VI

A MORE GENERAL STATEMENT

He [Euler] preferred instructing his pupils to the little satisfaction of amazing them. He would have thought not to have done enough for science if he should have failed to add to the discoveries, with which he enriched science, the candid exposition of the ideas that led him to those discoveries.
—CONDORCET

1. Euler. Of all mathematicians with whose work I am somewhat acquainted, Euler seems to be by far the most important for our inquiry. A master of inductive research in mathematics, he made important discoveries (on infinite series, in the Theory of Numbers, and in other branches of mathematics) by induction, that is, by observation, daring guess, and shrewd verification. In this respect, however, Euler is not unique; other mathematicians, great and small, used induction extensively in their work.

Yet Euler seems to me almost unique in one respect: he takes pains to present the relevant inductive evidence carefully, in detail, in good order. He presents it convincingly but honestly, as a genuine scientist should do. His presentation is "the candid exposition of the ideas that led him to those discoveries" and has a distinctive charm. Naturally enough, as any other author, he tries to impress his readers, but, as a really good author, he tries to impress his readers only by such things as have genuinely impressed himself.

The next section brings a sample of Euler's writing. The memoir chosen can be read with very little previous knowledge and is entirely devoted to the exposition of an inductive argument.

2. Euler's memoir is given here, in English translation, *in extenso*, except for a few unessential alterations which should make it more accessible to a modern reader.[1]

[1] The original is in French; see Euler's *Opera Omnia*, ser. 1, vol. 2, p. 241–253. The alterations consist in a different notation (footnote 2), in the arrangement of a table (explained in footnote 3), in slight changes affecting a few formulas, and in dropping a repetition of former arguments in the last No. 13 of the memoir. The reader may consult the easily available original.

DISCOVERY OF A MOST EXTRAORDINARY LAW OF THE NUMBERS CONCERNING THE SUM OF THEIR DIVISORS

1. Till now the mathematicians tried in vain to discover some order in the sequence of the prime numbers and we have every reason to believe that there is some mystery which the human mind shall never penetrate. To convince oneself, one has only to glance at the tables of the primes, which some people took the trouble to compute beyond a hundred thousand, and one perceives that there is no order and no rule. This is so much more surprising as the arithmetic gives us definite rules with the help of which we can continue the sequence of the primes as far as we please, without noticing, however, the least trace of order. I am myself certainly far from this goal, but I just happened to discover an extremely strange law governing the sums of the divisors of the integers which, at the first glance, appear just as irregular as the sequence of the primes, and which, in a certain sense, comprise even the latter. This law, which I shall explain in a moment, is, in my opinion, so much more remarkable as it is of such a nature that we can be assured of its truth without giving it a perfect demonstration. Nevertheless, I shall present such evidence for it as might be regarded as almost equivalent to a rigorous demonstration.

2. A prime number has no divisors except unity and itself, and this distinguishes the primes from the other numbers. Thus 7 is a prime, for it is divisible only by 1 and itself. Any other number which has, besides unity and itself, further divisors, is called composite, as for instance, the number 15, which has, besides 1 and 15, the divisors 3 and 5. Therefore, generally, if the number p is prime, it will be divisible only by 1 and p; but if p was composite, it would have, besides 1 and p, further divisors. Therefore, in the first case, the sum of its divisors will be $1 + p$, but in the latter it would exceed $1 + p$. As I shall have to consider the sum of divisors of various numbers, I shall use[2] the sign $\sigma(n)$ to denote the sum of the divisors of the number n. Thus, $\sigma(12)$ means the sum of all the divisors of 12, which are 1, 2, 3, 4, 6, and 12; therefore, $\sigma(12) = 28$. In the same way, one can see that $\sigma(60) = 168$ and $\sigma(100) = 217$. Yet, since unity is only divisible by itself, $\sigma(1) = 1$. Now, 0 (zero) is divisible by all numbers. Therefore, $\sigma(0)$ should be properly infinite. (However, I shall assign to it later a finite value, different in different cases, and this will turn out serviceable.)

3. Having defined the meaning of the symbol $\sigma(n)$, as above, we see clearly that if p is a prime $\sigma(p) = 1 + p$. Yet $\sigma(1) = 1$ (and not $1 + 1$); hence we see that 1 should be excluded from the sequence of the primes; 1 is the beginning of the integers, neither prime nor composite. If, however, n is composite, $\sigma(n)$ is greater than $1 + n$.

[2] Euler was the first to introduce a symbol for the sum of the divisors; he used $\int n$, not the modern $\sigma(n)$ of the text.

In this case we can easily find $\sigma(n)$ from the factors of n. If $a, b, c, d, \ldots$ are different primes, we see easily that

$$\sigma(ab) = 1 + a + b + ab = (1 + a)(1 + b) = \sigma(a)\sigma(b),$$
$$\sigma(abc) = (1 + a)(1 + b)(1 + c) = \sigma(a)\sigma(b)\sigma(c),$$
$$\sigma(abcd) = \sigma(a)\sigma(b)\sigma(c)\sigma(d)$$

and so on. We need particular rules for the powers of primes, as

$$\sigma(a^2) = 1 + a + a^2 = \frac{a^3 - 1}{a - 1}$$
$$\sigma(a^3) = 1 + a + a^2 + a^3 = \frac{a^4 - 1}{a - 1}$$

and, generally,

$$\sigma(a^n) = \frac{a^{n+1} - 1}{a - 1}.$$

Using this, we can find the sum of the divisors of any number, composite in any way whatever. This we see from the formulas

$$\sigma(a^2b) = \sigma(a^2)\sigma(b)$$
$$\sigma(a^3b^2) = \sigma(a^3)\sigma(b^2)$$
$$\sigma(a^3b^4c) = \sigma(a^3)\sigma(b^4)\sigma(c)$$

and, generally,

$$\sigma(a^\alpha b^\beta c^\gamma d^\delta e^\varepsilon) = \sigma(a^\alpha)\sigma(b^\beta)\sigma(c^\gamma)\sigma(d^\delta)\sigma(e^\varepsilon).$$

For instance, to find $\sigma(360)$ we set, since 360 factorized is $2^3 \cdot 3^2 \cdot 5$,

$$\sigma(360) = \sigma(2^3)\sigma(3^2)\sigma(5) = 15 \cdot 13 \cdot 6 = 1170.$$

4. In order to show the sequence of the sums of the divisors, I add the following table[3] containing the sums of the divisors of all integers from 1 up to 99.

