39. Prove Euler's theorem.

40. The initial remark of sect. 1 is vague, but can suggest several precise statements. Here is one that we have not considered in sect. 1: "If any one of the three quantities F, V, and E tends to ∞, also the other two must tend to ∞." Prove the following inequalities which hold generally for convex polyhedra and give still more precise information:

$$2E \geqq 3F, \quad 2V \geqq F + 4, \quad 3V \geqq E + 6,$$

$$2E \geqq 3V, \quad 2F \geqq V + 4, \quad 3F \geqq E + 6,$$

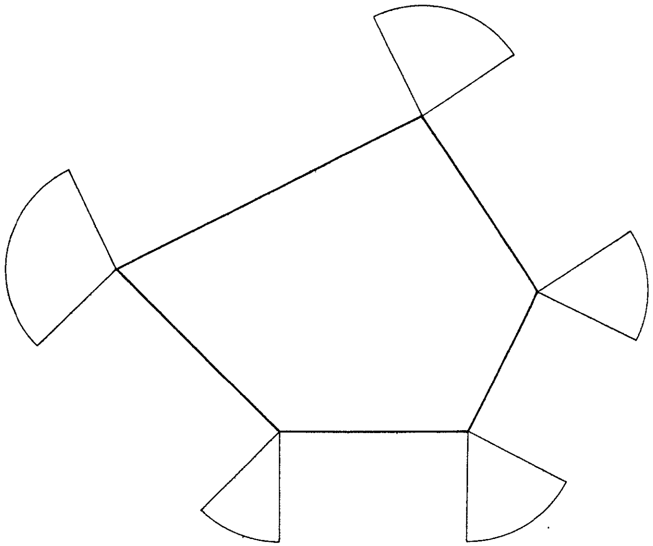

Fig. 3.7. Exterior angles of a polygon.

Can the case of equality be attained in these inequalities? For which kind of polyhedra can it be attained?

41. There are convex polyhedra all faces of which are polygons of the same kind, that is, polygons with the same number of sides. For example, all faces of a tetrahedron are triangles, all faces of a parallelepiped quadrilaterals, all faces of a regular dodecahedron pentagons. "And so on," you may be tempted to say. Yet such simple induction may be misguiding: there exists *no* convex polyhedron with faces which are all hexagons. Try to prove this. [Ex. 31.]

IV

INDUCTION IN THE THEORY OF NUMBERS

In the Theory of Numbers it happens rather frequently that, by some unexpected luck, the most elegant new truths spring up by induction.
—GAUSS[1]

1. Right triangles in integers.[2] The triangle with sides 3, 4, and 5 is a right triangle since

$$3^2 + 4^2 = 5^2.$$

This is the simplest example of a right triangle of which the sides are measured by integers. Such "right triangles in integers" have played a rôle in the history of the Theory of Numbers; even the ancient Babylonians discovered some of their properties.

One of the more obvious problems about such triangles is the following: *Is there a right triangle in integers, the hypotenuse of which is a given integer n?*

We concentrate upon this question. We seek a triangle the hypotenuse of which is measured by the given integer n and the legs by some integers x and y. We may assume that x denotes the longer of the two legs. Therefore, being given n, we seek two integers x and y such that

$$n^2 = x^2 + y^2, \quad 0 < y \leqq x < n.$$

We may attack the problem inductively and, unless we have some quite specific knowledge, we cannot attack it any other way. Let us take an example. We choose $n = 12$. Therefore, we seek two positive integers x and y, such that $x \geqq y$ and

$$144 = x^2 + y^2.$$

[1] *Werke*, vol. 2, p. 3.

[2] Parts of this chapter appeared already under the title "Let us teach guessing" in the volume *Études de philosophie des sciences en hommage à Ferdinand Gonseth.* Éditions du Griffon, 1950; see pp. 147–154.

Which values are available for x^2? The following:

$$1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121.$$

Is $x^2 = 121$? That is, is

$$144 - x^2 = 144 - 121 = y^2$$

a square? No, 23 is not a square. We should now try other squares but, in fact, we need not try too many of them. Since $y \leqq x$,

$$144 = x^2 + y^2 \leqq 2x^2$$
$$x^2 \geqq 72.$$

Therefore, $x^2 = 100$ and $x^2 = 81$ are the only remaining possibilities. Now, neither of the numbers

$$144 - 100 = 44, \quad 144 - 81 = 63$$

is a square and hence the answer: there is no right triangle in integers with hypotenuse 12.

