Euler's Pentagonal Number Theorem

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One of Euler's most profound discoveries, the *Pentagonal Number Theorem* [7], has been beautifully described by André Weil:

> Playing with series and products, he discovered a number of facts which to him looked quite isolated and very surprising. He looked at this infinite product
>
> $$(1-x)(1-x^2)(1-x^3)\cdots$$
>
> and just formally started expanding it. He had many products and series of that kind; in some cases he got something which showed a definite law, and in other cases things seemed to be rather random. But with this one, he was very successful. He calculated at least fifteen or twenty terms; the formula begins like this:
>
> $$\Pi(1-x^n)=1-x-x^2+x^5+x^7-x^{12}-x^{15}\cdots$$
>
> where the law, to your untrained eyes, may not be immediately apparent at first sight. In modern notation, it is as follows:
>
> $$\prod_{1}^{\infty}(1-q^n)=\sum_{-\infty}^{+\infty}(-1)^n q^{n(3n+1)/2} \tag{1}$$
>
> where I've changed x into q since q has become the standard notation in elliptic function-theory since Jacobi. The exponents make up a progression of a simple nature. This became immediately apparent to Euler after writing down some 20 terms; quite possibly he calculated about a hundred. He very reasonably says, "this is quite certain, although I cannot prove it;" ten years later he does prove it. He could not possibly guess that both series and product are part of the theory of elliptic modular functions. It is another tie-up between number-theory and elliptic functions [**22**, pp. 97–98].

G. Pólya [**16**, pp. 91–98] provides a more extensive account of Euler's wonderful discovery together with a translation of Euler's own description [**6**].

The numbers $n(3n-1)/2$ are called "pentagonal numbers" because of their relationship to pentagonal arrays of points. FIGURE 1 illustrates this. Legendre [**14**, pp. 131–133] observed that purely formal multiplication of the terms on the left side of (1) produces the term $\pm q^N$ precisely as often as N is representable as a sum of distinct positive integers; the "+" is taken when there is an even number of summands in the representation and the "−" when the number of summands is odd. For example,

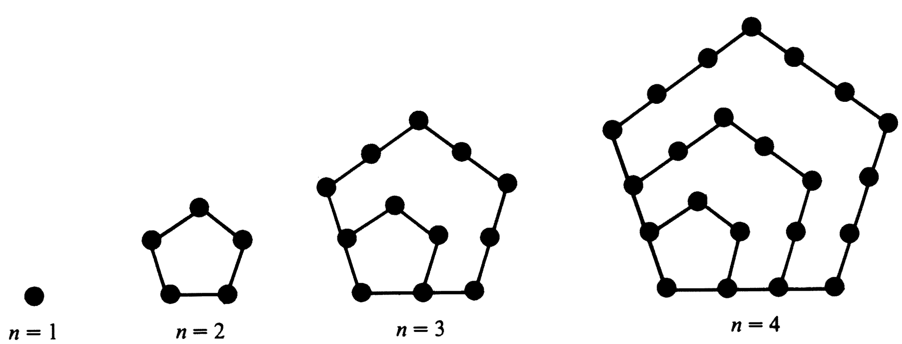

FIGURE 1. **The first four pentagonal numbers are 1, 5, 12, 22.**

$$(1-q)(1-q^2)(1-q^3)(1-q^4)\cdots$$
$$=1-q-q^2-q^3-q^4+q^{1+2}+q^{1+3}+q^{1+4}+q^{2+3}$$
$$+q^{2+4}+q^{3+4}-q^{1+2+3}-q^{1+2+4}-q^{1+3+4}-q^{2+3+4}+q^{1+2+3+4}+\cdots.$$

The term **partition** is usually used to describe the representation of a positive integer as the sum of positive integers. In this article, we are concerned with *unordered* partitions; two such partitions are considered the same if the terms in the sum are the same, e.g., $1+2$ and $2+1$ are considered as the same partition of 3. Thus Legendre's observation may be matched up with the actual infinite series expansion (1) as follows.

THEOREM. *Let $p_e(n)$ denote the number of partitions of n into an even number of distinct summands. Let $p_o(n)$ denote the number of partitions of n into an odd number of distinct summands. Then*

$$p_e(n)-p_o(n)=\begin{cases}(-1)^j & \text{if } n=j(3j\pm 1)/2\\ 0 & \text{otherwise.}\end{cases} \tag{2}$$

The impact of Euler's Pentagonal Number Theorem and Legendre's observations on subsequent developments in number theory is enormous. Both (1) and (2) are justly famous. Indeed, F. Franklin's purely arithmetic proof of (2) [**10**] (see also [**21**, pp. 261–263]) has been described by H. Rademacher as the first significant achievement of American mathematics. Franklin's proof is so elementary and lovely that it has been presented many times over in elementary algebra and number theory texts [**5**, pp. 563–564], [**11**], [**12**, pp. 206–207], [**13**, pp. 286–287], [**15**, pp. 221–222].