n	0	1	2	3	4	5	6	7	8	9
0	—	1	**3**	**4**	7	**6**	12	**8**	15	13
10	18	**12**	28	**14**	24	24	31	**18**	39	**20**
20	42	32	36	**24**	60	31	42	40	56	**30**
30	72	**32**	63	48	54	48	91	**38**	60	56
40	90	**42**	96	**44**	84	78	72	**48**	124	57
50	93	72	98	**54**	120	72	120	80	90	**60**
60	168	**62**	96	104	127	84	144	**68**	126	96
70	144	**72**	195	**74**	114	124	140	96	168	**80**
80	186	121	126	**84**	224	108	132	120	180	**90**
90	234	112	168	128	144	120	252	**98**	171	156

[3] The number in the intersection of the row marked 60 and the column marked 7, that is, 68, is $\sigma(67)$. If p is prime, $\sigma(p)$ is in heavy print. This arrangement of the table is a little more concise than the arrangement in the original.

If we examine a little the sequence of these numbers, we are almost driven to despair. We cannot hope to discover the least order. The irregularity of the primes is so deeply involved in it that we must think it impossible to disentangle any law governing this sequence, unless we know the law governing the sequence of the primes itself. It could appear even that the sequence before us is still more mysterious than the sequence of the primes.

5. Nevertheless, I observed that this sequence is subject to a completely definite law and could even be regarded as a *recurring* sequence. This mathematical expression means that each term can be computed from the foregoing terms, according to an invariable rule. In fact, if we let $\sigma(n)$ denote any term of this sequence, and $\sigma(n-1)$, $\sigma(n-2)$, $\sigma(n-3)$, $\sigma(n-4)$, $\sigma(n-5)$, . . . the preceding terms, I say that the value of $\sigma(n)$ can always be combined from some of the preceding as prescribed by the following formula:

$$\begin{aligned}\sigma(n) = {} & \sigma(n-1) + \sigma(n-2) - \sigma(n-5) - \sigma(n-7) \\ & + \sigma(n-12) + \sigma(n-15) - \sigma(n-22) - \sigma(n-26) \\ & + \sigma(n-35) + \sigma(n-40) - \sigma(n-51) - \sigma(n-57) \\ & + \sigma(n-70) + \sigma(n-77) - \sigma(n-92) - \sigma(n-100) \\ & + \cdots.\end{aligned}$$

On this formula we must make the following remarks.

I. In the sequence of the signs + and —, each arises twice in succession.

II. The law of the numbers 1, 2, 5, 7, 12, 15, . . . which we have to subtract from the proposed number n, will become clear if we take their differences:

Nrs.	1,	2,	5,	7,	12,	15,	22,	26,	35,	40,	51,	57,	70,	77,	92,	100, . . .
Diff.	**1**,	3,	**2**,	5,	**3**,	7,	**4**,	9,	**5**,	11,	**6**,	13,	**7**,	15,	**8**, . . .	

In fact, we have here, alternately, all the integers 1, 2, 3, 4, 5, 6, . . . and the odd numbers 3, 5, 7, 9, 11, . . . , and hence we can continue the sequence of these numbers as far as we please.

III. Although this sequence goes to infinity, we must take, in each case, only those terms for which the numbers under the sign σ are still positive and omit the σ for negative values.

IV. If the sign $\sigma(0)$ turns up in the formula, we must, as its value in itself is indeterminate, substitute for $\sigma(0)$ the number n proposed.

6. After these remarks it is not difficult to apply the formula to any given particular case, and so anybody can satisfy himself of its truth by as many examples as he may wish to develop. And since I must admit that I am not in a position to give it a rigorous demonstration, I will justify it by a sufficiently large number of examples.

$\sigma(1) = \sigma(0)$	$= 1$	$= 1$
$\sigma(2) = \sigma(1) + \sigma(0)$	$= 1 + 2$	$= 3$
$\sigma(3) = \sigma(2) + \sigma(1)$	$= 3 + 1$	$= 4$
$\sigma(4) = \sigma(3) + \sigma(2)$	$= 4 + 3$	$= 7$
$\sigma(5) = \sigma(4) + \sigma(3) - \sigma(0)$	$= 7 + 4 - 5$	$= 6$
$\sigma(6) = \sigma(5) + \sigma(4) - \sigma(1)$	$= 6 + 7 - 1$	$= 12$
$\sigma(7) = \sigma(6) + \sigma(5) - \sigma(2) - \sigma(0)$	$= 12 + 6 - 3 - 7$	$= 8$
$\sigma(8) = \sigma(7) + \sigma(6) - \sigma(3) - \sigma(1)$	$= 8 + 12 - 4 - 1$	$= 15$
$\sigma(9) = \sigma(8) + \sigma(7) - \sigma(4) - \sigma(2)$	$= 15 + 8 - 7 - 3$	$= 13$
$\sigma(10) = \sigma(9) + \sigma(8) - \sigma(5) - \sigma(3)$	$= 13 + 15 - 6 - 4$	$= 18$
$\sigma(11) = \sigma(10) + \sigma(9) - \sigma(6) - \sigma(4)$	$= 18 + 13 - 12 - 7$	$= 12$
$\sigma(12) = \sigma(11) + \sigma(10) - \sigma(7) - \sigma(5) + \sigma(0)$	$= 12 + 18 - 8 - 6 + 12$	$= 28$
$\sigma(13) = \sigma(12) + \sigma(11) - \sigma(8) - \sigma(6) + \sigma(1)$	$= 28 + 12 - 15 - 12 + 1$	$= 14$
$\sigma(14) = \sigma(13) + \sigma(12) - \sigma(9) - \sigma(7) + \sigma(2)$	$= 14 + 28 - 13 - 8 + 3$	$= 24$
$\sigma(15) = \sigma(14) + \sigma(13) - \sigma(10) - \sigma(8) + \sigma(3) + \sigma(0)$	$= 24 + 14 - 18 - 15 + 4 + 15$	$= 24$
$\sigma(16) = \sigma(15) + \sigma(14) - \sigma(11) - \sigma(9) + \sigma(4) + \sigma(1)$	$= 24 + 24 - 12 - 13 + 7 + 1$	$= 31$
$\sigma(17) = \sigma(16) + \sigma(15) - \sigma(12) - \sigma(10) + \sigma(5) + \sigma(2)$	$= 31 + 24 - 28 - 18 + 6 + 3$	$= 18$
$\sigma(18) = \sigma(17) + \sigma(16) - \sigma(13) - \sigma(11) + \sigma(6) + \sigma(3)$	$= 18 + 31 - 14 - 12 + 12 + 4$	$= 39$
$\sigma(19) = \sigma(18) + \sigma(17) - \sigma(14) - \sigma(12) + \sigma(7) + \sigma(4)$	$= 39 + 18 - 24 - 28 + 8 + 7$	$= 20$
$\sigma(20) = \sigma(19) + \sigma(18) - \sigma(15) - \sigma(13) + \sigma(8) + \sigma(5)$	$= 20 + 39 - 24 - 14 + 15 + 6$	$= 42$