We treat similarly the hypotenuse 13. Of the three numbers

$$169 - 144 = 25, \quad 169 - 121 = 48, \quad 169 - 100 = 69$$

only one is a square and so there is just one right triangle in integers with hypotenuse 13:

$$169 = 144 + 25.$$

Proceeding similarly, we can examine with a little patience all the numbers under a given not too high limit, such as 20. We find only five "hypotenuses" less than 20, the numbers 5, 10, 13, 15, and 17:

$$\begin{aligned} 25 &= 16 + 9 \\ 100 &= 64 + 36 \\ 169 &= 144 + 25 \\ 225 &= 144 + 81 \\ 289 &= 225 + 64. \end{aligned}$$

By the way, the cases 10 and 15 are not very interesting. The triangle with sides 10, 8, and 6 is similar to the simpler triangle with sides 5, 4, and 3, and the same is true of the triangle with sides 15, 12, and 9. The remaining three right triangles, with hypotenuse 5, 13, and 17, respectively, are essentially different, none is similar to another among them.

We may notice that all three numbers 5, 13, and 17 are *odd primes*. They are, however, not all the odd primes under 20; none of the other odd primes, 3, 7, 11, and 19 is a hypotenuse. Why that? What is the difference between

the two sets? *When, under which circumstances, is an odd prime the hypotenuse of some right triangle in integers, and when is it not?*

This is a modification of our original question. It may appear more hopeful; at any rate, it is new. Let us investigate it—again, inductively. With a little patience, we construct the following table (the dash indicates that there is no right triangle with hypotenuse p).

Odd prime p	*Right triangles with hypotenuse p*
3	—
5	$25 = 16 + 9$
7	—
11	—
13	$169 = 144 + 25$
17	$289 = 225 + 64$
19	—
23	—
29	$841 = 441 + 400$
31	—

When is a prime a hypotenuse; when is it not? What is the difference between the two cases? A physicist could easily ask himself some very similar questions. For instance, he investigates the double refraction of crystals. Some crystals do show double refraction; others do not. Which crystals are doubly refracting, which are not? What is the difference between the two cases?

The physicist looks at his crystals and we look at our two sets of primes

$$5, 13, 17, 29, \ldots \quad \text{and} \quad 3, 7, 11, 19, 23, 31, \ldots.$$

We are looking for some characteristic difference between the two sets. The primes in both sets increase by irregular jumps. Let us look at the lengths of these jumps, at the successive differences:

```
5   13   17   29          3   7   11   19   23   31
  8    4    12              4   4    8    4    8
```

Many of these differences are equal to 4, and, as it is easy to notice, *all are divisible by 4.* The primes in the first set, led by 5, leave the remainder 1 when divided by 4, are of the form $4n + 1$ with integral n. The primes in the second set, led by 3, are of the form $4n + 3$. Could this be the characteristic difference we are looking for? If we do not discard this possibility from the outset, we are led to the following conjecture: *A prime of the form $4n + 1$ is the hypotenuse of just one right triangle in integers; a prime of the form $4n + 3$ is the hypotenuse of no such triangle.*

2. Sums of squares. The problem of the right triangles in integers, one aspect of which we have just discussed (in sect. 1), played, as we have said, an important rôle in the history of the Theory of Numbers. It leads on, in fact, to many further questions. Which numbers, squares or not, can be decomposed into two squares? What about the numbers which cannot be decomposed into two squares? Perhaps, they are decomposable into three squares; but what about the numbers which are not decomposable into three squares?

We could go on indefinitely, but, and this is highly remarkable, we need not. Bachet de Méziriac (author of the first printed book on mathematical recreations) remarked that *any number* (that is, positive integer) *is either a square, or the sum of two, three, or four squares.* He did not pretend to possess a proof. He found indications pointing to his statement in certain problems of Diophantus and verified it up to 325.

In short, Bachet's statement was just a conjecture, found inductively. It seems to me that his main achievement was to put the question: *HOW MANY squares are needed to represent all integers?* Once this question is clearly put, there is not much difficulty in discovering the answer inductively. We construct a table beginning with

$$
\begin{aligned}
1 &= 1\\
2 &= 1+1\\
3 &= 1+1+1\\
4 &= 4\\
5 &= 4+1\\
6 &= 4+1+1\\
7 &= 4+1+1+1\\
8 &= 4+4\\
9 &= 9\\
10 &= 9+1.
\end{aligned}
$$

This verifies the statement up to 10. Only the number 7 requires four squares; the others are representable by one or two or three. Bachet went on tabulating up to 325 and found many numbers requiring four squares and none requiring more. Such inductive evidence satisfied him, it seems, at least to a certain degree, and he published his statement. He was lucky. His conjecture turned out to be true and so he became the discoverer of the "four-square theorem" which we can state also in the form: The equation

$$n = x^2 + y^2 + z^2 + w^2$$

where n is any given positive integer has a solution in which $x, y, z,$ and w are non-negative integers.

