Algorithms for D-finite Sequences

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Motivation

- Indefinite hypergeometric summation [Gosper78]

$$\sum_{k=0}^{n} \frac{(3k)!}{k!(k+1)!(k+2)!27^k} = \frac{(81n^2+261n+200)(3n+2)!}{40(n+2)!(n+1)!n!27^n} - \frac{9}{2}$$

- Definite hypergeometric summation [Zeilberger91]

$${}_2F_1\left(\left.\begin{matrix} a, b \\ c \end{matrix}\right| 1\right) = \sum_{k=0}^{\infty} \frac{(a)_k(b)_k}{(c)_k k!} = \frac{\Gamma(c-a-b)\Gamma(c)}{\Gamma(c-a)\Gamma(c-b)} \qquad \text{[Gauss1812]},$$

$$\sum_{k\in\mathbb{Z}} (-1)^k \binom{a+b}{a+k}\binom{a+c}{c+k}\binom{b+c}{b+k} = \frac{(a+b+c)!}{a!\,b!\,c!} \qquad \text{[Dixon1903]}.$$

I Polynomial Solutions

First Algorithm

Problem (Solutions such that $u_n = p(n)$, p unknown polynomial)

$$a_r(n)u_{n+r} + \cdots + a_0(n)u_n = 0$$

Algorithm

1. Find possible degrees;
2. indeterminate coefficients.

Example $3u(n+2) - nu(n+1) + (n-1)u(n) = 0.$

1. Let $u(n) = c_d n^d + c_{d-1}n^{d-1} + \cdots$, then

$$3u(n+2) = 3c_d n^d + \cdots,$$
$$-nu(n+1) = -c_d n^{d+1} - c_d d n^d - c_{d-1}n^d + \cdots,$$
$$(n-1)u(n) = c_d n^{d+1} + c_{d-1}n^d - c_d n^d + \cdots,$$

$$\rightarrow 3u(n+2) - nu(n+1) + (n-1)u(n) = c_d(3-d-1)n^d + \cdots$$

2. Set $u(n) = n^2 + an + b$

$$3u(n+2) - nu(n+1) + (n-1)u(n) = (a+11)n + 2(3a+b+6).$$

Solutions with Finite Support

Problem (Solutions such that $u_n = 0$ for all large n)

$a_r(n)u_{n+r} + \cdots + a_0(n)u_n = 0$ Wanted: initial conditions.

- Upper end of finite support: largest integer root N of a_0;
- Undetermined coefficients $\longrightarrow$

  $u_0, u_1, \ldots, u_N, u_{N+1}, \ldots, u_{N+r-1}$ in $\tilde{O}(N^2)$ bit ops;
- Matrix factorial $\rightarrow$ initial conditions in $\tilde{O}(N)$ bit ops and probabilistic test in $\tilde{O}(\sqrt{N})$.

Works for regular case ($0 \notin a_r(\mathbb{N})$) and irregular case (more technical).

Application: efficient polynomial solutions of linear differential equations (AB). [BoClSa05]

Polynomial Solutions

Problem (Solutions such that $u_n = p(n)$, p unknown polynomial)

$$a_r(n)u_{n+r} + \cdots + a_0(n)u_n = 0$$

In general, the coefficients of u do not satisfy a linear recurrence.

They do in a binomial basis [Boole1872].

$$P(n) = c_0 + c_1\binom{n}{1} + c_2\binom{n}{2} + \cdots + c_d\binom{n}{d}.$$

Solution in Binomial Basis

$$P(n) = c_0 + c_1\binom{n}{1} + c_2\binom{n}{2} + \cdots + c_d\binom{n}{d}.$$

Dictionary:

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1} \qquad S_n \mapsto 1 + S_k$$

$$n\binom{n}{k} = k\binom{n}{k} + (k+1)\binom{n}{k+1} \qquad n \mapsto k(1 + S_k^{-1}).$$

Example

$$3u(n+2) - nu(n+1) + (n-1)u(n) = (3S_n^2 - nS_n + n - 1)\cdot u(n) = 0.$$

$$3(1+S_k)^2 - k(1+S_k^{-1})(1+S_k) + k(1+S_k^{-1}) - 1 =$$
$$3S_k^2 + (6-k)S_k + (2-k)$$
$$3c_{k+2} + (6-k)c_{k+1} + (2-k)c_k = 0.$$

Upper end of support: $k = 2$;

$$c_4 = c_3 = 0, c_2 = 1, \quad 3c_3 + 5c_2 + c_1 = 0, \quad 3c_2 + 6c_1 + 2c_0 = 0,$$

Summary: Algorithm and Complexity

Algorithm (Polynomial Solutions)

1. Translate into recurrence for coefficients in binomial basis;
2. Find bound N on finite support;
3. Compute initial conditions (compact representation) or nonexistence of solution with finite support.