I think these examples are sufficient to discourage anyone from imagining that it is by mere chance that my rule is in agreement with the truth.

7. Yet somebody could still doubt whether the law of the numbers 1, 2, 5, 7, 12, 15, . . . which we have to subtract is precisely that one which I have indicated, since the examples given imply only the first six of these numbers. Thus, the law could still appear as insufficiently established and, therefore, I will give some examples with larger numbers.

I. Given the number 101, find the sum of its divisors. We have

$$\begin{aligned}\sigma(101) &= \sigma(100) + \sigma(99) - \sigma(96) - \sigma(94)\\ &+ \sigma(89) + \sigma(86) - \sigma(79) - \sigma(75)\\ &+ \sigma(66) + \sigma(61) - \sigma(50) - \sigma(44)\\ &+ \sigma(31) + \sigma(24) - \sigma(9) - \sigma(1)\\ &= 217 + 156 - 252 - 144\\ &+ 90 + 132 - 80 - 124\\ &+ 144 + 62 - 93 - 84\\ &+ 32 + 60 - 13 - 1\\ &= 893 - 791\\ &= 102\end{aligned}$$

and hence we could conclude, if we would not have known it before, that 101 is a prime number.

II. Given the number 301, find the sum of its divisors. We have

$$\begin{array}{rccccccc} \text{diff.} & 1 & & 3 & & 2 & & 5 \\ \sigma(301) = & \sigma(300) & + & \sigma(299) & - & \sigma(296) & - & \sigma(294) \; + \\ & 3 & & 7 & & 4 & & 9 \\ + & \sigma(289) & + & \sigma(286) & - & \sigma(279) & - & \sigma(275) \; + \\ & 5 & & 11 & & 6 & & 13 \\ + & \sigma(266) & + & \sigma(261) & - & \sigma(250) & - & \sigma(244) \; + \\ & 7 & & 15 & & 8 & & 17 \\ + & \sigma(231) & + & \sigma(224) & - & \sigma(209) & - & \sigma(201) \; + \\ & 9 & & 19 & & 10 & & 21 \\ + & \sigma(184) & + & \sigma(175) & - & \sigma(156) & - & \sigma(146) \; + \\ & 11 & & 23 & & 12 & & 25 \\ + & \sigma(125) & + & \sigma(114) & - & \sigma(91) & - & \sigma(79) \; + \\ & 13 & & 27 & & 14 & & \\ + & \sigma(54) & + & \sigma(41) & - & \sigma(14) & - & \sigma(0). \end{array}$$

We see by this example how we can, using the differences, continue the formula as far as is necessary in each case. Performing the computations, we find

$$\sigma(301) = 4939 - 4587 = 352.$$

We see hence that 301 is not a prime. In fact, $301 = 7 \cdot 43$ and we obtain

$$\sigma(301) = \sigma(7)\sigma(43) = 8 \cdot 44 = 352$$

as the rule has shown.

8. The examples that I have just developed will undoubtedly dispel any qualms which we might have had about the truth of my formula. Now, this beautiful property of the numbers is so much more surprising as we do not perceive any intelligible connection between the structure of my formula and the nature of the divisors with the sum of which we are here concerned. The sequence of the numbers 1, 2, 5, 7, 12, 15, . . . does not seem to have any relation to the matter in hand. Moreover, as the law of these numbers is "interrupted" and they are in fact a mixture of two sequences with a regular law, of 1, 5, 12, 22, 35, 51, . . . and 2, 7, 15, 26, 40, 57, . . . , we would not expect that such an irregularity can turn up in Analysis. The lack of demonstration must increase the surprise still more, since it seems wholly impossible to succeed in discovering such a property without being guided by some reliable method which could take the place of a perfect proof. I confess that I did not hit on this discovery by mere chance, but another proposition opened the path to this beautiful property—another proposition of the same nature which must be accepted as true although I am unable to

prove it. And although we consider here the nature of integers to which the Infinitesimal Calculus does not seem to apply, nevertheless I reached my conclusion by differentiations and other devices. I wish that somebody would find a shorter and more natural way, in which the consideration of the path that I followed might be of some help, perhaps.

9. In considering the partitions of numbers, I examined, a long time ago, the expression

$$(1-x)(1-x^2)(1-x^3)(1-x^4)(1-x^5)(1-x^6)(1-x^7)(1-x^8)\ldots,$$

in which the product is assumed to be infinite. In order to see what kind of series will result, I multiplied actually a great number of factors and found

$$1-x-x^2+x^5+x^7-x^{12}-x^{15}+x^{22}+x^{26}-x^{35}-x^{40}+\ldots.$$

The exponents of x are the same which enter into the above formula; also the signs + and − arise twice in succession. It suffices to undertake this multiplication and to continue it as far as it is deemed proper to become convinced of the truth of this series. Yet I have no other evidence for this, except a long induction which I have carried out so far that I cannot in any way doubt the law governing the formation of these terms and their exponents. I have long searched in vain for a rigorous demonstration of the equation between the series and the above infinite product $(1-x)(1-x^2)(1-x^3)\ldots$, and I have proposed the same question to some of my friends with whose ability in these matters I am familiar, but all have agreed with me on the truth of this transformation of the product into a series, without being able to unearth any clue of a demonstration. Thus, it will be a known truth, but not yet demonstrated, that if we put

$$s=(1-x)(1-x^2)(1-x^3)(1-x^4)(1-x^5)(1-x^6)\ldots$$

the same quantity s can also be expressed as follows:

$$s=1-x-x^2+x^5+x^7-x^{12}-x^{15}+x^{22}+x^{26}-x^{35}-x^{40}+\ldots.$$

For each of us can convince himself of this truth by performing the multiplication as far as he may wish; and it seems impossible that the law which has been discovered to hold for 20 terms, for example, would not be observed in the terms that follow.

