D-finiteness: Algorithms and Applications

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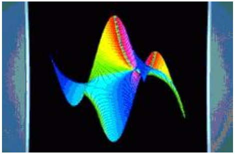


March 21, 2007

IV D-finiteness in Several Variables

Applications of Creative Telescoping

$$\sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 = \sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k} \sum_{j=0}^{k} \binom{k}{j}^3 \quad \text{[Strehl92]}$$

$$\int_0^{+\infty} x J_1(ax) I_1(ax) Y_0(x) K_0(x)\, dx = -\frac{\ln(1-a^4)}{2\pi a^2} \quad \text{[GlMo94]}$$

$$\frac{1}{2\pi i} \oint \frac{(1+2xy+4y^2)\exp\left(\frac{4x^2y^2}{1+4y^2}\right)}{y^{n+1}(1+4y^2)^{\frac{3}{2}}}\, dy = \frac{H_n(x)}{\lfloor n/2 \rfloor!} \quad \text{[Doetsch30]}$$

$$\sum_{k=0}^{n} \frac{q^{k^2}}{(q;q)_k (q;q)_{n-k}} = \sum_{k=-n}^{n} \frac{(-1)^k q^{(5k^2-k)/2}}{(q;q)_{n-k}(q;q)_{n+k}} \quad \text{[Andrews74]}$$

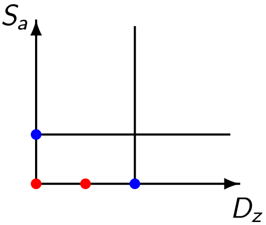
$$\sum_{j=0}^{n} \sum_{i=0}^{n-j} \frac{q^{(i+j)^2+j^2}}{(q;q)_{n-i-j}(q;q)_i(q;q)_j} = \sum_{k=-n}^{n} \frac{(-1)^k q^{7/2k^2+1/2k}}{(q;q)_{n+k}(q;q)_{n-k}} \quad \text{[Paule85]}$$

Example: Contiguity of Hypergeometric Series

$$F(a,b;c;z) = \sum_{n=0}^{\infty} \underbrace{\frac{(a)_n(b)_n}{(c)_n\, n!}}_{u_{a,n}} z^n, \qquad (x)_n := x(x+1)\cdots(x+n-1).$$

$$\frac{u_{a,n+1}}{u_{a,n}} = \frac{(a+n)(b+n)}{(c+n)(n+1)} \to \ z(1-z)F'' + (c-(a+b+1)z)F' - abF = 0,$$

$$\frac{u_{a+1,n}}{u_{a,n}} = \frac{n}{a} + 1 \to S_a \cdot F := F(a+1,b;c;z) = \frac{z}{a}F' + F.$$



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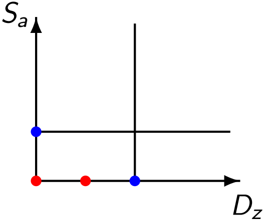
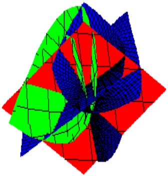


dim=2 $\Rightarrow S_a^2{\cdot}F, S_a{\cdot}F, F$ linearly dependent [Gauss1812]. Also:

- S_a^{-1} in terms of Id, D_z (step-down op.);
- relation between any three polynomials in $S_a, S_b, S_c, S_a^{-1}, S_b^{-1}, S_c^{-1}$;
  - generalizes to any ${}_pF_q$ and multivariate case [Takayama89].

0-dimensionality & D-finiteness



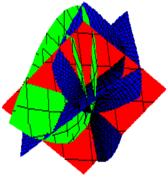
Polynomial algebra	Linear-operator algebra
0-dimensional ideal	0-dimensional left ideal
⇕	⇕

quotient is a finite-dimensional vector space

Exs: Orthogonal polynomials, hypergeometric series, their q-analogues, ...

system + ini. cond. = data structure

0-dimensionality & D-finiteness



Polynomial algebra	Linear-operator algebra
0-dimensional ideal	0-dimensional left ideal
$\Updownarrow$	$\Updownarrow$

quotient is a finite-dimensional vector space

$\Downarrow$	$\Downarrow$
polynomial expressions in sol. are algebraic	polynomials and ∂'s in sol. are D-finite

Tools: linear algebra, Gröbner bases. Implemented in `Mgfun` [Chyzak98]

Exs: Orthogonal polynomials, hypergeometric series, their q-analogues, ...