The decomposition of a number into a sum of squares has still other aspects. Thus, we may investigate the number of solutions of the equation

$$n = x^2 + y^2$$

in integers x and y. We may admit only positive integers, or all integers, positive, negative, and 0. If we choose the latter conception of the problem and take as example $n = 25$, we find 12 solutions of the equation

$$25 = x^2 + y^2,$$

namely the following

$$
\begin{aligned}
25 &= 5^2 + 0^2 = (-5)^2 + 0^2 = 0^2 + 5^2 = 0^2 + (-5)^2\\
&= 4^2 + 3^2 = (-4)^2 + 3^2 = 4^2 + (-3)^2 = (-4)^2 + (-3)^2\\
&= 3^2 + 4^2 = (-3)^2 + 4^2 = 3^2 + (-4)^2 = (-3)^2 + (-4)^2.
\end{aligned}
$$

By the way, these solutions have an interesting geometric interpretation, but we need not discuss it now. See ex. 2.

3. On the sum of four odd squares. Of the many problems concerned with sums of squares I choose one that looks somewhat far-fetched, but will turn out to be exceptionally instructive.

Let u denote a positive odd integer. Investigate inductively the number of the solutions of the equation

$$4u = x^2 + y^2 + z^2 + w^2$$

in positive odd integers $x, y, z,$ and w.

For example, if $u = 1$ we have the equation

$$4 = x^2 + y^2 + z^2 + w^2$$

and there is obviously just one solution, $x = y = z = w = 1$. In fact, we do not regard

$$x = -1, \quad y = 1, \quad z = 1, \quad w = 1$$

or

$$x = 2, \quad y = 0, \quad z = 0, \quad w = 0$$

as a solution, since we admit only positive odd numbers for $x, y, z,$ and w. If $u = 3$, the equation is

$$12 = x^2 + y^2 + z^2 + w^2$$

and the following two solutions:

$$
\begin{aligned}
x = 3, \quad y = 1, \quad z = 1, \quad w = 1\\
x = 1, \quad y = 3, \quad z = 1, \quad w = 1
\end{aligned}
$$

are different.

In order to emphasize the restriction laid upon the values of $x, y, z,$ and w, we shall avoid the term "solution" and use instead the more specific description: "representation of $4u$ as a sum of four odd squares." As this description is long, we shall abbreviate it in various ways, sometimes even to the one word "representation."

4. Examining an example. In order to familiarize ourselves with the meaning of our problem, let us consider an example. We choose $u = 25$. Then $4u = 100$, and we have to find all representations of 100 as a sum of four odd squares. Which odd squares are available for this purpose? The following:

$$1, \quad 9, \quad 25, \quad 49, \quad 81.$$

If 81 is one of the four squares the sum of which is 100, then the sum of the three others must be

$$100 - 81 = 19.$$

The only odd squares less than 19 are 1 and 9, and $19 = 9 + 9 + 1$ is evidently the only possibility to represent 19 as a sum of 3 odd squares if the terms are arranged in order of magnitude. We obtain

$$100 = 81 + 9 + 9 + 1.$$

We find similarly

$$100 = 49 + 49 + 1 + 1,$$
$$100 = 49 + 25 + 25 + 1,$$
$$100 = 25 + 25 + 25 + 25.$$

Proceeding systematically, by splitting off the largest square first, we may convince ourselves that we have exhausted all possibilities, provided that the 4 squares are arranged in descending order (or rather in non-ascending order). But there are more possibilities if we take into account, as we should, all arrangements of the terms. For example,

$$\begin{aligned} 100 &= 49 + 49 + 1 + 1 \\ &= 49 + 1 + 49 + 1 \\ &= 49 + 1 + 1 + 49 \\ &= 1 + 49 + 49 + 1 \\ &= 1 + 49 + 1 + 49 \\ &= 1 + 1 + 49 + 49. \end{aligned}$$

These 6 sums have the same terms, but the order of the terms is different; they are to be considered, according to the statement of our problem, as 6 different representations; the one representation

$$100 = 49 + 49 + 1 + 1$$

with non-increasing terms is a source of 5 other representations, of 6 representations in all. We have similarly

Non-increasing terms	*Number of arrangements*
81 + 9 + 9 + 1	12
49 + 49 + 1 + 1	6
49 + 25 + 25 + 1	12
25 + 25 + 25 + 25	1

To sum up, we found in our case where $u = 25$ and $4u = 100$

$$12 + 6 + 12 + 1 = 31$$

representations of $4u = 100$ as a sum of 4 odd squares.