10. As we have thus discovered that those two infinite expressions are equal even though it has not been possible to demonstrate their equality, all the conclusions which may be deduced from it will be of the same nature, that is, true but not demonstrated. Or, if one of these conclusions could be demonstrated, one could reciprocally obtain a clue to the demonstration of that equation; and it was with this purpose in mind that I maneuvered those two expressions in many ways, and so I was led among other discoveries to that which I explained above; its truth, therefore, must be as certain as

that of the equation between the two infinite expressions. I proceeded as follows. Being given that the two expressions

I. $s=(1-x)(1-x^2)(1-x^3)(1-x^4)(1-x^5)(1-x^6)(1-x^7)\ldots$

II. $s=1-x-x^2+x^5+x^7-x^{12}-x^{15}+x^{22}+x^{26}-x^{35}-x^{40}+\ldots$

are equal, I got rid of the factors in the first by taking logarithms

$$\log s=\log(1-x)+\log(1-x^2)+\log(1-x^3)+\log(1-x^4)+\ldots.$$

In order to get rid of the logarithms, I differentiate and obtain the equation

$$\frac{1}{s}\frac{ds}{dx}=-\frac{1}{1-x}-\frac{2x}{1-x^2}-\frac{3x^2}{1-x^3}-\frac{4x^3}{1-x^4}-\frac{5x^4}{1-x^5}-\ldots$$

or

$$-\frac{x}{s}\frac{ds}{dx}=\frac{x}{1-x}+\frac{2x^2}{1-x^2}+\frac{3x^3}{1-x^3}+\frac{4x^4}{1-x^4}+\frac{5x^5}{1-x^5}+\ldots.$$

From the second expression for s, as infinite series, we obtain another value for the same quantity

$$-\frac{x}{s}\frac{ds}{dx}=\frac{x+2x^2-5x^5-7x^7+12x^{12}+15x^{15}-22x^{22}-26x^{26}+\ldots}{1-x-x^2+x^5+x^7-x^{12}-x^{15}+x^{22}+x^{26}-\ldots}.$$

11. Let us put

$$-\frac{x}{s}\frac{ds}{dx}=t.$$

We have above two expressions for the quantity t. In the first expression, I expand each term into a geometric series and obtain

$$\begin{aligned} t = x + x^2 &+ x^3 + x^4 + x^5 + x^6 + x^7 + x^8 + \ldots \\ &+ 2x^2 \quad + 2x^4 \quad + 2x^6 \quad + 2x^8 + \ldots \\ &+ 3x^3 \quad + 3x^6 \quad + \ldots \\ &+ 4x^4 \quad + 4x^8 + \ldots \\ &+ 5x^5 \quad + \ldots \\ &+ 6x^6 \quad + \ldots \\ &+ 7x^7 \quad + \ldots \\ &+ 8x^8 + \ldots. \end{aligned}$$

Here we see easily that each power of x arises as many times as its exponent has divisors, and that each divisor arises as a coefficient of the same power of x. Therefore, if we collect the terms with like powers, the coefficient of each power of x will be the sum of the divisors of its exponent. And, therefore, using the above notation $\sigma(n)$ for the sum of the divisors of n, I obtain

$$t=\sigma(1)x+\sigma(2)x^2+\sigma(3)x^3+\sigma(4)x^4+\sigma(5)x^5+\ldots.$$

The law of the series is manifest. And, although it might appear that some induction was involved in the determination of the coefficients, we can easily satisfy ourselves that this law is a necessary consequence.

12. By virtue of the definition of t, the last formula of No. 10 can be written as follows:

$$t(1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + x^{22} + x^{26} - \dots)$$
$$- x - 2x^2 + 5x^5 + 7x^7 - 12x^{12} - 15x^{15} + 22x^{22} + 26x^{26} - \dots = 0.$$

Substituting for t the value obtained at the end of No. 11, we find

$$\begin{aligned} 0 = \sigma(1)x + \sigma(2)x^2 + \sigma(3)x^3 + \sigma(4)x^4 + \sigma(5)x^5 + \sigma(6)x^6 + \dots \\ - x - \sigma(1)x^2 - \sigma(2)x^3 - \sigma(3)x^4 - \sigma(4)x^5 - \sigma(5)x^6 - \dots \\ - 2x^2 - \sigma(1)x^3 - \sigma(2)x^4 - \sigma(3)x^5 - \sigma(4)x^6 - \dots \\ + 5x^5 + \sigma(1)x^6 + \dots . \end{aligned}$$

Collecting the terms, we find the coefficient for any given power of x. This coefficient consists of several terms. First comes the sum of the divisors of the exponent of x, and then sums of divisors of some preceding numbers, obtained from that exponent by subtracting successively 1, 2, 5, 7, 12, 15, 22, 26, ... Finally, if it belongs to this sequence, the exponent itself arises. We need not explain again the signs assigned to the terms just listed. Therefore, generally, the coefficient of x^n is

$$\sigma(n) - \sigma(n-1) - \sigma(n-2) + \sigma(n-5) + \sigma(n-7) - \sigma(n-12)$$
$$- \sigma(n-15) + \dots .$$

This is continued as long as the numbers under the sign σ are not negative. Yet, if the term $\sigma(0)$ arises, we must substitute n for it.

13. Since the sum of the infinite series considered in the foregoing No. 12 is 0, whatever the value of x may be, the coefficient of each single power of x must necessarily be 0. Hence we obtain the law that I explained above in No. 5; I mean the law that governs the sum of the divisors and enables us to compute it recursively for all numbers. In the foregoing development, we may perceive some reason for the signs, some reason for the sequence of the numbers

$$1, 2, 5, 7, 12, 15, 22, 26, 35, 40, 51, 57, 70, 77, \dots$$

and, especially, a reason why we should substitute for $\sigma(0)$ the number n itself, which could have appeared the strangest feature of my rule. This reasoning, although still very far from a perfect demonstration, will certainly lift some doubts about the most extraordinary law that I explained here.