It is, however, interesting to note that Euler's proof of (1) alluded to by Weil remains almost unknown. In recent years only Rademacher's book has contained an exposition of it [**17**, pp. 224–226]. This book and earlier books [**4**, pp. 23–24] have presented it more or less as Euler did. While the idea behind Euler's proof is ingenious (as one would expect), the mathematical notation of Euler's day hides the fact that other results of significance are either transparent corollaries of Euler's proof or lie just below the surface. The remainder of this article is devoted to a long overdue modern exposition of Euler's proof and an examination of its consequences.

To begin, we define a function of two variables:

$$f(x,q)=1-\sum_{n=1}^{\infty}(1-xq)(1-xq^2)\cdots(1-xq^{n-1})x^{n+1}q^n. \tag{3}$$

(Absolute convergence of all series and products considered is ensured by $|q|<1$, $|x|<|q|^{-1}$). We first note that

$$f(1,q)=\prod_{n=1}^{\infty}(1-q^n). \tag{4}$$

This is because we may easily establish the identity

$$1-\sum_{n=1}^{N}(1-q)(1-q^2)\cdots(1-q^{n-1})q^n=\prod_{n=1}^{N}(1-q^n) \tag{5}$$

by mathematical induction on N (a nice exercise for the reader). Thus (4) is the limiting case of (5) as $N\to\infty$.

The main step in Euler's proof is essentially the verification of the following functional equation:

$$f(x,q)=1-x^2q-x^3q^2f(xq,q). \tag{6}$$

Actually Euler does equation (6) over and over, first with $x=1$, then $x=q$, then $x=q^2$, and so on [7, pp. 473–475]; page 473 of his paper in *Opera Omnia* is shown on the next page. This repetition of special cases of (6) tends to hide what exactly is happening. To prove (6), we take the defining equation (3) through a sequence of algebraic manipulations:

$$\begin{aligned}
f(x,q) &= 1-x^2q-\sum_{n=2}^{\infty}(1-xq)(1-xq^2)\cdots(1-xq^{n-1})x^{n+1}q^n\\
&= 1-x^2q-\sum_{n=1}^{\infty}(1-xq)(1-xq^2)\cdots(1-xq^{n})x^{n+2}q^{n+1}\\
&= 1-x^2q-\sum_{n=1}^{\infty}(1-xq^2)\cdots(1-xq^{n})x^{n+2}q^{n+1}(1-xq)\\
&= 1-x^2q-\sum_{n=1}^{\infty}(1-xq^2)\cdots(1-xq^{n})x^{n+2}q^{n+1}\\
&\quad+\sum_{n=1}^{\infty}(1-xq^2)\cdots(1-xq^{n})x^{n+3}q^{n+2}\\
&= 1-x^2q-x^3q^2-\sum_{n=2}^{\infty}(1-xq^2)\cdots(1-xq^{n})x^{n+2}q^{n+1}\\
&\quad+\sum_{n=1}^{\infty}(1-xq^2)\cdots(1-xq^{n})x^{n+3}q^{n+2}\\
&= 1-x^2q-x^3q^2-\sum_{n=1}^{\infty}(1-xq^2)\cdots(1-xq^{n+1})x^{n+3}q^{n+2}\\
&\quad+\sum_{n=1}^{\infty}(1-xq^2)\cdots(1-xq^{n})x^{n+3}q^{n+2}\\
&= 1-x^2q-x^3q^2-\sum_{n=1}^{\infty}(1-xq^2)\cdots(1-xq^{n})x^{n+3}q^{n+2}\{(1-xq^{n+1})-1\}\\
&= 1-x^2q-x^3q^2\left\{1-\sum_{n=1}^{\infty}(1-xq^2)\cdots(1-xq^{n})x^{n+1}q^{2n+1}\right\}\\
&= 1-x^2q-x^3q^2f(xq,q).
\end{aligned}$$

If you followed the above sequence of steps carefully, you see how at each stage things seem to fit together magically at just the right moment. Also you can appreciate the complication of repeated presentation of the same steps first with $x=1$, then $x=q$, then $x=q^2$, etc. However, the empirical discovery of (6) by Euler must have come precisely in this repetitive manner.