N is exponential in the bit-size of the recurrence

Example: $nu(n+1)-(n+100)u(n)=0$

	Direct [AbBrPe95]	With matrix factorials [BoChClSa06]
nb of coeff ops	$O(N)$	$O(\sqrt{N})$
nb of bit ops	$O(N^2)$	$\tilde{O}(N)$.

Philosophy: such polynomials with high degree have special coefficients

Operations on Compact Representations

Lemma

If u_n and v_n satisfy linear recurrences, so does $w_N := \sum_{n=0}^{N} u_n v_n$.

Corollary

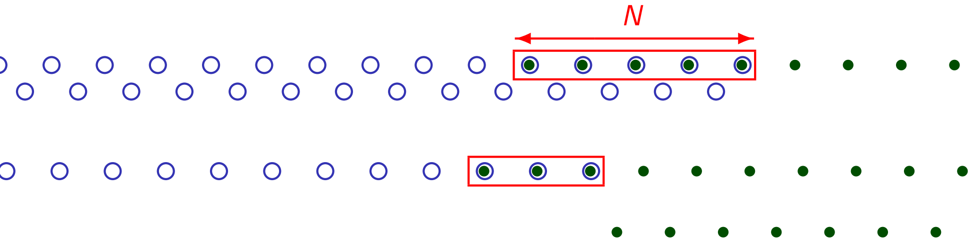
P polynomial in compact representation, a algebraic number, then $(\Delta^k P)(a)$ can be computed in $\tilde{O}(N)$ bit operations (with $k = O(N)$).

$\rightarrow$ Compact representations can be evaluated and shifted to another point efficiently.

II Rational Solutions

Poles of Rational Solutions

$$a_0(n)a_0(n)u(n+d)+\cdots+a_d(n)a_d(n)u(n)=0.$$



Algorithm (Rational Solutions, Abramov 89)

1. bound the poles;
2. compute new equation for numerator;
3. look for polynomial solutions.

Rational Solutions of Linear Recurrences

$$a_r(n)u(n+r)+\cdots+a_0(n)u(n)=f(n).$$

Theorem (Abramov 95)

The polynomial $C(n)$ of the Gosper-Petkovšek form of $(a_r(n-r+1), a_0(n))$ is a multiple of the denominator of all rational solutions.

Definition (Gosper-Petkovšek normal form of $(P(n), Q(n))$)

$$\frac{P(n)}{Q(n)}=\frac{A(n)}{B(n)}\frac{C(n+1)}{C(n)},\qquad \begin{cases}A(n)\wedge C(n)=B(n)\wedge C(n+1)=1,\\ A(n)\wedge B(n+h)=1\quad (h\in\mathbb{N}).\end{cases}$$

deg C potentially exponential, but admits a compact form:

$$\frac{C(n+1)}{C(n)}=\frac{g_1(n)}{g_1(n-h_1)}\cdots\frac{g_s(n)}{g_s(n-h_s)},$$

h_i's roots of the dispersion polynomial, g_i factors of P.

Compact Rational Solutions of Linear Recurrences

$$a_r(n)u(n+r)+\cdots+a_0(n)u(n)=f(n).$$

1. Compute a compact representation of $C(n)$.
2. Change unknown: $u(n)=v(n)/C(n)$.
3. Homogeneous case ($f=0$):
   normalize and find compact polynomial solutions;
4. Non-homogeneous case (f non-zero polynomial):
   make homogeneous first. (Otherwise, rhs of expl degree.)

Corollary: Compact Gosper algorithm for indefinite summation.

III Indefinite Hypergeometric Summation

Problem

Definition (Hypergeometric Sequence)

$u(n+1)/u(n)$ rational function in n.

Examples C^n, $n!$, $1/n!$, $A^n \prod_j (n+c_j)!/\prod_k (n+d_k)!$.

Problem (Indefinite Hypergeometric Summation)

Given $u(n)$ hypergeometric, find $v(n)$ hypergeometric such that $v(n+1)-v(n)=u(n)$, if possible.

Motivation: $\sum_{n=a}^{n=b} u(n) = v(b+1)-v(a)$ (a discrete primitive).
Examples:

$$\sum_n 3^n = \frac{1}{2}3^n,$$

$$\sum_n \frac{(3n)!}{n!(n+1)!(n+2)!27^n} = \frac{9}{40}\frac{(81n^2+99n+20)(n+1)(n+2)(3n)!}{n!(n+1)!(n+2)!27^n}.$$

Algorithm for $v(n+1)-v(n)=u(n)$, u, v hypergeometric

Lemma

There exists a rational function $r(n)$ such that $v(n)=r(n)u(n)$.