$$\boxed{\text{system} + \text{ini. cond.} = \text{data structure}}$$

Example: Jacobi's Orthogonal Polynomials

$$\mathbb{O} = \mathbb{Q}(n, x, a, b)\langle S_n, D_x \rangle, \qquad \text{Jacobi} = P_n^{(a,b)}(x)$$

Annihilating ideal Ann P generated by

$$G_1 = p_{11}(n, a, b)S_n^2 + p_{12}(n, x, a, b)S_n + p_{13}(n, a, b)\mathrm{Id},$$
$$G_2 = p_{21}(n, x, a, b)D_x + p_{22}(n, x, a, b)S_n + p_{23}(n, a, b)\mathrm{Id}, \quad (p_{ij} \text{ poly.})$$
$$G_3 = p_{31}(x, a, b)D_x^2 + p_{32}(x, a, b)D_x + p_{33}(n, a, b)\mathrm{Id}.$$

$\Rightarrow$ P is D-finite: $\dim \mathbb{O}/\operatorname{Ann} P = 2$.

$$G_4 = p_{41}(n, x, a, b)S_a + p_{42}(n)S_n + p_{43}(n, a)\mathrm{Id},$$
$$G_5 = p_{51}(n, x, a, b)S_b + p_{52}(n)S_n + p_{53}(n, b)\mathrm{Id}.$$

In $\mathbb{O}' = \mathbb{Q}(n, x, a, b)\langle S_n, D_x, S_a, S_b \rangle$, $\dim \mathbb{O}'/\operatorname{Ann} P = 2$ as well.

Creative Telescoping [Zeilberger 90]

$$F_n = \sum_{k=\omega_1}^{\omega_2} u_{n,k} = ?$$

IF ONE KNOWS $A(n, S_n)$ and $B(n, k, S_n, S_k)$ such that

$$\big(A(n, S_n) + \Delta_k B(n, k, S_n, S_k)\big) \cdot u_{n,k} = 0,$$

i.e.,

$$A(n, S_n) \cdot u_{n,k} + B(n, k+1, S_n, S_k) \cdot u_{n,k+1} - B(n, k, S_n, S_k) \cdot u_{n,k} = 0,$$

then the sum “telescopes,” leading to $A(n, S_n) \cdot F_n = 0$.

Creative Telescoping [Zeilberger 90]

$$I(x) = \int_\Omega u(x, y)\, dy = ?$$

IF ONE KNOWS $A(x, \partial_x)$ and $B(x, y, \partial_x, \partial_y)$ such that

$$\big(A(x, \partial_x) + \partial_y B(x, y, \partial_x, \partial_y)\big) \cdot u(x, y) = 0,$$

i.e.,

$$A(x, \partial_x) \cdot u(x, y) + \frac{\partial}{\partial y}\big(B(x, y, \partial_x, \partial_y) \cdot u(x, y)\big) = 0,$$

then the integral "telescopes," leading to $A(x, \partial_x) \cdot I(x) = 0.$

Creative Telescoping [Zeilberger 90]

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IF ONE KNOWS $A(x, \partial_x)$ and $B(x, y, \partial_x, \partial_y)$ such that

$$\big(A(x,\partial_x) + \partial_y B(x,y,\partial_x,\partial_y)\big)\cdot u(x,y) = 0,$$

then the integral “telescopes,” leading to $A(x, \partial_x)\cdot I(x) = 0$.

Then I come along and try differentating under the integral sign, and often it worked. So I got a great reputation for doing integrals.

Richard P. Feynman, 1985

Creative telescoping=“differentiation” under integral+“integration” by parts

Generating Series of the Jacobi Polynomials

$$\mathbb{O} = \mathbb{Q}(n, x, a, b, z)\langle S_n, D_x, S_a, S_b, D_z\rangle$$

1. Ann $P_n^{(a,b)}(x)$ generated by $G_1, \dots, G_5, D_z$ (dim 2);
2. Ann z^n generated by $S_n - z\,\mathrm{Id}, D_x, S_a - \mathrm{Id}, S_b - \mathrm{Id}, zD_z - n\,\mathrm{Id}$ (dim 1);
3. Closure by product (dim 2);
4. Creative telescoping:

$$q_{11}\mathrm{Id} + q_{12}D_z + q_{13}D_x, \quad q_{21}\mathrm{Id} + q_{22}D_z + q_{23}S_b,$$
$$q_{31}\mathrm{Id} + q_{32}D_z + q_{33}S_a, \quad q_{41}\mathrm{Id} + q_{42}D_z + q_{43}D_z^2 \qquad (\text{dim } 2).$$