5. Tabulating the observations. The special case $u = 25$ where $4u = 100$ and the number of representations is 31 has shown us clearly the meaning of the problem. We may now explore systematically the simplest cases, $u = 1, 3, 5, \ldots$ up to $u = 25$. We construct Table I. (See below; the reader should construct the table by himself, or at least check a few items.)

Table I

u	$4u$	*Non-increasing*	*Arrangements*	*Representations*
1	4	1 + 1 + 1 + 1	1	1
3	12	9 + 1 + 1 + 1	4	4
5	20	9 + 9 + 1 + 1	6	6
7	28	25 + 1 + 1 + 1	4	8
		9 + 9 + 9 + 1	4	
9	36	25 + 9 + 1 + 1	12	13
		9 + 9 + 9 + 9	1	
11	44	25 + 9 + 9 + 1	12	12
13	52	49 + 1 + 1 + 1	4	14
		25 + 25 + 1 + 1	6	
		25 + 9 + 9 + 9	4	
15	60	49 + 9 + 1 + 1	12	24
		25 + 25 + 9 + 1	12	
17	68	49 + 9 + 9 + 1	12	18
		25 + 25 + 9 + 9	6	
19	76	49 + 25 + 1 + 1	12	20
		49 + 9 + 9 + 9	4	
		25 + 25 + 25 + 1	4	
21	84	81 + 1 + 1 + 1	4	32
		49 + 25 + 9 + 1	24	
		25 + 25 + 25 + 9	4	
23	92	81 + 9 + 1 + 1	12	24
		49 + 25 + 9 + 9	12	
25	100	81 + 9 + 9 + 1	12	31
		49 + 49 + 1 + 1	6	
		49 + 25 + 25 + 1	12	
		25 + 25 + 25 + 25	1	

6. What is the rule? Is there any recognizable law, any simple connection between the odd number u and the number of different representations of $4u$ as a sum of four odd squares?

This question is the kernel of our problem. We have to answer it on the basis of the observations collected and tabulated in the foregoing section. We are in the position of the naturalist trying to extract some rule, some general formula from his experimental data. Our experimental material available at this moment consists of two parallel series of numbers

1	3	5	7	9	11	13	15	17	19	21	23	25
1	4	6	8	13	12	14	24	18	20	32	24	31.

The first series consists of the successive odd numbers, but what is the rule governing the second series?

As we try to answer this question, our first feeling may be close to despair. That second series looks quite irregular, we are puzzled by its complex origin, we can scarcely hope to find any rule. Yet, if we forget about the complex origin and concentrate upon what is before us, there is a point easy enough to notice. It happens rather often that a term of the second series exceeds the corresponding term of the first series by just one unit. Emphasizing these cases by heavy print in the first series, we may present our experimental material as follows:

1	**3**	**5**	**7**	9	**11**	**13**	15	**17**	**19**	21	**23**	25
1	4	6	8	13	12	14	24	18	20	32	24	31.

The numbers in heavy print attract our attention. It is not difficult to recognize them: they are *primes*. In fact, they are *all* the primes in the first row as far as our table goes. This remark may appear very surprising if we remember the origin of our series. We considered squares, we made no reference whatever to primes. Is it not strange that the prime numbers play a rôle in our problem? It is difficult to avoid the impression that our observation is significant, that there is something remarkable behind it.

What about those numbers of the first series which are not in heavy print? They are odd numbers, but not primes. The first, 1, is unity, the others are composite

$$9 = 3 \times 3, \quad 15 = 3 \times 5, \quad 21 = 3 \times 7, \quad 25 = 5 \times 5.$$

What is the nature of the corresponding numbers in the second series?

If the odd number u is a prime, the corresponding number is $u + 1$; if u is not a prime, the corresponding number is not $u + 1$. This we have observed already. We may add one little remark. If $u = 1$, the corresponding number is also 1, and so *less* than $u + 1$, but in all other cases in which u is not a prime the corresponding number is *greater* than $u + 1$. That is, the number corresponding to u is less than, equal to, or greater than $u + 1$ accordingly as u is unity, a prime, or a composite number. There is some regularity.

