D-finiteness: Algorithms and Applications

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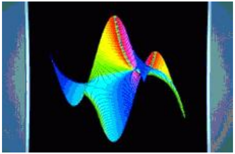

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I Introduction

Undecidability Issues

Theorem (Richardson 68)

In the class obtained from $\mathbb{Q}(x)$, π, $\log 2$ *by the operations* $+, -, \times$ *and composition with* exp, sin *and* $|\cdot|$, *testing for zero-equivalence is undecidable.*

Consequences:

1. "Simplification" is always difficult;
2. Computer algebra isolates classes for which it can provide algorithms.

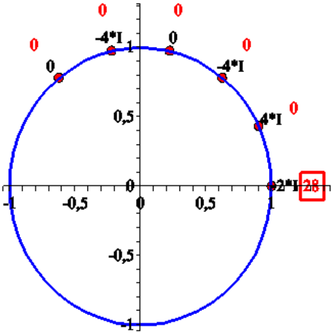
Example: Algebraic Numbers

An irreducible polynomial is a good data-struc

$$x = \frac{\sin\frac{2\pi}{7}}{\sin^2\frac{3\pi}{7}} - \frac{\sin\frac{\pi}{7}}{\sin^2\frac{2\pi}{7}} + \frac{\sin\frac{3\pi}{7}}{\sin^2\frac{\pi}{7}} = 2\sqrt{7}.$$

$\mathbb{Q}(\exp(i\pi/7))$ has dim 6 over $\mathbb{Q}$

Coordinates of x^2



Definition

A number $x \in \mathbb{C}$ is *algebraic* when its powers generate a finite-dimensional vector space over $\mathbb{Q}$.

Tools: Euclidean division, (extended) Euclidean algorithm, linear algebra.

Starting point:

```
> sin(2*a)/sin(3*a)^2-sin(a)/sin(2*a)^2+sin(3*a)/sin(a)^2;
```

$$\frac{\sin(2\,a)}{\sin(3\,a)^2}-\frac{\sin(a)}{\sin(2\,a)^2}+\frac{\sin(3\,a)}{\sin(a)^2}$$

Convert into rational function in T=$\exp\left(\frac{i\cdot\pi}{7}\right)$:

```
> expand(convert(%,exp));
```

$$\frac{2\,\mathrm{I}\left(\mathrm{e}^{\mathrm{I}a}\right)^2}{\left(\left(\mathrm{e}^{\mathrm{I}a}\right)^3-\frac{1}{\left(\mathrm{e}^{\mathrm{I}a}\right)^3}\right)^2}-\frac{2\,\mathrm{I}}{\left(\left(\mathrm{e}^{\mathrm{I}a}\right)^3-\frac{1}{\left(\mathrm{e}^{\mathrm{I}a}\right)^3}\right)^2\left(\mathrm{e}^{\mathrm{I}a}\right)^2}-\frac{2\,\mathrm{I}\,\mathrm{e}^{\mathrm{I}a}}{\left(\left(\mathrm{e}^{\mathrm{I}a}\right)^2-\frac{1}{\left(\mathrm{e}^{\mathrm{I}a}\right)^2}\right)^2}+\frac{2\,\mathrm{I}}{\left(\left(\mathrm{e}^{\mathrm{I}a}\right)^2-\frac{1}{\left(\mathrm{e}^{\mathrm{I}a}\right)^2}\right)^2\mathrm{e}^{\mathrm{I}a}}+\frac{2\,\mathrm{I}\left(\mathrm{e}^{\mathrm{I}a}\right)^3}{\left(\mathrm{e}^{\mathrm{I}a}-\frac{1}{\mathrm{e}^{\mathrm{I}a}}\right)^2}-\frac{2\,\mathrm{I}}{\left(\mathrm{e}^{\mathrm{I}a}-\frac{1}{\mathrm{e}^{\mathrm{I}a}}\right)^2\left(\mathrm{e}^{\mathrm{I}a}\right)^3}$$

```
> newval:=normal(subs(exp(I*a)=T,%));
```

$$newval := \frac{2\,\mathrm{I}\left(T^{16}+5\,T^{14}+12\,T^{12}+T^{11}+20\,T^{10}+3\,T^9+23\,T^8+3\,T^7+20\,T^6+T^5+12\,T^4+5\,T^2+1\right)}{T\left(T^2-1\right)\left(T^2+1\right)^2\left(T^4+T^2+1\right)^2}$$

```
> polmin:=normal((T^7+1)/(T+1));
```