3. Transition to a more general viewpoint. Euler's foregoing text is extraordinarily instructive. We can learn from it a great deal about mathematics, or the psychology of invention, or inductive reasoning. The examples and comments at the end of this chapter provide for opportunity to examine some of Euler's mathematical ideas, but now we wish to concentrate on his inductive argument.

The theorem investigated by Euler is remarkable in several respects and is of great mathematical interest even today. However, we are concerned here not so much with the mathematical content of this theorem, but rather with the reasons which induced Euler to believe in the theorem when it was still unproved. In order to understand better the nature of these reasons, I shall ignore the mathematical content of Euler's memoir and give a schematic outline of it, emphasizing a certain general aspect of his inductive argument.

As we shall disregard the mathematical content of the various theorems that we must discuss, we shall find it advantageous to designate them by letters, as T, T^*, C_1, C_2, . . . , C_1^*, C_2^*, ... The reader may ignore the meaning of these letters completely. Yet, in case he wishes to recognize them in Euler's text, here is the key.

T is the theorem

$$(1-x)(1-x^2)(1-x^3)\dots = 1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + \dots .$$

The law of the numbers 1, 2, 5, 7, 12, 15, . . . is explained in sect. 2, No. 5, II.

C_n is the assertion that the coefficient of x^n is the same on both sides of the foregoing equation. For example, C_6 asserts that expanding the product on the left hand side, we shall find that the coefficient of x^6 is 0. Observe that C_n is a consequence of the theorem T.

C_n^* is the equation

$$\sigma(n) = \sigma(n-1) + \sigma(n-2) - \sigma(n-5) - \sigma(n-7) + \dots$$

explained at length in sect. 2, No. 5. For example, C_6^* asserts that

$$\sigma(6) = \sigma(5) + \sigma(4) - \sigma(1).$$

T^* is the "most extraordinary law," asserting that $C_1^*, C_2^*, C_3^*, \dots$ are all true. Observe that C_n^* is a consequence (a particular case) of the theorem T^*.

4. Schematic outline of Euler's memoir.[4] *Theorem* T is of such a nature that we can be assured of its truth without giving it a perfect demonstration. Nevertheless, I shall present such evidence for it as might be regarded as almost equivalent to a rigorous demonstration.

Theorem T *includes an infinite number of particular cases:* C_1, C_2, C_3, ... *Conversely, the infinite set of these particular cases* C_1, C_2, C_3, . . . *is equivalent to theorem* T. *We can find out by a simple calculation whether* C_1 *is true or not.*

[4] This outline was first published in my paper, "Heuristic Reasoning and the Theory of Probability," *Amer. Math. Monthly*, vol. 48, 1941, p. 450–465. The italics indicate phrases which are not due to Euler.

Another simple calculation determines whether C_2 is true or not, and similarly for C_3, and so on. I have made these calculations and I find that $C_1, C_2, C_3, \ldots, C_{40}$ are all true. It suffices to undertake *these calculations* and to continue *them* as far as is deemed proper to become convinced of the truth of this sequence *continued indefinitely.* Yet I have no other evidence for this, except a long induction which I have carried out so far that I cannot in any way doubt the law of which $C_1, C_2, \ldots$ *are the particular cases.* I have long searched in vain for a rigorous demonstration of *theorem T*, and I have proposed the same question to some of my friends with whose ability in these matters I am familiar, but all have agreed with me on the truth of *theorem T* without being able to unearth any clue of a demonstration. Thus it will be a known truth, but not yet demonstrated; for each of us can convince himself of this truth by the *actual calculation of the cases* $C_1, C_2, C_3, \ldots$ as far as he may wish; and it seems impossible that the law which has been discovered to hold for 20 terms, for example, would not be observed in the terms that follow.

As we have thus discovered the truth of *theorem T* even though it has not been possible to demonstrate it, all the conclusions which may be deduced from it will be of the same nature, that is, true but not demonstrated. Or, if one of these conclusions could be demonstrated, one could reciprocally obtain a clue to the demonstration of *theorem T*; and it was with this purpose in mind that I maneuvered *theorem T* in many ways and so discovered among others *theorem T^** whose truth must be as certain as that of *theorem T*.

Theorems T and T^* are equivalent; they are both true or false; they stand or fall together. Like T, theorem T^* includes an infinity of particular cases $C_1^*, C_2^*, C_3^*, \ldots$, and this sequence of particular cases is equivalent to theorem T^*. Here again, a simple calculation shows whether C_1^* is true or not. Similarly, it is possible to determine whether C_2^* is true or not, and so on. It is not difficult to apply theorem T^* to any given particular case, and so anybody can satisfy himself of its truth by as many examples as he may wish to develop. And since I must admit that I am not in a position to give it a rigorous demonstration, I will justify it by a sufficiently large number of examples, by $C_1^*, C_2^*, \ldots, C_{20}^*$. I think these examples are sufficient to discourage anyone from imagining that it is by mere chance that my rule is in agreement with the truth.

If one still doubts that the law is precisely that one which I have indicated, I will give some examples with larger numbers. *By examination, I find that C_{101}^* and C_{301}^* are true, and so I find that theorem T^* is valid even for these cases which are far removed from those which I examined earlier.* These examples which I have just developed undoubtedly will dispel any qualms which we might have had about the truth of *theorems T and T^*.*

EXAMPLES AND COMMENTS ON CHAPTER VI

In discovering his "Most Extraordinary Law of the Numbers" Euler "reached his conclusion by differentiations and other devices" although "the Infinitesimal Calculus does not seem to apply to the nature of integers." In order to understand Euler's method, we apply it to similar examples. We begin by giving a name to his principal "device" or mathematical tool.

1. *Generating functions.* We restate the result of No. 11 of Euler's memoir in modern notation:

$$\sum_{n=1}^{\infty} \frac{nx^n}{1 - x^n} = \sigma(1)x + \sigma(2)x^2 + \ldots + \sigma(n)x^n + \ldots .$$

The right hand side is a power series in x. The coefficient of x^n in this power series is $\sigma(n)$, the sum of the divisors of n. Both sides of the equation represent the same function of x. The expansion of this function in powers of x "generates" the sequence $\sigma(1), \sigma(2), \ldots \sigma(n), \ldots$ and so we call this function the *generating function* of $\sigma(n)$. Generally, if

$$f(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n + \ldots$$

we say that $f(x)$ is the generating function of a_n, or the function generating the sequence $a_0, a_1, a_2, \ldots a_n, \ldots$.