The rest of Euler's proof is now almost mechanical. Equation (6) is repeatedly iterated; thus

$$\begin{aligned}
f(x,q) &= 1-x^2q-x^3q^2(1-x^2q^3-x^3q^5f(xq^2,q))\\
&= 1-x^2q-x^3q^2+x^5q^5+x^6q^7(1-x^2q^5-x^3q^8f(xq^3,q))\\
&\ \ \vdots\\
&= 1+\sum_{n=1}^{N-1}(-1)^n\left(x^{3n-1}q^{n(3n-1)/2}+x^{3n}q^{n(3n+1)/2}\right)\\
&\quad+(-1)^Nx^{3N-1}q^{N(3N-1)/2}+(-1)^Nx^{3N}q^{N(3N+1)/2}f(xq^N,q),
\end{aligned}\tag{7}$$

47—48] EVOLUTIO PRODUCTI INFINITI $(1-x)(1-xx)(1-x^3)(1-x^4)$ etc. 473

2. Quoniam hi termini omnes factorem habent communem $1-x$, eo evoluto singuli termini discerpentur in binas partes, quas ita repraesentemus

$$\begin{aligned} A = {} & xx + x^3(1-xx) + x^4(1-xx)(1-x^3) + x^5(1-xx)(1-x^3)(1-x^4) + \text{etc.} \\ & - x^3 - x^4(1-xx) - x^5(1-xx)(1-x^3) - x^6(1-xx)(1-x^3)(1-x^4) - \text{etc.} \end{aligned}$$

Hinc iam binae partes eadem potestate ipsius x affectae in unam contrahantur ac resultabit pro A sequens forma

$$A = xx - x^5 - x^7(1-xx) - x^9(1-xx)(1-x^3) - x^{11}(1-x^2)(1-x^3)(1-x^4) - \text{etc.},$$

ubi duo termini primi $xx - x^5$ iam sunt evoluti; sequentes autem procedunt per has potestates $x^7, x^9, x^{11}, x^{13}, x^{15}$, quarum exponentes binario crescunt.

3. Ponamus nunc simili modo ut ante

$$A = xx - x^5 - B,$$

ita ut sit

$$B = x^7(1-xx) + x^9(1-xx)(1-x^3) + x^{11}(1-xx)(1-x^3)(1-x^4) + \text{etc.},$$

cuius omnes termini habent factorem communem $1-xx$, quo evoluto singuli termini in binas partes discerpantur, uti sequitur,

$$\begin{aligned} B = {} & x^7 + x^9(1-x^3) + x^{11}(1-x^3)(1-x^4) + x^{13}(1-x^3)(1-x^4)(1-x^5) + \text{etc.} \\ & - x^9 - x^{11}(1-x^3) - x^{13}(1-x^3)(1-x^4) - x^{15}(1-x^3)(1-x^4)(1-x^5) - \text{etc.} \end{aligned}$$

Hic iterum bini termini, qui eandem potestatem ipsius x habent praefixam, in unam colligantur et prodibit

$$B = x^7 - x^{12} - x^{15}(1-x^3) - x^{18}(1-x^3)(1-x^4) - x^{21}(1-x^3)(1-x^4)(1-x^5) - \text{etc.}$$

ubi iam potestates ipsius x crescunt ternario.

4. Statuatur nunc porro

$$B = x^7 - x^{12} - C,$$

ita ut sit

$$C = x^{15}(1-x^3) + x^{18}(1-x^3)(1-x^4) + x^{21}(1-x^3)(1-x^4)(1-x^5) + \text{etc.},$$

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another fine exercise in mathematical induction on N. In the limit as $N \to \infty$ we find

$$f(x,q) = 1 + \sum_{n=1}^{\infty} (-1)^n \left(x^{3n-1} q^{n(3n-1)/2} + x^{3n} q^{n(3n+1)/2} \right). \tag{8}$$

Therefore by (4) and (8),

$$\begin{aligned} \prod_{n=1}^{\infty} (1-q^n) = f(1,q) &= 1 + \sum_{n=1}^{\infty} (-1)^n \left(q^{n(3n-1)/2} + q^{n(3n+1)/2} \right) \\ &= \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n+1)/2}, \end{aligned}$$

which completes the proof of (1).

It should be noted that several authors (L. J. Rogers [**18**, pp. 334–335], G. W. Starcher [**19**], and N. J. Fine [**9**]) also found formula (8) essentially in the way we have; however, none has noted that he was, in fact, rediscovering Euler's proof in simpler clothing. M. V. Subbarao [**20**] has also shown the connection between (8) and Franklin's arithmetic proof [**10**].