5. (optional) Resolution:

$$\sum_{n=0}^{\infty} P_n^{(a,b)}(x)z^n = \frac{2^{a+b}}{R(x,z)\,(1 - z + R(x,z))^a\,(1 + z + R(x,z))^b},$$
$$R(x,z) = \frac{1}{\sqrt{1 - 2xz + z^2}}.$$

Extended Gosper Decision Algorithm [Chyzak2000]

Input: $t(k) \in V = \mathrm{vect}_{\mathbb{Q}(k)}\langle b_1(k), \dots, b_N(k)\rangle$, V closed under S_k
Output: $T(k) \in V$ such that $T(k+1) - T(k) = t(k)$ or '$\nexists T$'

1. let $T = \sum_{i=1}^{N} \phi_i b_i$ for undetermined coefficients $\phi_i \in \mathbb{Q}(k)$;
2. extract the coefficients of $\Delta_k \cdot T - t$ in the b_i's to obtain a first-order functional system in the ϕ_i's;
3. solve for rational solutions (Abramov's decision algorithms);
4. if solvable return T; otherwise return '$\nexists T$'.

Parametrized extension: $t(k) = \eta_1 b_1(k) + \cdots + \eta_N b_N(k)$ for unknown η_i's $\rightarrow$ returns η_i's as well.

Extended Zeilberger Fast Algorithm [Chyzak2000]

Input: $t(n,k) \in V = \mathrm{vect}_{\mathbb{Q}(n,k)}\langle b_1(n,k), \dots, b_N(n,k)\rangle$, V closed under S_n, S_k
Output: $P(n, S_n)$, $T(n,k) \in V$ s.t. $P{\cdot}t(n,k) = \Delta_k{\cdot}T(n,k)$

For $r = 0, 1, \dots$:

1. let $P = \sum_{i=0}^{r} \eta_i S_n^i$ and $t = P{\cdot}u$ for undetermined coefficients $\eta_i = \eta_i(n)$;
2. compute T and the η_i's by the Parametrized Extended Gosper algorithm;
3. if $T \neq$ '$\nexists T$' return (P, T).

Termination guaranteed only for so-called "holonomic" terms.

Neumann's Addition Theorem for Bessel Functions

$$J_0(z)^2 + 2\sum_{k=1}^{\infty} J_k(z)^2 = 1, \qquad J_k(z) := \left(\frac{z}{2}\right)^k \sum_{k=0}^{\infty} \frac{(-z^2/4)^n}{n!\,(n+k)!}.$$

1. Bessel J_k defined by

$$z^2 D_z^2 + zD_z + (z^2 - k^2)\,\mathrm{Id}, \quad S_k + D_z - (k/z)\,\mathrm{Id} \qquad (\text{dim } 2)$$

2. Square (dim 3) by closure algorithm (lin. alg.)

$$z^2 D_z^3 + 3zD_z^2 + (4z^2 - 4k^2 + 1)D_z + 4z\,\mathrm{Id}, \quad 2zS_k + zD_z^2 + (2k-1)D_z - z\,\mathrm{Id}\,.$$

3. Look for $P + (S_k - \mathrm{Id})Q$ with P free of k and of order 1 in D_z:

$$P = D_z, \qquad Q = \frac{1}{2}D_z + \frac{k}{z}\,\mathrm{Id}\,.$$

4. Conclusion:

$$D_z \cdot \sum_{k=-\infty}^{\infty} J_k^2 + \left[Q \cdot J_k^2\right]_{k=-\infty}^{\infty} = 0.$$

Applications of Creative Telescoping

Generating series: Multiplication by z^n using closure by $\times$, then definite summation.

Extraction of coefficients: $\times\, z^{-n-1}$, then Cauchy integral.

Diagonals: If $f(x,y) = \sum_{n,k} a_{n,k} x^n y^k$, its *diagonal*

$\sum_n a_{n,n} x^n = \frac{1}{2i\pi} \oint f(x/s, s) \frac{ds}{s}$. Generalizes to more variables.

Hadamard product: $f(\mathbf{x}) \odot g(\mathbf{x})$ is a diagonal of $f(\mathbf{x}) g(\mathbf{y})$.