The name "generating function" is due to Laplace. Yet, without giving it a name, Euler used the device of generating functions long before Laplace, in several memoirs of which we have seen one in sect. 2. He applied this mathematical tool to several problems in Combinatory Analysis and the Theory of Numbers.

A generating function is a device somewhat similar to a bag. Instead of carrying many little objects detachedly, which could be embarrassing, we put them all in a bag, and then we have only one object to carry, the bag. Quite similarly, instead of handling each term of the sequence $a_0, a_1, a_2, \ldots a_n, \ldots$ individually, we put them all in a power series Σa_nx^n, and then we have only one mathematical object to handle, the power series.

2. Find the generating function of n. Or, what is the same, find the sum of the series Σnx^n.

3. Being given that $f(x)$ generates the sequence $a_0, a_1, a_2, \ldots a_n, \ldots$ find the function generating the sequence

$$0a_0, 1a_1, 2a_2, \ldots na_n, \ldots .$$

4. Being given that $f(x)$ generates the sequence $a_0, a_1, a_2, \ldots a_n, \ldots$, find the function generating the sequence

$$0, a_0, a_1, \ldots a_{n-1}, \ldots .$$

5. Being given that $f(x)$ is the generating function of a_n, find the generating function of

$$s_n = a_0 + a_1 + a_2 + \ldots + a_n.$$

6. Being given that $f(x)$ and $g(x)$ are the generating functions of a_n and b_n, respectively, find the generating function of

$$c_n = a_0 b_n + a_1 b_{n-1} + a_2 b_{n-2} + \cdots + a_n b_0.$$

7. *A combinatorial problem in plane geometry.* A convex polygon with n sides is dissected into $n - 2$ triangles by $n - 3$ diagonals; see fig. 6.1. Call D_n the number of different dissections.

Find D_n for $n = 3, 4, 5, 6$.

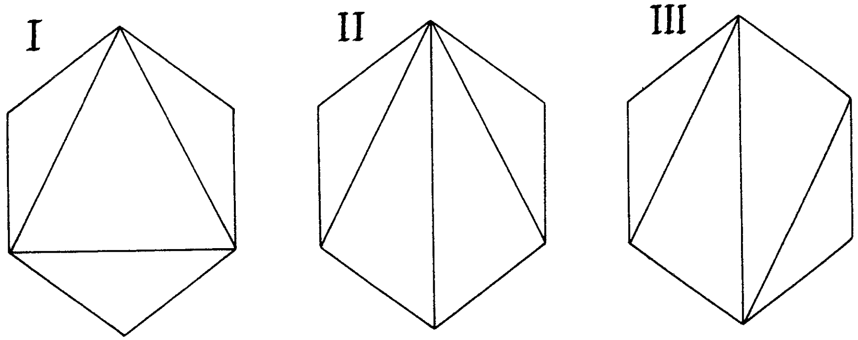

Fig. 6.1. Three types of dissection for a hexagon.

8 (continued). It is not easy to guess a general, explicit expression for D_n on the basis of the numerical values considered in ex. 7. Yet the sequence $D_3, D_4, D_5, \ldots$ is a "recurring" sequence in the following, very general sense: each term can be computed from the foregoing terms according to an invariable rule, a "recursion formula." (See Euler's memoir, No. 5.)

Define $D_2 = 1$, and show that for $n \geqq 3$

$$D_n = D_2 D_{n-1} + D_3 D_{n-2} + D_4 D_{n-3} + \cdots + D_{n-1} D_2.$$

[Check the first cases. Refer to fig. 6.2.]

9 (continued). The derivation of an explicit expression for D_n from the recursion formula of ex. 8 is not obvious. Yet consider the generating function

$$g(x) = D_2 x^2 + D_3 x^3 + D_4 x^4 + \cdots + D_n x^n + \cdots.$$

Show that $g(x)$ satisfies a quadratic equation and derive hence that for $n = 3, 4, 5, 6, \ldots$

$$D_n = \frac{2}{2}\,\frac{6}{3}\,\frac{10}{4}\,\frac{14}{5} \cdots \frac{4n-10}{n-1}.$$

10. *Sums of squares.* Recall the definition of $R_k(n)$ (ex. 4.1), extend it to $n = 0$ in setting $R_k(0) = 1$ (a reasonable extension), introduce the generating function

$$\sum_{n=0}^{\infty} R_k(n) x^n = R_k(0) + R_k(1)x + R_k(2)x^2 + \cdots,$$

and show that

$$\sum_{n=0}^{\infty} R_3(n) x^n = (1 + 2x + 2x^4 + 2x^9 + \cdots)^3.$$

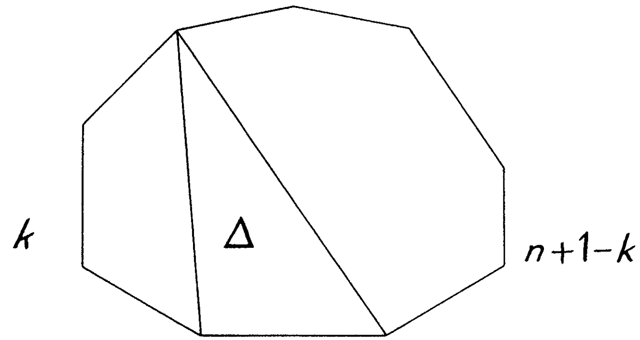

Fig. 6.2. Starting the dissection of a polygon with n sides.

[What is $R_3(n)$? The number of solutions of the equation

$$u^2 + v^2 + w^2 = n$$

in integers u, v, and w, positive, negative, or 0.

What may be the rôle of the series on the right-hand side of the equation that you are required to prove?

$$1 + 2x + 2x^4 + 2x^9 + \cdots = \sum_{u=-\infty}^{\infty} x^{u^2} = \sum_{n=0}^{\infty} R_1(n) x^n.$$

How should you conceive the right-hand side of the equation you are aiming at? Perhaps so:

$$\Sigma x^{u^2} \cdot \Sigma x^{u^2} \cdot \Sigma x^{u^2}.]$$

11. Generalize the result of ex. 10.

12. Recall the definition of $S_k(n)$ (ex. 4.1) and express the generating function

$$\sum_{n=1}^{\infty} S_k(n) x^n.$$

13. Use ex. 11 to prove that, for $n \geqq 1$, $R_2(n)$ is divisible by 4, $R_4(n)$ by 8, and $R_8(n)$ by 16. (The result was already used in ch. IV, Tables II and III.)