Proof.

$$v(n) \text{ hypergeometric} \Leftrightarrow v(n+1)=s(n)v(n), \quad s \text{ rational},$$
$$\Rightarrow v(n+1)-v(n)=(s(n)-1)v(n)=u(n).$$

Consequence: $r(n)$ must obey

$$r(n+1)\underbrace{\frac{u(n+1)}{u(n)}}_{\text{rational}}-r(n)=1.$$

Conclusion: Indefinite hypergeometric summation reduces to finding rational solutions.

Example

$$u(n) = \frac{(3n)!}{n!(n+1)!(n+2)!27^n}$$

1. $\frac{u(n+1)}{u(n)} = \frac{(3n+1)(3n+2)}{9(n+2)(n+3)}$;
2. Recurrence for r:

$$(3n+1)(3n+2)r(n+1) - 9(n+2)(n+3)r(n) = 9(n+2)(n+3)$$

3. roots do not differ by an integer $\rightarrow$ no pole;
4. bound on degree of polynomial solution: 4;
5. indeterminate coefficients (or matrix factorials):

$$v(n) = \frac{9}{40}(81n^2 + 99n + 20)(n+1)(n+2)u(n).$$

IV Definite Hypergeometric Summation

Definite Hypergeometric Summation

Problem

Given $t_{n,k}$, such that both $t_{n+1,k}/t_{n,k}$ and $t_{n,k+1}/t_{n,k}$ are rational functions in n and k, find a linear recurrence for

$$S(n) = \sum_k t_{n,k}.$$

Example 1 $S(n) = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$

Input: $\dfrac{t_{n+1,k}}{t_{n,k}} = \dfrac{(n+1)y}{n-k+1}, \quad \dfrac{t_{n,k+1}}{t_{n,k}} = \dfrac{(n-k)x}{(k+1)y}$

Output: $S(n+1) - (x+y)S(n) = 0.$

Definite Hypergeometric Summation

Problem

Given $t_{n,k}$, such that both $t_{n+1,k}/t_{n,k}$ and $t_{n,k+1}/t_{n,k}$ are rational functions in n and k, find a linear recurrence for

$$S(n) = \sum_k t_{n,k}.$$

Example 2 $S(n) = \sum_{k=0}^{n} \frac{(n+k)!}{(n-k)!k!^2}$

Input: $\frac{t_{n+1,k}}{t_{n,k}} = \frac{n+k+1}{n-k+1}, \quad \frac{t_{n,k+1}}{t_{n,k}} = \frac{(n+k+1)(n-k)}{(k+1)^2}$

Output: $(n+2)S(n+2) - 3(2n+3)S(n+1) + (n+1)S(n) = 0.$

Example: $\zeta(3)$ is irrational [Apéry78]

$$a_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2, \quad b_n = a_n \sum_{k=1}^{n} \frac{1}{k^3} + \sum_{k=1}^{n} \sum_{m=1}^{k} \frac{(-1)^{m+1}\binom{n}{k}^2\binom{n+k}{k}^2}{2m^3\binom{n}{m}\binom{n+m}{m}}.$$

1. $m^3 \binom{n}{m}\binom{n+m}{m} \geq n^2 \Rightarrow \lim_{n\to\infty} b_n/a_n = \zeta(3)$.
2. $a_n \in \mathbb{N}^\star$, $d_n^3 b_n \in \mathbb{Z}$, where $d_n := \operatorname{lcm}(1,\dots,n)$
$$\frac{()^2()^2}{2m^3()()} = \frac{\binom{n}{k}\binom{n+k}{k}\binom{n-m}{n-k}\binom{n+k}{k-m}}{2m^3\binom{k}{m}^2} \quad \text{and} \quad m\binom{k}{m} \Big| \, d_k.$$
3. By creative telescoping, both a_n and b_n satisfy
$$(n+1)^3 u_{n+1} = (34n^3 + 51n^2 + 27n + 5)u_n - n^3 u_{n-1}, \quad n \geq 1.$$

Example: $\zeta(3)$ is irrational [Apéry78]

1. $\lim_{n\to\infty} b_n/a_n = \zeta(3)$.
2. $a_n \in \mathbb{N}^\star$, $d_n^3 b_n \in \mathbb{Z}$, where $d_n := \operatorname{lcm}(1,\dots,n)$
3. By creative telescoping, both a_n and b_n satisfy

$$(n+1)^3 u_{n+1} = (34n^3 + 51n^2 + 27n + 5)u_n - n^3 u_{n-1}, \quad n \geq 1.$$