Coefficients in Chebyshev or Neumann series:

$$\int_{-1}^{1} \frac{f(t) T_n(t)}{\sqrt{1-t^2}}\, dt, \quad \int_0^1 f(r) J_n(r) r\, dr.$$

But termination/success not guaranteed!

Operators & Commutation Rules

Operator type	Commutation
Differential	$Df = fD + (D \cdot f)\,\mathrm{Id}$

Leibniz Rule: $D \cdot (fg)(x) = f(x)(D \cdot g(x)) + (D \cdot f(x))g(x)$

Operators & Commutation Rules

Operator type	Commutation
Differential	$Df = fD + (D \cdot f)\,\mathrm{Id}$
Shift	$Sf = (S \cdot f)S$

Leibniz Rule: $S \cdot (fg)(x) = (S \cdot f(x))(S \cdot g(x))$

Operators & Commutation Rules

Operator type	Commutation
Differential	$Df = fD + (D{\cdot}f)\,\mathrm{Id}$
Shift	$Sf = (S{\cdot}f)S$
Difference	$\Delta f = (S{\cdot}f)\Delta + (\Delta{\cdot}f)\,\mathrm{Id}$

Leibniz Rule: $\Delta{\cdot}(fg)(x) = (S{\cdot}f(x))(\Delta{\cdot}g(x)) + (\Delta{\cdot}f(x))g(x)$

Operators & Commutation Rules

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Shift	$Sf = (S\cdot f)S$
Difference	$\Delta f = (S\cdot f)\Delta + (\Delta\cdot f)\,\mathrm{Id}$
q-shift	$Qf = (Q\cdot f)Q$

Leibniz Rule: $Q\cdot(fg)(x) = (Q\cdot f(x))(Q\cdot g(x))$ where $Q\cdot f(x) = f(qx)$

Operators & Commutation Rules

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Differential	$Df = fD + (D{\cdot}f)\,\mathrm{Id}$
Shift	$Sf = (S{\cdot}f)S$
Difference	$\Delta f = (S{\cdot}f)\Delta + (\Delta{\cdot}f)\,\mathrm{Id}$
q-shift	$Qf = (Q{\cdot}f)Q$
Mahler	$Mf = (M{\cdot}f)M$

Leibniz Rule: $M{\cdot}(fg)(x) = (M{\cdot}f(x))(M{\cdot}g(x))$ where $M{\cdot}f(x) = f(x^k)$

Operators & Commutation Rules

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Common commutation pattern: $\partial f = \sigma(f)\partial + \delta(f)$

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Natural condition: $\partial(fg) = (\partial f)g$, so that

$$\sigma(fg)\partial + \delta(fg) = (\sigma(f)\partial + \delta(f))g = \sigma(f)\sigma(g)\partial + \sigma(f)\delta(g) + \delta(f)g.$$

Skew Polynomial Rings

Definition (Skew polynomial ring)

$\mathbb{A}[\partial;\sigma,\delta]$: set of polynomials in ∂ with coefficients in the (non-commutative) cancellation ring $\mathbb{A}$, with commutation rule $\partial f = \sigma(f)\partial + \delta(f)$, where σ is an injective ring endomorphism of $\mathbb{A}$ and δ is a σ-derivation.

Definition (General univariate D-finiteness)

When $\mathbb{A}$ is a field, f is D-finite if there exists $P \in \mathbb{A}[\partial;\sigma,\delta]$ such that $P\cdot f = 0$.

Theorem (Ore, 1933)

When $\mathbb{A}$ is a field, Euclidean division and extended Euclidean algorithm yield greatest common right divisors (GCRD), least common left multiples (LCLM), Bézout identities.

Closure Properties of Univariate D-Finite Functions

Prop. Closure under $+$.

Proof. $P \cdot f = Q \cdot g = 0 \Rightarrow \mathrm{LCLM}(P, Q) \cdot (f + g) = 0$

Prop. Closure under $\times$: provided there exist polynomials A and B such that $\sigma = A(\partial)|_{\mathbb{A}}$ and $\delta = B(\partial)|_{\mathbb{A}}$, with, for all $w \in \mathbb{A}$, $((A - 1) \cdot w)B = (B \cdot w)(A - 1)$.

Proof. $\partial \cdot (f \otimes g) = (A \cdot f) \otimes (\partial \cdot g) + (B \cdot f) \otimes g$ + Linear algebra.

Ore Algebras & Their Ideals

Prop. $\mathbb{A}[\partial; \sigma, \delta]$ is a cancellation ring.