$$polmin := T^6-T^5+T^4-T^3+T^2-T+1$$

Simplify denominator by Bezout

```
> G:=gcdex(denom(newval),polmin,T,'U','V'):U,V,G;
```

$$\frac{10}{7}+\frac{1}{7}\,T-\frac{5}{7}\,T^2-\frac{5}{7}\,T^3+\frac{1}{7}\,T^4+\frac{10}{7}\,T^5,$$

$$1+\frac{17}{7}\,T+\frac{24}{7}\,T^5-\frac{25}{7}\,T^9+\frac{26}{7}\,T^3+\frac{11}{7}\,T^2+\frac{25}{7}\,T^6+\frac{23}{7}\,T^4-\frac{2}{7}\,T^8-\frac{26}{7}\,T^{12}-\frac{24}{7}\,T^{10}-\frac{11}{7}\,T^{13}-\frac{23}{7}\,T^{11}-\frac{5}{7}\,T^7-\frac{10}{7}\,T^{14},\ 1$$

```
> rem(U*numer(newval),polmin,T);
```

$$-2\,\mathrm{I}-4\,\mathrm{I}\,T^4-4\,\mathrm{I}\,T^2+4\,\mathrm{I}\,T$$

Compute square

```
> rem(%^2,polmin,T);
```

$$28$$

Automatic Univariate Identities

$$F_nF_{n+2} - F_{n+1}^2 = (-1)^n$$ Cassini

$$\frac{1-\sqrt{1-4z}}{2z} = \sum_{n=0}^{\infty} \binom{2n}{n} \frac{z^n}{n+1}$$ Catalan num

$${}_2F_1\left(\begin{matrix} a, b \\ a+b+1/2 \end{matrix} \middle| z\right) = {}_2F_1\left(\begin{matrix} 2a, 2b \\ a+b+1/2 \end{matrix} \middle| \frac{1-\sqrt{1-z}}{2}\right)$$ Legendre

$$\sum_{n=0}^{\infty} H_n(x)H_n(y)\frac{u^n}{n!} = \frac{\exp\left(\frac{4u(xy-u(x^2+y^2))}{1-4u^2}\right)}{\sqrt{1-4u^2}}$$ Mehler

Automatic Multivariate Identities

$$\sum_{k=0}^{n}\binom{n}{k}^2\binom{n+k}{k}^2=\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}\sum_{j=0}^{k}\binom{k}{j}^3,$$

$$\sum_{n=0}^{+\infty}P_n(x)y^n=\frac{1}{\sqrt{1-2xy+y^2}},\quad P_n(x)=\frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}^2x^k$$

$$\int_0^{+\infty}xJ_1(ax)I_1(ax)Y_0(x)K_0(x)\,dx=-\frac{\ln(1-a^4)}{2\pi a^2},$$

$$\oint_0\frac{(1+2xy+4y^2)\exp\left(\frac{4x^2y^2}{1+4y^2}\right)}{y^{n+1}(1+4y^2)^{\frac{3}{2}}}\,dy=\frac{n!H_n(x)}{\lfloor n/2\rfloor!},$$

$$\sum_{k=0}^{n}\frac{q^{k^2}}{(q;q)_k(q;q)_{n-k}}=\sum_{k=-n}^{n}\frac{(-1)^kq^{(5k^2-k)/2}}{(q;q)_{n-k}(q;q)_{n+k}}$$

$$\sum_{j=0}^{n}\sum_{i=0}^{n-j}\frac{q^{(i+j)^2+j^2}}{(q;q)_{n-i-j}(q;q)_i(q;q)_j}=\sum_{k=-n}^{n}\frac{(-1)^kq^{7/2k^2+1/2k}}{(q;q)_{n+k}(q;q)_{n-k}}.$$

Plan

1. Definitions and closure properties (BS)
2. Fast algorithms for functions (AB)
3. Algorithms for sequences (BS)
4. Multivariate case (FC)

Context

http://algo.inria.fr/esf

II Definitions

D-finite Series

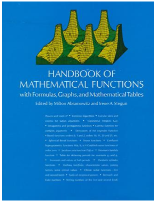
Definition

A *series* $f(x) \in \mathbb{Q}[[x]]$ is differentially finite (D-finite) when its derivatives generate a finite-dimensional vector space over $\mathbb{Q}(x)$. (LDE)