14. Use ex. 12 to prove that

$$S_2(n) = 0 \text{ if } n \text{ is not of the form } 8m + 2,$$
$$S_4(n) = 0 \text{ if } n \text{ is not of the form } 8m + 4,$$
$$S_8(n) = 0 \text{ if } n \text{ is not of the form } 8m.$$

15. Use ex. 11 to prove that

$$R_{k+l}(n) = R_k(0)R_l(n) + R_k(1)R_l(n-1) + \ldots + R_k(n)R_l(0).$$

16. Prove that

$$S_{k+l}(n) = S_k(1)S_l(n-1) + S_k(2)S_l(n-2) + \ldots + S_k(n-1)S_l(1).$$

17. Propose a simple method for computing Table III of ch. IV from the Tables I and II of the same chapter.

18. Let $\sigma_k(n)$ stand for the sum of the kth powers of the divisors of n. For example,

$$\sigma_3(15) = 1^3 + 3^3 + 5^3 + 15^3 = 3528;$$

$\sigma_1(n) = \sigma(n)$.

(1) Show that the conjectures found in sect. 4.6 and ex. 4.23 imply

$$\sigma(1)\sigma(2u-1) + \sigma(3)\sigma(2u-3) + \ldots + \sigma(2u-1)\sigma(1) = \sigma_3(u)$$

where u denotes an odd integer.

(2) Test particular cases of the relation found in (1) numerically.

(3) How does such a verification influence your confidence in the conjectures from which the relation verified has been derived?

19. *Another recursion formula.* We consider the generating functions

$$G = \sum_{m=1}^{\infty} S_1(m)x^m, \qquad H = \sum_{m=1}^{\infty} S_4(m)x^m.$$

We set

$$S_4(4u) = s_u$$

where u is an odd integer. Then

$$G = x + x^9 + x^{25} + \ldots + x^{(2n-1)^4} + \ldots,$$
$$H = s_1x^4 + s_3x^{12} + s_5x^{20} + \ldots + s_{2n-1}x^{8n-4} + \ldots,$$
$$G^4 = H$$

by ex. 14 and 12. We derive from the last equation, by taking the logarithms and differentiating, that

$$4 \log G = \log H,$$
$$\frac{4G'}{G} = \frac{H'}{H},$$
$$G \cdot xH' = 4 \cdot xG' \cdot H,$$

$$(x + x^9 + x^{25} + \ldots)(4s_1x^4 + 12s_3x^{12} + 20s_5x^{20} + \ldots)$$
$$= 4(x + 9x^9 + 25x^{25} + \ldots)(s_1x^4 + s_3x^{12} + s_5x^{20} + \ldots).$$

Comparing the coefficients of x^5, x^{13}, x^{21}, . . . on both sides of the foregoing equation, we find, after some elementary work, the following relations:

$$\begin{array}{l}
0s_1 = 0\\
1s_3 - 4s_1 = 0\\
2s_5 - 3s_3 = 0\\
3s_7 - 2s_5 - 12s_1 = 0\\
4s_9 - 1s_7 - 11s_3 = 0\\
5s_{11} \quad - 10s_5 = 0\\
6s_{13} + 1s_{11} - 9s_7 - 24s_1 = 0\\
7s_{15} + 2s_{13} - 8s_9 - 23s_3 = 0\\
8s_{17} + 3s_{15} - 7s_{11} - 22s_5 = 0\\
9s_{19} + 4s_{17} - 6s_{13} - 21s_7 = 0\\
10s_{21} + 5s_{19} - 5s_{15} - 20s_9 - 40s_1 = 0\\
11s_{23} + 6s_{21} - 4s_{17} - 19s_{11} - 39s_3 = 0\\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{array}$$

The very first equation of this system is vacuous and is displayed here only to emphasize the general law. Yet we know that $s_1 = 1$. Knowing this, we obtain from the next equation s_3. Knowing s_3, we obtain from the following equation s_5. And so on, we can compute from the system the terms of the sequence s_1, s_3, s_5, . . . as far as we wish, one after the other, *recurrently.*

The system has a remarkable structure. There is 1 equation containing 1 of the quantities s_1, s_3, s_5, . . . , 2 equations containing 2 of them, 3 equations containing 3 of them, and so on. The coefficients in each column are increased by 1 and the subscripts by 2 as we pass from one row to the next. The subscript at the head of each column is 1 and the coefficient is -4 multiplied by the first coefficient in the same row.

We can concentrate the whole system in one equation (recursion formula); write it down.

20. *Another Most Extraordinary Law of the Numbers Concerning the Sum of their Divisors.* If the conjecture of sect. 4.6 stands

$$s_{2n-1} = S_4(4(2n-1)) = \sigma(2n-1)$$

and so ex. 19 yields a recursion formula connecting the terms of the sequence $\sigma(1)$, $\sigma(3)$, $\sigma(5)$, $\sigma(7)$, . . . which is in many ways strikingly similar to Euler's formula.

Write out in detail and verify numerically the first cases of the indicated recursion formula.

21. For us there is also a heuristic similarity between Euler's recursion formula for $\sigma(n)$ (sect. 2) and the foregoing recursion formula for $\sigma(2n-1)$ (ex. 20). For us this latter is a conjecture. We derived this conjecture, as Euler has derived his, "by differentiation and other devices" from another conjecture.

Show that the recursion formula for $\sigma(2n-1)$ indicated by ex. 20 is equivalent to the equation

$$S_4(4(2n-1)) = \sigma(2n-1)$$

to which we arrived in sect. 4.6. That is, if one of the two assertions is true, the other is necessarily also true.

22. Generalize ex. 19.

23. Devise a method for computing $R_8(n)$ independently of $R_4(n)$.

24. *How Euler missed a discovery.* The method illustrated by ex. 19 and ex. 23, and generally stated in ex. 22, is due to Euler.[5] In inventing his method, Euler aimed at the problem of four squares and some related problems. In fact, he applied his method to the problem of four squares and investigated inductively the number of representations, but failed to discover the remarkable law governing $R_4(n)$, which is after all not so difficult to discover inductively (ex. 4.10–4.15). How did it happen?