Let us concentrate upon the composite numbers in the upper line and the corresponding numbers in the lower line:

$$\begin{array}{cccc} 3 \times 3 & 3 \times 5 & 3 \times 7 & 5 \times 5 \\ 13 & 24 & 32 & 31 \end{array} .$$

There is something strange. Squares in the first line correspond to primes in the second line. Yet we have too few observations; probably we should not attach too much weight to this remark. Still, it is true that, conversely, under the composite numbers in the first line which are not squares, we find numbers in the second line which are not primes:

$$\begin{array}{cc} 3 \times 5 & 3 \times 7 \\ 4 \times 6 & 4 \times 8. \end{array}$$

Again, there is something strange. Each factor in the second line exceeds the corresponding factor in the first line by just one unit. Yet we have too few observations; we had better not attach too much weight to this remark. Still, our remark shows some parallelism with a former remark. We noticed before

$$\begin{array}{c} p \\ p + 1 \end{array}$$

and we notice now

$$\begin{array}{c} pq \\ (p + 1)(q + 1) \end{array}$$

where p and q are primes. There is some regularity.

Perhaps we shall see more clearly if we write the entry corresponding to pq differently:

$$(p + 1)(q + 1) = pq + p + q + 1.$$

What can we see there? What are these numbers pq, p, q, 1? At any rate, the cases

$$\begin{array}{cc} 9 & 25 \\ 13 & 31 \end{array}$$

remain unexplained. In fact, the entries corresponding to 9 and 25 are greater than $9 + 1$ and $25 + 1$, respectively, as we have already observed:

$$13 = 9 + 1 + 3 \qquad 31 = 25 + 1 + 5 .$$

What are these numbers?

If one more little spark comes from somewhere, we may succeed in combining our fragmentary remarks into a coherent whole, our scattered indications into an illuminating view of the full correspondence:

$$\begin{array}{ccccc} p & pq & 9 & 25 & 1 \\ p + 1 & pq + p + q + 1 & 9 + 3 + 1 & 25 + 5 + 1 & 1 . \end{array}$$

DIVISORS! The second line shows the divisors of the numbers in the first line. This may be the desired rule, and a discovery, a real discovery: *To each number in the first line corresponds the sum of its divisors.*

And so we have been led to a conjecture, perhaps to one of those "most elegant new truths" of Gauss: *If u is an odd number, the number of representations of $4u$ as a sum of four odd squares is equal to the sum of the divisors of u.*

7. On the nature of inductive discovery. Looking back at the foregoing sections (3 to 6) we may find many questions to ask.

What have we obtained? Not a proof, not even the shadow of a proof, just a conjecture: a simple description of the facts within the limits of our experimental material, and a certain hope that this description may apply beyond the limits of our experimental material.

How have we obtained our conjecture? In very much the same manner that ordinary people, or scientists working in some non-mathematical field, obtain theirs. We collected relevant observations, examined and compared them, noticed fragmentary regularities, hesitated, blundered, and eventually succeeded in *combining the scattered details into an apparently meaningful whole.* Quite similarly, an archaeologist may reconstitute a whole inscription from a few scattered letters on a worn-out stone, or a palaeontologist may reconstruct the essential features of an extinct animal from a few of its petrified bones. In our case the meaningful whole appeared at the same moment when we recognized the appropriate unifying concept (the divisors).

8. On the nature of inductive evidence. There remain a few more questions.

How strong is the evidence? Your question is incomplete. You mean, of course, the inductive evidence for our conjecture stated in sect. 6 that we can derive from Table I of sect. 5; this is understood. Yet what do you mean by "strong"? The evidence is strong if it is convincing; it is convincing if it convinces somebody. Yet you did not say whom it should convince—me, or you, or Euler, or a beginner, or whom?

Personally, I find the evidence pretty convincing. I feel sure that Euler would have thought very highly of it. (I mention Euler because he came very near to discovering our conjecture; see ex. 6.24.) I think that a beginner who knows a little about the divisibility of numbers ought to find the evidence pretty convincing, too. A colleague of mine, an excellent mathematician who however was not familiar with this corner of the Theory of Numbers, found the evidence "hundred per cent convincing."

I am not concerned with subjective impressions. What is the precise, objectively evaluated degree of rational belief, justified by the inductive evidence? You give me one thing (A), you fail to give me another thing (B), and you ask me a third thing (C).

(A) You give me exactly the inductive evidence: the conjecture has been verified in the first thirteen cases, for the numbers 4, 12, 20, . . . , 100. This is perfectly clear.

