0}^{\lfloor h/3 \rfloor} (-1)^{h-b} \binom{h-b-1}{2b-1} U^b.$$

It is thus enough to prove that both $(p_h)_{h \geq 1}$ and $(q_h)_{h \geq 1}$ satisfy recurrence (1). We prove this for $(p_h)_{h \geq 1}$, the proof for $(q_h)_{h \geq 1}$ being similar. Extracting the coefficient of U^n on both sides of (1) with (s_h) replaced by (p_h) is equivalent to

$$\binom{h+3-n}{2\,n} - 2\binom{h+2-n}{2\,n} + \binom{h+1-n}{2\,n} = \binom{h+1-n}{2\,n-2}$$

and this identity is an immediate consequence of the Pascal triangle rule. $\square$

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Corollary 2. *For any $h \geq 1$, the rational number*

$$\sum_{b=0}^{\lfloor h/3 \rfloor} \frac{(-1)^{h-b}}{h-b} \binom{h-b}{2\,b} \left(\frac{2}{3}\right)^{b}$$

is non-zero, except for $h = 3$.

Proof. With previous notation, we need to prove that $a_h(2/3) = 0$ if and only if $h = 3$. By Theorem 1, the sequence

$$(u_h)_{h\geq 1} := \left(3^{h-1} \cdot h \cdot a_h(2/3)\right)_{h\geq 1} = (-1, 3, 0, -45, 324, \dots)$$

satisfies the linear recurrence relation

$$u_{n+3} + 6\,u_{n+2} + 9\,u_{n+1} = 18\,u_n, \quad \text{for all } n \geq 1. \tag{2}$$

It is clearly enough to prove that $u_h = 0$ if and only if $h = 3$. First, the terms u_n are all integers, by induction. Recurrence (2) shows that u_{n+3} and u_{n+1} have the same parity for all $n \geq 1$; since $u_2 = 3$, this implies that u_{2n} is an odd integer, and in particular it is nonzero, for all $n \geq 1$. It remains to consider the odd subsequence $(v_h)_{h\geq 1} := (u_{2h-1})_{h\geq 1} = (-1, 0, 324, 5508, 2916, \dots)$. From (2) it follows that the sequence $(v_h)_{h\geq 1}$ satisfies the recurrence relation

$$v_{n+3} - 18\,v_{n+2} + 297\,v_{n+1} = 324\,v_n \quad \text{for all } n \geq 1. \tag{3}$$

The same recurrence is also satisfied with $(v_h)_{h\geq 1}$ replaced by the sequence $(w_h)_{h\geq 1} := (v_h/4)_{h\geq 3} = (81, 1377, 729, -369603, \dots)$. In particular, w_{n+3} and w_{n+1} have the same parity for all $n \geq 1$, hence w_h is odd for any $h \geq 1$. It follows that v_h is nonzero for all $h \geq 3$, which concludes the proof. $\square$

Remark 3. An equivalent, equally simple, but slightly more "conceptual" proof of Theorem 1 is expressed in terms of generating functions. One starts with the Pascal triangle rule in the equivalent form $\sum_{a,b} \binom{a}{b} U^b z^a = 1/(1-(1+U)z)$, then extracts odd and even parts from it,

$$\sum_{a,b} \binom{a}{2b} U^b z^a = \frac{1-z}{(1-z)^2 - Uz^2}, \quad \sum_{a,b} \binom{a-1}{2b-1} U^b z^a = \frac{Uz^2}{(1-z)^2 - Uz^2},$$

and finally substitutes successively $a \leftarrow h-b$, $z \leftarrow -z$, $U \leftarrow Uz$; this yields

$$\sum_{h\geq 1} s_h z^h = \left(\frac{z+1}{(1+z)^2 - Uz^3} - 1\right) + \frac{1}{2} \cdot \frac{Uz^3}{(1+z)^2 - Uz^3} = \frac{z+1+Uz^3/2}{(1+z)^2 - Uz^3} - 1.$$

Recurrence (1) is now read off the denominator of the last rational function.

Remark 4. We leave it as an open problem to prove that the polynomials $a_h(U)$ and $s_h(U)$ are irreducible in $\mathbb{Q}[U]$ for all $h \geq 1$. (We checked this for $h \leq 5000$.) If true, this would imply a generalization of Corollary 2.

References

[CKW09] Aldo Conca, Christian Krattenthaler, and Junzo Watanabe. Regular sequences of symmetric polynomials. *Rendiconti del Seminario Matematico della Università di Padova*, 121:179–199, 2009.