Equivalent definition A power series $y(z)$ is D-finite if there exist polynomials $a_i \in \mathbb{Q}[z]$ such that

$$a_k(z)y^{(k)}(z) + \cdots + a_0(z)y(z) = 0.$$

Examples of functions: exp, log, sin, cos, sinh, cosh, arccos, arccosh, arcsin, arcsinh, arctan, arctanh, arccot, arccoth, arccsc, arccsch, arcsec, arcsech, ${}_pF_q$ (includes Bessel J, Y, I and K, Airy Ai and Bi and polylogarithms), Struve, Weber and Anger fcns, the large class of algebraic functions,...



About 60% of Abramowitz & Stegun.

Coefficients ↔ Series

Theorem

A series is D-finite if and only if its sequence of coefficients satisfies a linear recurrence.

Proof (Dictionary)

$$y = \sum_n a_n z^n$$

$$y' = \sum_n n a_n z^{n-1} = \sum_n (n+1) a_{n+1} z^n$$

$$z \cdot y = \sum_n a_{n-1} z^n$$

$$z^i y^{(j)} = z^i D_z^j \cdot y = \sum_n (n-i+1)(n-i+2)\cdots(n+j-i) a_{n+j-i} z^n$$

$$(zD_z)^k z^{-i} \cdot y = \sum_n a_{n+i} n^k z^n.$$

Warning: orders differ

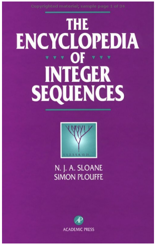
D-finite Sequences

Definition

A *sequence* u_n is D-finite when its shifts $(u_n, u_{n+1}, \dots)$ generate a finite-dimensional vector space over $\mathbb{Q}(n)$. (LRE)

Examples of sequences: rational sequences, hypergeometric sequences (includes $n!$, multinomials,...), classical orthogonal polynomials.

About 25% of Sloane & Plouffe.



eqn+ini. cond.=data structure

Example: Generalized Hypergeometric Series

$$y(z) := {}_pF_q\left(\left.\begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix}\right| z\right) = \sum_{n=0}^{\infty} \underbrace{\frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n n!}}_{u_n} z^n,$$

$$(x)_n := x(x+1)\cdots(x+n-1).$$

$\Longleftrightarrow$ first order linear recurrences (hypergeometric sequences)

$$\frac{u_n}{u_{n-1}} = \frac{(n+a_1-1)\cdots(n+a_p-1)}{n(n+b_1-1)\cdots(n+b_q-1)} \Longrightarrow$$

$$((\theta+a_1-1)\cdots(\theta+a_p-1)z - \theta(\theta+b_1-1)\cdots(\theta+b_q-1))\, y(z) = 0,$$

$$(\theta = zD_z).$$

Special Cases

$$\exp(z) = {}_0F_0\left(\begin{matrix} - \\ - \end{matrix}\middle| z\right) = 1 + z + \frac{z^2}{2!} + \cdots,$$

$$(1-z)^{-a} = {}_1F_0\left(\begin{matrix} a \\ - \end{matrix}\middle| z\right) = 1 + az + a(a+1)\frac{z^2}{2!} + \cdots,$$

$$\log\frac{1}{1-z} = z\,{}_2F_1\left(\begin{matrix} 1,1 \\ 2 \end{matrix}\middle| z\right) = z + \frac{z^2}{2} + \cdots,$$

$$\arcsin(z) = z\,{}_2F_1\left(\begin{matrix} \frac{1}{2},\frac{1}{2} \\ \frac{3}{2} \end{matrix}\middle| z\right), \quad \arctan(z) = z\,{}_2F_1\left(\begin{matrix} \frac{1}{2},1 \\ \frac{3}{2} \end{matrix}\middle| z\right),$$

$$J_\nu(z) = \frac{1}{\Gamma(\nu+1)}\left(\frac{z}{2}\right)^\nu {}_0F_1\left(\begin{matrix} - \\ \nu+1 \end{matrix}\middle| -\frac{z^2}{4}\right),$$

$$\mathrm{AGM}(a,b) = \frac{a}{{}_2F_1\left(\begin{matrix} \frac{1}{2},\frac{1}{2} \\ 1 \end{matrix}\middle| 1-\frac{b^2}{a^2}\right)}, \ldots$$

III Closure Properties

Proof of Identities: $\sin^2 + \cos^2 = 1$

```
> series(sin(x)^2+cos(x)^2,x,4);
```

$$1 + O(x^4)$$

Why is this a proof?