In examining the equation

$$n = x^2 + y^2 + z^2 + w^2$$

we may choose various standpoints, especially the following:

(1) We admit for x, y, z, and w only non-negative integers.

(2) We admit for x, y, z, and w all integers, positive, negative, and null.

The second standpoint may be less obvious, but leads to $R_4(n)$ and to the remarkable connection between $R_4(n)$ and the divisors of n. The first standpoint is more obvious, but the number of solutions does not seem to have any simple remarkable property. Euler chose the standpoint (1), not the standpoint (2), he applied his method explained in ex. 22 to

$$(1 + x + x^4 + x^9 + \dots)^4,$$

not to

$$(1 + 2x + 2x^4 + 2x^9 + \dots)^4,$$

and so he bypassed a great discovery. It is instructive to compare two lines of inquiry which look so much alike at the outset, but one of which is wonderfully fruitful and the other almost completely barren.

[5] *Opera Omnia*, ser. 1, vol. 4, p. 125–135.

The properties of $R_4(n)$, $S_4(n)$, $R_8(n)$, and $S_8(n)$ investigated in ch. IV (ex. 4.10–4.15, sect. 4.3–4.6, ex. 4.18–4.23) have been discovered by Jacobi, not inductively, but as incidental corollaries of his researches on elliptic functions. Several proofs of these theorems have been found since, but no known proof is quite elementary and straightforward.[6]

25. *A generalization of Euler's theorem on $\sigma(n)$.* Given k, set

$$\prod_{n=1}^{\infty} (1 - x^n)^k = 1 - \sum_{n=1}^{\infty} a_n x^n$$

and show that, for $n = 1, 2, 3, \dots$

$$\sigma(n) = \sum_{m=1}^{n-1} a_m \sigma(n-m) + n a_n / k.$$

Which particular case yields Euler's theorem of sect. 2?

[6] See also for further references G. H. Hardy and E. M. Wright, *An introduction to the theory of numbers*, Oxford, 1938, chapter XX.

confidence we would have scarcely found the courage to undertake the proof which did not look at all a routine job. "When you have satisfied yourself that the theorem is true, you start proving it"—the traditional mathematics professor is quite right.

Third, the examples in which the familiar limit formula for e popped up again and again, gave us reasonable ground for introducing that limit formula into the statement of the theorem. And introducing it turned out the crucial step toward the solution.

On the whole, it seems natural and reasonable that the inductive phase precedes the demonstrative phase. First guess, then prove.

EXAMPLES AND COMMENTS ON CHAPTER V.

1. By multiplying the series

$$(1-x^2)^{-1/2} = 1 + \frac{1}{2}x^2 + \frac{1}{2}\frac{3}{4}x^4 + \dots$$

$$\arcsin x = \frac{x}{1} + \frac{1}{2}\frac{x^3}{3} + \frac{1}{2}\frac{3}{4}\frac{x^5}{5} + \dots$$

you find the first terms of the expansion

$$y = (1-x^2)^{-1/2}\arcsin x = x + \frac{2}{3}x^3 + \dots .$$

(a) Compute a few more terms and try to guess the general term.

(b) Show that y satisfies the differential equation

$$(1-x^2)y' - xy = 1$$

and use this equation to prove your guess.

2. By multiplying the series

$$e^{x^2/2} = 1 + \frac{x^2}{2} + \frac{x^4}{2\cdot 4} + \dots$$

$$\int_0^x e^{-t^2/2}\,dt = \frac{x}{1} - \frac{1}{2}\frac{x^3}{3} + \frac{1}{2\cdot 4}\frac{x^5}{5} - \dots$$

you find the first terms of the expansion

$$y = e^{x^2/2}\int_0^x e^{-t^2/2}\,dt = x + \frac{1}{3}x^3 + \dots .$$

(a) Compute a few more terms and try to guess the general term.

(b) Your guess, if correct, suggests that y satisfies a simple differential equation. By establishing this equation, prove your guess.

3. The functional equation

$$f(x) = \frac{1}{1+x} f\left(\frac{2\sqrt{x}}{1+x}\right)$$

is satisfied by the power series

$$f(x) = 1 + \frac{1}{4}x^2 + \frac{9}{64}x^4 + \frac{25}{256}x^6 + \frac{1225}{16384}x^8 + \dots .$$

Verify these coefficients, derive a few more, if necessary, and try to guess the general term.

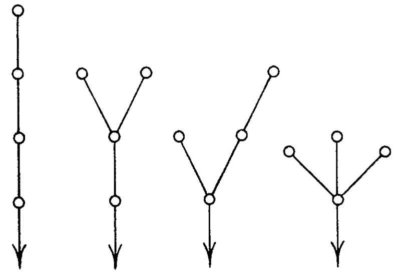

Fig. 5.1. Compounds C_4H_9OH.

4. The functional equation

$$f(x) = 1 + \frac{x}{6}\left[f(x)^3 + 3f(x)f(x^2) + 2f(x^3)\right]$$

is satisfied by the power series

$$f(x) = 1 + x + x^2 + 2x^3 + 4x^4 + \dots + a_n x^n + \dots .$$

It is asserted that a_n is the number of the structurally different chemical compounds (aliphatic alcohols) having the same chemical formula $C_nH_{2n+1}OH$. In the case $n = 4$, the answer is true. There are $a_4 = 4$ alcohols C_4H_9OH; they are represented in fig. 5.1, each compound as a "tree," each C as a little circle or "knot," and the radical —OH as an arrow; the H's are dropped. Test other values of n.

5. $$\sum_{k=0}^{n/2} (-1)^k \binom{n-k}{k} = 1, 1, 0, -1, -1, 0$$

according as $n \equiv 0, 1, 2, 3, 4, 5 \pmod{6}$.

6. An ellipse describes a prolate, or an oblate, spheroid according as it rotates about its major, or minor, axis.

For the area of the surface of the prolate spheroid

$$E = 2\pi ab[(1-\varepsilon^2)^{1/2} + (\arcsin \varepsilon)/\varepsilon], \quad P = 4\pi(a^2 + 2b^2)/3$$