Def. $\mathbb{F}$ a field, $\mathbb{O} = \mathbb{F}[\partial_1; \sigma_1, \delta_1] \cdots [\partial_n; \sigma_n, \delta_n]$, such that $\partial_i \partial_j = \partial_j \partial_i$ for all i, j, is an Ore algebra.

Def. Left ideal: $\mathcal{I} \subset \mathbb{O}$ such that $\mathcal{I} + \mathcal{I} = \mathbb{O}\mathcal{I} = \mathcal{I}$.

Prop. $\operatorname{Ann} f$ is a left ideal; $\mathbb{O}/\operatorname{Ann} f \simeq \mathbb{O}\cdot f$. $\quad (\bar{1} \leftrightarrow f)$

Def. f is D-finite w.r.t. $\mathbb{O}$ if the quotient $\mathbb{O}/\operatorname{Ann} f$ is a *finite-dimensional* vector space over $\mathbb{F}$.

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Def. f is D-finite w.r.t. $\mathbb{O}$ if the quotient $\mathbb{O}/\operatorname{Ann} f$ is a *finite-dimensional* vector space over $\mathbb{F}$.

$\rightarrow$ Algorithms based on noncommutative Gröbner bases [ChSa98]:

- The leading monomial of a product is the product of the leading monomials (+ adjust coefficients).
- Buchberger's algorithm for Gröbner bases works in Ore algebras [Kredel93].

Properties of Multivariate D-Finite Functions [ChSa98]

Prop. 1 Closure under $+$.

Prop. 2 Closure under $\times$: if $\sigma_i = A_i(\partial_i)$ and $\delta_i = B_i(\partial_i)$, with $((A_i - 1)\cdot w)B_i = (B_i\cdot w)(A_i - 1)$, for all $w \in \mathbb{F}$ and all i.

Prop. 3 f D-finite w.r.t. $\mathbb{O}$, then $P\cdot f$ D-finite for any $P \in \mathbb{O}$.

Prop. 4 f D-finite w.r.t. $\mathbb{O}$, then for any $P \in \mathbb{O}$, f satisfies an equation $\sum_{i=0}^{k} a_i P^i\cdot f = 0$, with $k \leq \dim \mathbb{O}/\operatorname{Ann} f$.

Prop. 5 $f(\mathbf{x}, \mathbf{y})$ D-finite w.r.t. $\mathbb{K}(\mathbf{x}, \mathbf{y})[\boldsymbol{\partial}_\mathbf{x}; \boldsymbol{\sigma}_\mathbf{x}, \boldsymbol{\delta}_\mathbf{x}][\boldsymbol{\partial}_\mathbf{y}; \boldsymbol{\sigma}_\mathbf{y}, \boldsymbol{\delta}_\mathbf{y}]$, then for any $\mathbf{a} \in \mathbb{K}^m$ where the specialization is well-defined, $f(\mathbf{x}, \mathbf{a})$ is D-finite w.r.t. $\mathbb{K}(\mathbf{x})[\boldsymbol{\partial}_\mathbf{x}; \boldsymbol{\sigma}_\mathbf{x}, \boldsymbol{\delta}_\mathbf{x}]$.

Properties of Multivariate D-Finite Functions [ChSa98]

Prop. 1 Closure under $+$.
Proof $\mathbb{O}\cdot f \oplus \mathbb{O}\cdot g$ of finite dimension.

Prop. 2 Closure under $\times$: if $\sigma_i = A_i(\partial_i)$ and $\delta_i = B_i(\partial_i)$, with $((A_i - 1)\cdot w)B_i = (B_i\cdot w)(A_i - 1)$, for all $w \in \mathbb{F}$ and all i.
Proof $\mathbb{O}\cdot f \otimes \mathbb{O}\cdot g$ of finite dimension.

Prop. 3 f D-finite w.r.t. $\mathbb{O}$, then $P\cdot f$ D-finite for any $P \in \mathbb{O}$.
Proof $\boldsymbol{\partial}^{\mathbf{m}}\cdot(P\cdot f) \in \mathbb{O}\cdot f$ which is finite-dimensional.

Prop. 4 f D-finite w.r.t. $\mathbb{O}$, then for any $P \in \mathbb{O}$, f satisfies an equation $\sum_{i=0}^{k} a_i P^i\cdot f = 0$, with $k \leq \dim \mathbb{O}/\operatorname{Ann} f$.