(B) You wish me to evaluate the degree of rational belief justified by this evidence. Yet such belief must depend, if not on the whims and the temperament, certainly on the *knowledge* of the person receiving the evidence. He may know a proof of the conjectural theorem or a counter-example exploding it. In these two cases the degree of his belief, already firmly established, will remain unchanged by the inductive evidence. Yet if he knows something that comes very close to a complete proof, or to a complete refutation, of the theorem, his belief is still capable of modification and will be affected by the inductive evidence here produced, although different degrees of belief will result from it according to the kind of knowledge he has. Therefore, if you wish a definite answer, you should specify a definite level of knowledge on which the proposed inductive evidence (A) should be judged. You should give me a definite set of relevant known facts (an explicit list of known elementary propositions in the Theory of Numbers, perhaps).

(C) You wish me to evaluate the degree of rational belief justified by the inductive evidence exactly. Should I give it to you perhaps expressed in percentages of "full credence"? (We may agree to call "full credence" the degree of belief justified by a complete mathematical proof of the theorem in question.) Do you expect me to say that the given evidence justifies a belief amounting to 99% or to 2.875% or to .000001% of "full credence"?

In short, you wish me to solve a problem: Given (A) the inductive evidence and (B) a definite set of known facts or propositions, compute the percentage of full credence rationally resulting from both (C).

To solve this problem is much more than I can do. I do not know anybody who could do it, or anybody who would dare to do it. I know of some philosophers who promise to do something of this sort in great generality. Yet, faced with the concrete problem, they shrink and hedge and find a thousand excuses why not to do just this problem.

Perhaps the problem is one of those typical philosophical problems about which you can talk a lot in general, and even worry genuinely, but which fade into nothingness when you bring them down to concrete terms.

Could you compare the present case of inductive inference with some standard case and so arrive at a reasonable estimate of the strength of the evidence? Let us compare the inductive evidence for our conjecture with Bachet's evidence for his conjecture.

Bachet's conjecture was: For $n = 1, 2, 3, \ldots$ the equation

$$n = x^2 + y^2 + z^2 + w^2$$

has at least one solution in non-negative integers $x, y, z,$ and w. He verified this conjecture for $n = 1, 2, 3, \ldots, 325$. (See sect. 2, especially the short table.)

Our conjecture is: For a given odd u, the number of solutions of the equation

$$4u = x^2 + y^2 + z^2 + w^2$$

in positive odd integers x, y, z, and w is equal to the sum of the divisors of u. We verified this conjecture for $u = 1, 3, 5, 7, \ldots, 25$ (13 cases). (See sect. 3 to 6.)

I shall compare these two conjectures and the inductive evidence yielded by their respective verifications in three respects.

Number of verifications. Bachet's conjecture was verified in 325 cases, ours in 13 cases only. The advantage in this respect is clearly on Bachet's side.

Precision of prediction. Bachet's conjecture predicts that the number of solutions is $\geqq 1$; ours predicts that the number of solutions is exactly equal to such and such a quantity. It is obviously reasonable to assume, I think, that *the verification of a more precise prediction carries more weight* than that of a less precise prediction. The advantage in this respect is clearly on our side.

Rival conjectures. Bachet's conjecture is concerned with the maximum number of squares, say M, needed in representing an arbitrary positive integer as sum of squares. In fact, Bachet's conjecture asserts that $M = 4$. I do not think that Bachet had any *a priori* reason to prefer $M = 4$ to, say, $M = 5$, or to any other value, as $M = 6$ or $M = 7$; even $M = \infty$ is not excluded *a priori*. (Naturally, $M = \infty$ would mean that there are larger and larger integers demanding more and more squares. On the face, $M = \infty$ could appear as the most likely conjecture.) In short, Bachet's conjecture has many obvious rivals. Yet ours has none. Looking at the irregular sequence of the numbers of representations (sect. 6) we had the impression that we might not be able to find any rule. Now we did find an admirably clear rule. We hardly expect to find any other rule.

It may be difficult to choose a bride if there are many desirable young ladies to choose from; if there is just one eligible girl around, the decision may come much quicker. It seems to me that our attitude toward conjectures is somewhat similar. Other things being equal, a conjecture that has many obvious rivals is more difficult to accept than one that is unrivalled. If you think as I do, you should find that in this respect the advantage is on the side of our conjecture, not on Bachet's side.

Please observe that the evidence for Bachet's conjecture is stronger in one respect and the evidence for our conjecture is stronger in other respects, and do not ask unanswerable questions.