1. sin and cos satisfy a 2nd order LDE: $y'' + y = 0$;
2. their squares (and their sum) satisfy a 3rd order LDE (basis $\sin^2, \cos^2, \sin\cos$);
3. the constant 1 satisfies a 1st order LDE: $y' = 0$;
4. $\rightarrow \sin^2 + \cos^2 - 1$ satisfies a LDE of order at most 4;
5. it is not singular at 0;
6. Cauchy's theorem concludes.

Another identity (same idea): $F_n^2 - F_{n+1}F_{n-1} = (-1)^{n+1}$

```
> for n to 5 do
   fibonacci(n)^2-fibonacci(n+1)*fibonacci(n-1)+(-1)^n od;
```

Euclidean Division & Finite Dimension

Theorem (XIXth century)

D-finite series and sequences form $\mathbb{Q}$-algebras.

Proof. (for product of series)

$$f^{(n)}=a_0f+a_1f'+\cdots+a_{n-1}f^{(n-1)},\quad g^{(m)}=b_0g+b_1g'+\cdots+b_{m-1}g^{(m-1)}.$$

$$h=fg,$$
$$h'=f'g+fg'$$
$$\vdots$$
$$h^{(k)}=\sum_{\substack{0\le i<n\\0\le j<m}}c_{ijk}f^{(i)}g^{(j)}$$

$\Longrightarrow h,h',\dots,h^{(mn)}$ linearly dependent.

This proof is an algorithm.

Everything implemented in gfun [SaZi94]

Euclidean Division & Finite Dimension

Theorem (XIXth century)

D-finite series and sequences form $\mathbb{Q}$-algebras.

Corollary

D-finite series are closed under Hadamard (termwise) product, Laplace transform, Borel transform (ogf$\leftrightarrow$egf).

Everything implemented in `gfun` [SaZi94]

Mehler's Identity for Hermite Polynomials

$$\sum_{n=0}^{\infty} H_n(x)H_n(y)\frac{u^n}{n!} = \frac{\exp\left(\frac{4u(xy-u(x^2+y^2))}{1-4u^2}\right)}{\sqrt{1-4u^2}}$$

1. Definition of Hermite polynomials (D-finite over $\mathbb{Q}(x)$): recurrence of order 2
2. Product by linear algebra: $H_{n+k}(x)H_{n+k}(y)/(n+k)!, k \in \mathbb{N}$ generated over $\mathbb{Q}(x,n)$ by

   $$\frac{H_n(x)H_n(y)}{n!}, \frac{H_{n+1}(x)H_n(y)}{n!}, \frac{H_n(x)H_{n+1}(y)}{n!}, \frac{H_{n+1}(x)H_{n+1}(y)}{n!}$$

   $\longrightarrow$ recurrence of order at most 4;
3. Translation into a differential equation

I. Definition

> $R_1 := \{H(n+2) = (-2n-2)H(n) + 2H(n+1)x, H(0)=1, H(1)=2x\}$:

> $R_2 := subs(H=H_2, x=y, R_1)$;

$$R_2 := \{H_2(0)=1, H_2(n+2) = (-2n-2)H_2(n) + 2H_2(n+1)y, H_2(1)=2y\}$$

II. Product

> $R_3 := gfun:\text{-}poltorec(H(n)\cdot H_2(n)\cdot v(n), [R_1, R_2, \{v(n+1)\cdot(n+1)=v(n), v(1)=1\}], [H(n), H_2(n), v(n)], c(n))$;

$$R_3 := \Big\{c(0)=1, c(1)=4xy, c(2)=8x^2y^2+2-4y^2-4x^2, c(3)=\frac{32}{3}x^3y^3+24xy-16xy^3-16x^3y, (16n+16)c(n)-16xyc(n+1)+(-8n-20+8y^2+8x^2)c(n+2)-4xc(n+3)y+(n+4)c(n+4)\Big\}$$

III. Differential Equation

> $gfun:\text{-}rectodiffeq(R_3, c(n), f(u))$;

$$\left\{(16u^3-16u^2yx-4u+8uy^2+8ux^2-4xy)f(u)+(16u^4-8u^2+1)\left(\frac{d}{du}f(u)\right), f(0)=1\right\}$$

> $dsolve(\%, f(u))$;

$$f(u) = \frac{I\, e^{\left(\frac{-4xyu+y^2+x^2}{(2u-1)(2u+1)}\right)}}{e^{(-y^2-x^2)}\sqrt{2u+1}\sqrt{2u-1}}$$

>

Algebraic Series

Theorem (Cockle1860)

D-finite series composed with algebraic power series are D-finite.