Prop. 5 $f(\mathbf{x}, \mathbf{y})$ D-finite w.r.t. $\mathbb{K}(\mathbf{x}, \mathbf{y})[\boldsymbol{\partial}_{\mathbf{x}}; \boldsymbol{\sigma}_{\mathbf{x}}, \boldsymbol{\delta}_{\mathbf{x}}][\boldsymbol{\partial}_{\mathbf{y}}; \boldsymbol{\sigma}_{\mathbf{y}}, \boldsymbol{\delta}_{\mathbf{y}}]$, then for any $\mathbf{a} \in \mathbb{K}^m$ where the specialization is well-defined, $f(\mathbf{x}, \mathbf{a})$ is D-finite w.r.t. $\mathbb{K}(\mathbf{x})[\boldsymbol{\partial}_{\mathbf{x}}; \boldsymbol{\sigma}_{\mathbf{x}}, \boldsymbol{\delta}_{\mathbf{x}}]$.
Proof Use Prop. 4 for each of the $\partial_{\mathbf{x}}$ and specialize coefficients.

General Creative Telescoping

$$I(x) = \int_\Omega u(x,y)\,dy = ?$$

IF ONE KNOWS $A(x,\partial_x)$ and $B(x,y,\partial_x,\partial_y)$ such that

$$\big(A(x,\partial_x) + \partial_y B(x,y,\partial_x,\partial_y)\big)\cdot u(x,y) = 0,$$

then the integral "telescopes," leading to $A(x,\partial_x)\cdot I(x) = 0$.

Creative telescoping = "differentiation" under integral + "integration" by parts

- General case: Find annihilators of

$$I(x_1,\dots,x_{n-1}) = \partial_n^{-1}\big|_\Omega f(x_1,\dots,x_n)$$

  knowing generators of $\operatorname{Ann} f$ in $\mathbb{O}_n = \mathbb{K}(x_1,\dots,x_n)[\partial_1;\sigma_1,\delta_1]\cdots[\partial_n;\sigma_n,\delta_n]$;
- Crucial step: compute $(\mathbb{O}_n \operatorname{Ann} f + \partial_n \mathbb{O}_n) \cap \mathbb{O}_{n-1}$.

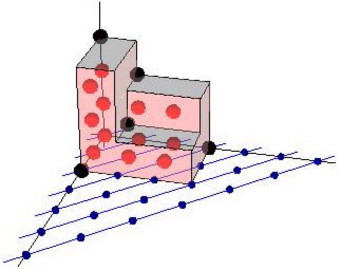
(Partial) Algorithms for Creative Telescoping

Aim: $\mathcal{I} = (\mathbb{O}_n \operatorname{Ann} f + \partial_n \mathbb{O}_n) \cap \mathbb{O}_{n-1}$

- By Gröbner bases, eliminate x_n and set ∂_n to 0 [ChSa98]
  $\to (\mathbb{O}_n \operatorname{Ann} f \cap \mathbb{O}_{n-1}[\partial_n] + \partial_n \mathbb{O}_{n-1}) \cap \mathbb{O}_{n-1} \subset \mathcal{I}$
- Differential case: algorithms from $\mathcal{D}$-module theory [SaStTa00,Tsai00], Gröbner bases with negative weights.
- Shift case, $n = 2$, dim 1 (= hypergeometric): [Zeilberger91]
  For increasing k, search for a_i and B

$$\mathbb{O}_{n-1} \ni \sum_{i=0}^{k} a_i \partial_{n-1}^i f = \partial_n B f$$

  Termination [Abramov03].



- Arbitrary n and $\mathbb{O}_n$: [Chyzak2000]

$$\mathbb{O}_{n-1} \ni \sum_{\lambda} a_\lambda \partial^\lambda \in \partial_n B + \operatorname{Ann} f$$

  B is given by rational solutions of a linear system in σ_n, δ_n.

Open Problems

Efficiency

- Faster Gröbner bases;
- Other elimination techniques (adapt geometric resolution [GiHe93,GiLeSa01] to Ore algebras);
- Structured Padé-Hermite approximants + bounds.

Understand non-minimality

- Remove apparent singularities by Ore closure, a generalization of Weyl closure [Tsai00], and of [AbBavH05] ([ChDuLeMaMiSa05] in progress);
- Exploit symmetry (explain [Paule94]).

Easy-to-use Implementations

- Improve `gfun` and `Mgfun`. Make the ESF interactive.