4. Using closure properties, $0 < \zeta(3) - \frac{b_n}{a_n} = \sum_{k\geq n+1} \frac{b_k}{a_k} - \frac{b_{k-1}}{a_{k-1}}$:

$$b_k a_{k-1} - b_{k-1} a_k = \frac{6}{k^3};$$

5. $\lambda a_n + \mu b_n \approx \alpha_\pm^n$, with $\alpha_\pm^2 = 34\alpha_\pm - 1$;
6. Conclusion: $0 < \underbrace{a_n d_n^3}_{\in\mathbb{N}} \zeta(3) - \underbrace{d_n^3 b_n}_{\in\mathbb{N}} \simeq C\alpha_-^n e^{3n} \to 0.$

Creative Telescoping

$$I_n := \sum_{k=0}^{n} \underbrace{\binom{n}{k}}_{u_{n,k}} = 2^n.$$

IF one knows Pascal's triangle:

$$\binom{n+1}{k+1} = \binom{n}{k+1} + \binom{n}{k},$$

then summing over k gives

$$I_{n+1} = I_n + I_n = 2I_n.$$

The initial condition $I_0 = 1$ concludes the proof.

Creative Telescoping

$$I_n := \sum_{k=0}^{n} \underbrace{\binom{n}{k}}_{u_{n,k}} = 2^n.$$

Zeilberger's idea: look for rational $\lambda(n)$ such that

$$v_{n,k+1} - v_{n,k} = u_{n+1,k} + \lambda(n) u_{n,k}, \quad \text{indefinite sum wrt } k$$

When summing over k, the left-hand side telescopes.

1. Dividing out by $u_{n,k}$ gives a recurrence for r:

$$\frac{n-k}{k+1} r(n,k+1) - r(n,k) = \frac{n+1}{n+1-k} + \lambda(n).$$

2. rational solution $r(n,k) = k/(k-n)$ (linear in λ).

The rational function $r(n,k)$ is the certificate: checking reduced to

$$r(n,k+1) - r(n,k) = \frac{u_{n+1,k}}{u_{n,k}} + \lambda(n) \in \mathbb{Q}(n,k).$$

Zeilberger's Algorithm

Input: a hypergeometric term $u_{n,k}$, i.e., $u_{n+1,k}/u_{n,k}$ and $u_{n,k+1}/u_{n,k}$ rational functions;
Output: a linear recurrence satisfied by $\sum_k u_{n,k}$ and a certificate.
For $m = 1, 2, 3, \ldots$

1. Set up the recurrence for $v_{n,k} = r(n, k)u_{n,k}$

$$v_{n,k+1} - v_{n,k} = u_{n+m,k} + \lambda_1(n)u_{n+m-1,k} + \cdots + \lambda_{m-1}(n)u_{n,k}$$

   with unknown r and λ_i;
2. Find rational solution r:
   1. locate poles (as in Abramov);
   2. equation for numerator;
   3. solve system linear in the coefficients and the λ_i's.
3. If a nonzero solution is found, break.

V Hypergeometric Solutions

Petkovšek's Algorithm Hyper

Problem ($a_0(n)u_{n+k} + \cdots + a_k(n)u_n = b(n)$)

a_i polynomials, $b = 0$ or (linear combination of) hypergeometric sequences, find all solutions that are linear combinations of hypergeometric sequences, or a proof that none exists.

Example
$$S(n) = \sum_{k=0}^{n} \binom{3k+1}{k}\binom{3n-3k}{n-k}.$$

1. Zeilberger's algorithm yields:

$$(n+2)(2n+3)(2n+5)S(n+2)+$$
$$(9n^2+27n+22)(2n+3)S(n+1)-81(n+1)(3n+2)(3n+4)S(n)=0.$$

2. Petkovšek's algorithm finds special solution:

$$T(n) = \binom{3n+1}{n}.$$

3. Conclusion $S(0) = T(0), S(1) = T(1) \Rightarrow S(n) = T(n)$.

Beyond Hypergeometric

$$u_n = (n-1)u_{n-1} + (n-1)u_{n-2} \qquad \text{(derangements)}$$

1. Hyper $\to n!$
2. Reduction of order: $u_n =: n!\sum^{n-1} v_k, \quad (n+2)v_{n+1} + v_n = 0.$
3. Conclusion: $u_n = n!\sum_{k=0}^{n}\frac{(-1)^k}{k!}$, and it is not hypergeometric.

Theorem (Petkovšek92)

If a LRE has a solution of the form

$$h_0(n)\sum_{k_1=s_1}^{n-1} h_1(k_1)\sum_{k_2=s_2}^{k_1-1} h_2(k_2)\cdots\sum_{k_m=s_m}^{k_{m-1}-1} h_m(k_m),$$

then it is found by applying Hyper and reduction of order.

Summary

1. Polynomial and rational solutions at the heart of everything;
2. complexity (still) exponential (size of result is);
3. indefinite and definite sums can be computed in hypergeometric case (rec. of order 1).

After lunch

1. Beyond order 1;
2. not only sums;
3. more variables.