Proof.

$$P(x,y)=0 \text{ and } AP+BP_y=1 \Rightarrow y'=-\frac{P_x}{P_y}=-BP_x \bmod P$$

$$\Rightarrow y^{(k)} \in \underbrace{\bigoplus_{i<\deg_y P} \mathbb{Q}(x)y^i}_{\text{finite dim}}.$$

$$(f\circ y)^{(p)} \text{ linear combination of } (f^{(j)}\circ y)y^k.$$

□

Also, $\exp\int y$.

Motzkin Numbers (Unary-Binary Trees)

$$M(z) - \left(1 + zM(z) + zM(z)^2\right) =: P(M) = 0.$$

Bézout: $\quad AP + BP_M = 1 \Rightarrow M' = -BP_z \bmod P =: cM + d1.$

Vector space of dimension 2

$\texttt{gfun[algeqtodiffeq]}(M = 1 + zM + zM^2, M(z));$

$$-1 - z + (-3\,z + 1)\,M\,(z) + \left(z - 6z^2 + z^3\right)\,M'\,(z)$$

$\texttt{gfun[diffeqtorec]}(\%, M(z), u(n));$

$\{nu(n) + (-9 - 6n)u(1 + n) + (3 + n)u(n + 2), u(0) = 1, u(1) = 2\}$

$\rightarrow$ fast computation.

Works for arbitrary degree.

Forests of Catalan Trees

$$Y(z) = \exp\left(\frac{1-\sqrt{1-4z}}{2}\right).$$

Same computation $\rightarrow$

$$\{(n+2)(n+1)u(n+2) = u(n) + 2(n+1)(2n+1)u(n+1),$$
$$u(0) = 1, u(1) = 1\}$$

Example: Airy Ai at Infinity

$$\mathrm{Ai}(z) = \frac{\sqrt{z}e^{-\xi}}{2\pi}\int_{-\infty}^{\infty} e^{-\xi[(u-1)(4u^2+4u+1)]}\,dv, \quad \xi = \frac{2}{3}z^{3/2}, u = \sqrt{1+\frac{v^2}{3}}$$

$$\sim \frac{1}{2}\pi^{-1/2}z^{-1/4}e^{-\xi}\sum_{n=0}^{\infty}(-1)^n\xi^{-n}\frac{\Gamma(3n+\frac{1}{2})}{54^n n!\Gamma(n+\frac{1}{2})}.$$

Computation:

1. algebraic change of variables $t^2 = (u-1)(4u^2+4u+1)$;

$$\to \int_{-\infty}^{\infty} e^{-\xi t^2} f(t)\,dt, \quad f(t) = \frac{dv}{dt},$$

2. recurrence satisfied by the coefficients of f (generating series);
3. termwise integration (Hadamard product).

I. Algebraic change of variables

$eq_u := u^2 - \left(1 + \frac{v^2}{3}\right)$:

$eq_t := t^2 - (u-1)\cdot\left(4\cdot u^2 + 4\cdot u + 1\right)$:

$res := resultant\left(eq_u, eq_t, u\right)$;

$$t^4 + 2\,t^2 - 3\,v^2 - \frac{8}{3}\,v^4 - \frac{16}{27}\,v^6$$

gfun :- algeqtodiffeq$\left(res, v(t), \left\{v(0)=0, D(v)(0)=sqrt\left(\frac{2}{3}\right)\right\}\right)$;

$$\left\{-4\,v(t) + 9\,t\left(\frac{d}{dt}\,v(t)\right) + \left(9\,t^2 + 18\right)\left(\frac{d^2}{dt^2}\,v(t)\right), v(0)=0, (D(v))(0) = \frac{1}{3}\sqrt{6}\right\}$$