EXAMPLES AND COMMENTS ON CHAPTER IV

1. *Notation.* We assume that n and k are positive integers and consider the Diophantine equation

$$n = x_1^2 + x_2^2 + \ldots + x_k^2.$$

We say that two solutions $x_1, x_2, \ldots x_k$ and $x_1', x_2', \ldots x_k'$ are equal if, and only if, $x_1 = x_1', x_2 = x_2', \ldots x_k = x_k'$. If we admit for $x_1, x_2, \ldots x_k$ *all* integers, positive, negative, or null, we call the number of solutions $R_k(n)$. If

we admit only *positive odd* integers, we call the number of solutions $S_k(n)$. This notation is important in the majority of the following problems.

Bachet's conjecture (sect. 2) is expressed in this notation by the inequality

$$R_4(n) > 0 \text{ for } n = 1, 2, 3, \ldots .$$

The conjecture that we discovered in sect. 6 affirms that $S_4(4(2n-1))$ equals the sum of the divisors of $2n - 1$, for $n = 1, 2, 3, \ldots$.

Find $R_2(25)$ and $S_3(11)$.

2. Let x and y be rectangular coordinates in a plane. The points for which both x and y are integers are called the "lattice points" of the plane. Lattice points in space are similarly defined.

Interpret $R_2(n)$ and $R_3(n)$ geometrically, in terms of lattice points.

3. Express the conjecture encountered in sect. 1 in using the symbol $R_2(n)$.

4. When is an odd prime the sum of two squares? Try to answer this question inductively, by examining the table

3	—
5 =	4 + 1
7	—
11	—
13 =	9 + 4
17 =	16 + 1
19	—
23	—
29 =	25 + 4
31	—

Extend this table if necessary and compare it with the table in sect. 1.

5. Could you verify by mathematical deduction some part of your answer to ex. 4 obtained by induction? After such a verification, would it be reasonable to change your confidence in the conjecture?

6. Verify Bachet's conjecture (sect. 2) up to 30 inclusively. Which numbers require actually four squares?

7. In order to understand better Table I in sect. 5, let a^2, b^2, c^2, and d^2 denote four different odd squares and consider the sums

$$(1)\ a^2 + b^2 + c^2 + d^2$$
$$(2)\ a^2 + a^2 + b^2 + c^2$$
$$(3)\ a^2 + a^2 + b^2 + b^2$$
$$(4)\ a^2 + a^2 + a^2 + b^2$$
$$(5)\ a^2 + a^2 + a^2 + a^2 .$$

How many different representations (in the sense of sect. 3) can you derive from each by permuting the terms?

8. The number of representations of $4u$, as a sum of four odd squares is odd if, and only if, u is a square. (Following the notation of sect. 3, we assume that u is odd.) Prove this statement and show that it agrees with the conjecture of sect. 6. How does this remark influence your confidence in the conjecture?

9. Now, let a, b, c, and d denote different positive integers (odd or even). Consider the five sums mentioned in ex. 7 and also the following:

$$(6)\ a^2 + b^2 + c^2 \qquad (9)\ a^2 + b^2$$
$$(7)\ a^2 + a^2 + b^2 \qquad (10)\ a^2 + a^2$$
$$(8)\ a^2 + a^2 + a^2 \qquad (11)\ a^2 .$$

Find in each of these eleven cases the contribution to $R_4(n)$. You derive from each sum all possible representations by the following obvious operations: you add 0^2 as many times as necessary to bring the number of terms to 4, you change the arrangement, and you replace some (or none, or all) of the numbers a, b, c, d by $-a$, $-b$, $-c$, $-d$, respectively. (Check examples in Table II.)

10. Investigate inductively the number of solutions of the equation $n = x^2 + y^2 + z^2 + w^2$ in integers x, y, z, and w, positive, negative, or 0. Start by constructing a table analogous to Table I.

11 (continued). Try to use the method, or the result, of sect. 6.

12 (continued). Led by the analogy of sect. 6 or by your observation of Table II, distinguish appropriate classes of integers and investigate each class by itself.

13 (continued). Concentrate upon the most stubborn class.

14 (continued). Try to summarize all fragmentary regularities and express the law in one sentence.

15 (continued). Check the rule found in the first three cases not contained in Table II.

16. Find $R_8(5)$ and $S_8(40)$.

17. Check at least two entries of Table III, p. 75, not yet given in Tables I and II.

18. Using Table III, investigate inductively $R_8(n)$ and $S_8(8n)$.

19 (continued). Try to use the method, or the result, of sect. 6 and ex. 10–15.