Now the reader may naturally ask: Was anything gained in this general formulation of Euler's proof besides simplicity of presentation? Is any new information available that was hidden before? We can answer a resounding YES just by setting $x=-1$ in (3) and (8). This substitution gives the equation

$$\begin{aligned}1+\sum_{n=1}^{\infty}(1+q)(1+q^2)\cdots(1+q^{n-1})(-1)^n q^n \\ =f(-1,q)=1+\sum_{n=1}^{\infty}\left(-q^{n(3n-1)/2}+q^{n(3n+1)/2}\right).\end{aligned} \tag{9}$$

Equation (9) yields a corollary as appealing as Legendre's Theorem. It implies that *the product*

$$(1+q)(1+q^2)\cdots(1+q^{n-1})q^n$$

when multiplied out produces the term q^N exactly as often as N can be partitioned into distinct summands with largest part equal to n. For example, when $n=4$,

$$\begin{aligned}(1+q)(1+q^2)(1+q^3)q^4 = q^4+q^{4+1}+q^{4+2}+q^{4+3}+q^{4+2+1} \\ +q^{4+3+1}+q^{4+3+2}+q^{4+3+2+1}\end{aligned}$$

Hence the series on the left side of (9) when expanded out yields the term $\pm q^N$ for each partition of N into distinct summands; the "+" occurs if the largest summand is even, and the "−" occurs if the largest summand is odd. In the same manner that (1) yielded Legendre's Theorem, we see that (9) yields the following equally elegant but little publicized result found by N. J. Fine [**8**] more than 118 years after Legendre's observation.

THEOREM. *Let $\pi_e(n)$ denote the number of partitions of n with distinct summands the largest of which is even. Let $\pi_o(n)$ denote the number of partitions of n with distinct summands the largest of which is odd. Then*

$$\pi_e(n)-\pi_o(n)=\begin{cases}1 & \text{if } n=j(3j+1)/2,\\ -1 & \text{if } n=j(3j-1)/2,\\ 0 & \text{otherwise}.\end{cases}$$

Let us check out the theorems with an example. The partitions of $n=12$ into distinct parts are: 12, 11+1, 10+2, 9+3, 9+2+1, 8+4, 8+3+1, 7+5, 7+4+1, 7+3+2, 6+5+1, 6+4+2, 6+3+2+1, 5+4+3, 5+4+2+1. The partitions enumerated by each of $p_e(12)$, $p_o(12)$, $\pi_e(12)$ and $\pi_o(12)$ are listed in the following table:

$p_e(12)$	$p_o(12)$	$\pi_e(12)$	$\pi_o(12)$
11 + 1	12	12	11 + 1
10 + 2	9 + 2 + 1	10 + 2	9 + 3
9 + 3	8 + 3 + 1	8 + 4	9 + 2 + 1
8 + 4	7 + 4 + 1	8 + 3 + 1	7 + 5
7 + 5	7 + 3 + 2	6 + 5 + 1	7 + 4 + 1
6 + 3 + 2 + 1	6 + 5 + 1	6 + 4 + 2	7 + 3 + 2
5 + 4 + 2 + 1	6 + 4 + 2	6 + 3 + 2 + 1	5 + 4 + 3
	5 + 4 + 3		5 + 4 + 2 + 1.

Thus $p_e(12)-p_o(12)=7-8=-1$ and $\pi_e(12)-\pi_o(12)=7-8=-1$, as predicted by our theorems.

Beyond this immediate pleasant discovery that Euler's approach, properly modernized, yields Fine's Theorem, we may ask: Are there interesting extensions of Euler's method that yield more than equation (8)? Here again the answer is positive. L. J. Rogers [**18**, p. 334], apparently unaware of how Euler's proof worked, showed in almost precisely the way we proved (8) that

$$1+\sum_{n=1}^{\infty}\frac{(1-a)(1-aq)\cdots(1-aq^{n-1})t^n}{(1-b)(1-bq)\cdots(1-bq^{n-1})}=\frac{1-at}{1-t}+$$

$$\sum_{n=1}^{\infty}\frac{(1-a)(1-aq)\cdots(1-aq^{n-1})\left(1-\frac{atq}{b}\right)\left(1-\frac{atq^2}{b}\right)\cdots\left(1-\frac{atq^n}{b}\right)b^nt^nq^{n^2-n}(1-atq^{2n})}{(1-b)(1-bq)\cdots(1-bq^n)(1-t)(1-tq)\cdots(1-tq^{n+1})}. \tag{10}$$

Rogers in fact showed that if $f(a,b,t)$ denotes the left side of (10), then

$$f(a,b,t)=\frac{1-at}{1-t}+t\frac{(1-a)(b-atq)}{(1-b)(1-t)}f(aq,bq,tq). \tag{11}$$

This result and deeper extensions of it that require much more than Euler's method have had a major impact in the theory of partitions [**1**, Secs. 3 and 4], [**2**, Ch. 7], [**3**, Ch. 3]. N. J. Fine [**9**] also independently rediscovered (10).

Surely the story unfolded here emphasizes how valuable it is to study and understand the central ideas behind major pieces of mathematics produced by giants like Euler. The discoveries of theorems as appealing as the two we have described would not be separated by 118 years if students of additive number theory had followed this advice.

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