II. Recurrence satisfied by the coefficients of f

gfun :- poltodiffeq$(diff(v(t), t), [\%], [v(t)], f(t))$;

$$\left\{5\,f(t) + 27\,t\left(\frac{d}{dt}\,f(t)\right) + \left(9\,t^2 + 18\right)\left(\frac{d^2}{dt^2}\,f(t)\right), f(0) = \frac{1}{3}\sqrt{6}, (D(f))(0) = 0\right\}$$

$R_f :=$ gfun :- diffeqtorec$(\%, f(t), c(n))$;

$$\left\{\left(5 + 18\,n + 9\,n^2\right)c(n) + \left(18\,n^2 + 54\,n + 36\right)c(n+2), c(0) = \frac{1}{3}\sqrt{6}, c(1) = 0\right\}$$

III. Hadamard product

assume(ξ > 0); *s* := *Int*(exp(-ξ * *t*^2)* *t*^*n*, *t* = -∞ .. ∞): *s* = *student*[*intparts*](*s*, exp(-ξ * *t*^2));

$$\int_{-\infty}^{\infty} e^{(-\xi\sim t^2)}\, t^n \, dt = -\int_{-\infty}^{\infty} -\frac{2\,\xi\sim t\, e^{(-\xi\sim t^2)}\, t^{(n+1)}}{n+1}\, dt$$

$R_i := \left\{ c(n) = \frac{2\cdot\xi}{(n+1)}\cdot c(n+2),\ c(0) = value(eval(s, n=0)),\ c(1) = value(eval(s, n=1)) \right\}$;

$$\left\{ c(n) = \frac{2\,\xi\sim c(n+2)}{n+1},\ c(0) = \frac{\sqrt{\pi}}{\sqrt{\xi\sim}},\ c(1) = 0 \right\}$$

```
> FinalRec:=gfun:-`rec*rec`(R[i],R[f],c(n));
```

$$FinalRec := \left\{ \left(5 + 18\,n + 9\,n^2\right) c(n) + (36\,\xi\sim n + 72\,\xi\sim)\, c(n+2),\ c(1) = 0,\ c(0) = \frac{1}{3}\,\frac{\sqrt{\pi}\,\sqrt{6}}{\sqrt{\xi\sim}} \right\}$$

Sol := *rsolve*(*FinalRec*, *c*(*n*));

$$\begin{cases} \frac{1}{3}\,\frac{(-1)^{\left(\frac{1}{2}n\right)}\, 2^{\left(-1-\frac{1}{2}n\right)}\, \Gamma\left(\frac{1}{2}n+\frac{5}{6}\right)\Gamma\left(\frac{1}{2}n+\frac{1}{6}\right)\xi\sim^{\left(-\frac{1}{2}n\right)}\sqrt{6}}{\sqrt{\pi}\,\Gamma\left(\frac{1}{2}n+1\right)\sqrt{\xi\sim}} & n::even \\ 0 & n::odd \end{cases}$$

D-finite Series in Arithmetic

Definition (For $k \in \mathbb{Z}$, modular form of weight k)

f defined on $\Im z > 0$ such that
$f((az+b)/(cz+d)) = (cz+d)^k f(z)$, for all matrices $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ in $SL(2,\mathbb{Z})$ or one of its subgroups of finite index.
Modular function: modular form of weight 0.

Theorem (XIXth century)

Let $f(z)$ be a meromorphic modular form of weight $k > 0$ and $t(z)$ a modular function. Then $F(t)$ defined by $F(t(z)) = f(z)$ satisfies a LDE of order $k+1$ with algebraic coefficients.

Ex. [Apéry] $t(z) = \left(\frac{\eta(z)\eta(6z)}{\eta(2z)\eta(3z)}\right)^{12}, \quad f(z) = \frac{(\eta(2z)\eta(3z))^7}{(\eta(z)\eta(6z))^5},$

$$F(t) = 1 + 5t + 73t^2 + \cdots = \sum_{n\geq 0}\sum_k \binom{n}{k}^2\binom{n+k}{k}^2 t^n.$$

Summary

1. LDEs and LREs can be viewed as a data-structure;
2. algorithms for $+$, $\times$, translation LDE$\leftrightarrow$LRE;
3. several sources of D-finiteness (algebraic series, hypergeometric series, series of arithmetical origin);
4. applications to proofs of identities, to computations of coefficients of expansions.

Next episodes: efficient algorithms, more applications, generalization to the multivariate case.