20 (continued). Led by analogy or observation, distinguish appropriate classes of integers and investigate each class by itself.

21 (continued). Try to discover a cue in the most accessible case.

22 (continued). Try to find some unifying concept that could summarize the fragmentary regularities.

23 (continued). Try to express the law in one sentence.

24. Which numbers can and which numbers cannot be expressed in the form $3x + 5y$, where x and y are non-negative integers?

25. Try to guess the law of the following table:

a	b	Last integer not expressible in form $ax + by$
2	3	1
2	5	3
2	7	5
2	9	7
3	4	5
3	5	7
3	7	11
3	8	13
4	5	11
5	6	19

It is understood that x and y are non-negative integers. Check a few items and extend the table, if necessary. [Observe the change in the last column when just one of the two numbers a and b changes.]

26. *Dangers of induction.* Examine inductively the following assertions:

(1) $(n - 1)! + 1$ is divisible by n when n is a prime, but not divisible by n when n is composite.

(2) $2^{n-1} - 1$ is divisible by n when n is an odd prime, but not divisible by n when n is composite.

Table II

n	Non-increasing	Repres.	$R_4(n)/8$
1	1	4 × 2	1
2	1 + 1	6 × 4	3
3	1 + 1 + 1	4 × 8	4
4	4 1 + 1 + 1 + 1	4 × 2 1 × 16	3
5	4 + 1	12 × 4	6
6	4 + 1 + 1	12 × 8	12
7	4 + 1 + 1 + 1	4 × 16	8
8	4 + 4	6 × 4	3
9	9 4 + 4 + 1	4 × 2 12 × 8	13
10	9 + 1 4 + 4 + 1 + 1	12 × 4 6 × 16	18
11	9 + 1 + 1	12 × 8	12
12	9 + 1 + 1 + 1 4 + 4 + 4	4 × 16 4 × 8	12
13	9 + 4 4 + 4 + 4 + 1	12 × 4 4 × 16	14
14	9 + 4 + 1	24 × 8	24
15	9 + 4 + 1 + 1	12 × 16	24
16	16 4 + 4 + 4 + 4	4 × 2 1 × 16	3
17	16 + 1 9 + 4 + 4	12 × 4 12 × 8	18
18	16 + 1 + 1 9 + 9 9 + 4 + 4 + 1	12 × 8 6 × 4 12 × 16	39
19	16 + 1 + 1 + 1 9 + 9 + 1	4 × 16 12 × 8	20
20	16 + 4 9 + 9 + 1 + 1	12 × 4 6 × 16	18
21	16 + 4 + 1 9 + 4 + 4 + 4	24 × 8 4 × 16	32
22	16 + 4 + 1 + 1 9 + 9 + 4	12 × 16 12 × 8	36
23	9 + 9 + 4 + 1	12 × 16	24
24	16 + 4 + 4	12 × 8	12
25	25 16 + 9 16 + 4 + 4 + 1	4 × 2 12 × 4 12 × 16	31

Table II (*continued*)

n	Non-increasing	Repres.	$R_4(n)/8$
26	25 + 1 16 + 9 + 1 9 + 9 + 4 + 4	12 × 4 24 × 8 6 × 16	42
27	25 + 1 + 1 16 + 9 + 1 + 1 9 + 9 + 9	12 × 8 12 × 16 4 × 8	40
28	25 + 1 + 1 + 1 16 + 4 + 4 + 4 9 + 9 + 9 + 1	4 × 16 4 × 16 4 × 16	24
29	25 + 4 16 + 9 + 4	12 × 4 24 × 8	30
30	25 + 4 + 1 16 + 9 + 4 + 1	24 × 8 24 × 16	72

Table III

n	$R_4(n)/8$	$R_8(n)/16$	$S_8(8n)$	$S_4(4(2n-1))$	$2n-1$
1	1	1	1	1	1
2	3	7	8	4	3
3	4	28	28	6	5
4	3	71	64	8	7
5	6	126	126	13	9
6	12	196	224	12	11
7	8	344	344	14	13
8	3	583	512	24	15
9	13	757	757	18	17
10	18	882	1008	20	19
11	12	1332	1332	32	21
12	12	1988	1792	24	23
13	14	2198	2198	31	25
14	24	2408	2752	40	27
15	24	3528	3528	30	29
16	3	4679	4096	32	31
17	18	4914	4914	48	33
18	39	5299	6056	48	35
19	20	6860	6860	38	37
20	18	8946	8064	56	39