Efficient experimental mathematics for combinatorics and number theory

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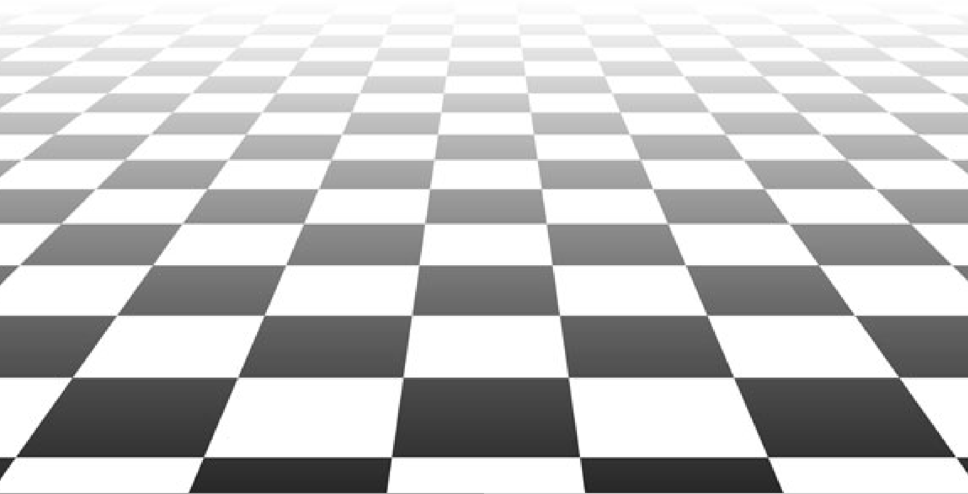
Overview

Lecture 1: Context, Motivation, Examples

Lecture 2: Exp. Math. for Combinatorics

Lecture 3: Inside the Exp. Math. Toolbox

Lecture 1: Context, Motivation, Examples



CONTEXT

A 3-step process

1. Compute a high-order approximation

   —*high precision numerical approximation*,
   power series truncated to high order,
   large number of terms in a sequence,...—

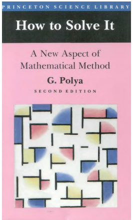
2. Guess/conjecture a general formula

   —*with the help of a computer*—

3. Prove it

   —*using computer-algebra algorithms*—

Guess-and-Prove



Guessing and Proving

George Pólya

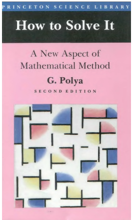
What is "scientific method"? Philosophers and non-philosophers have discussed this question and have not yet finished discussing it. Yet as a first introduction it can be described in three syllables:

Guess and test.

Mathematicians too follow this advice in their research although they sometimes refuse to confess it. They have, however, something which the other scientists cannot really have. For mathematicians the advice is

First guess, then prove.

Guess-and-Prove



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First guess, then prove.

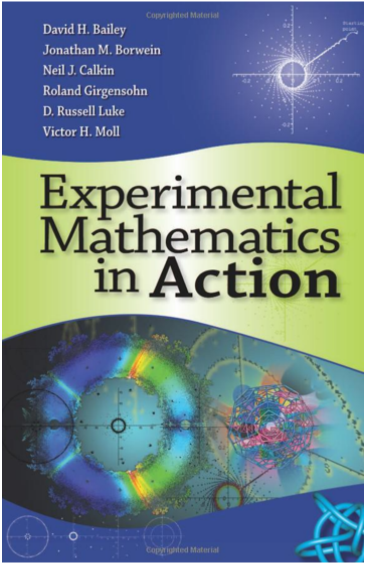
generate data → make conjectures → prove them

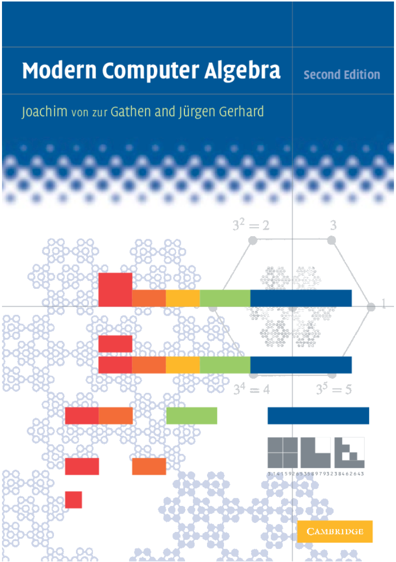
Effective mathematics: what can we compute exactly?

And complexity: how fast? (also, how big is the result?)

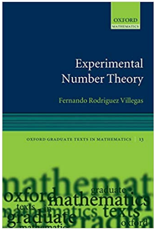
Systems with several million users

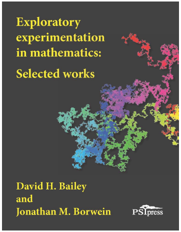
50+ years of algorithmic progress
in computational mathematics!

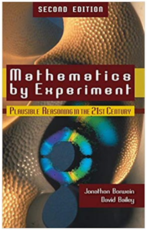

David H. Bailey
Jonathan M. Borwein
Neil J. Calkin
Roland Girgensohn
D. Russell Luke
Victor H. Moll
Experimental Mathematics in Action

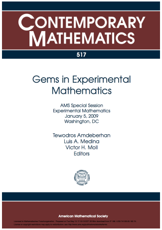

Modern Computer Algebra
Second Edition
Joachim von zur Gathen and Jürgen Gerhard
CAMBRIDGE

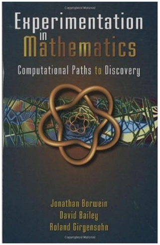
Experimental mathematics books

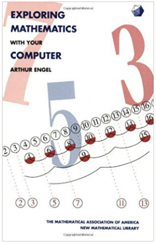


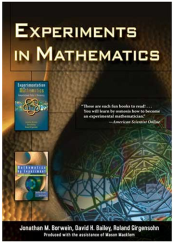


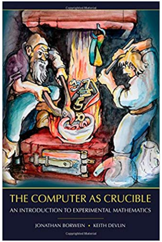








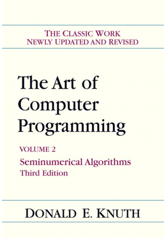


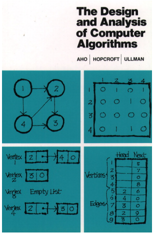


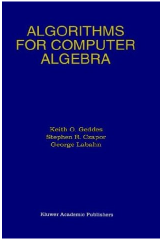
Experimental mathematics journal

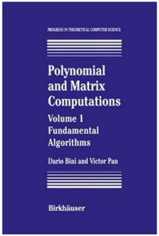
https://www.tandfonline.com/loi/uexm

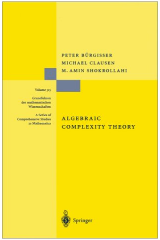
Computer algebra books

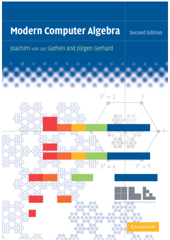


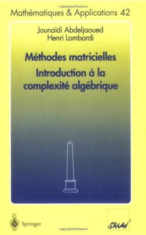




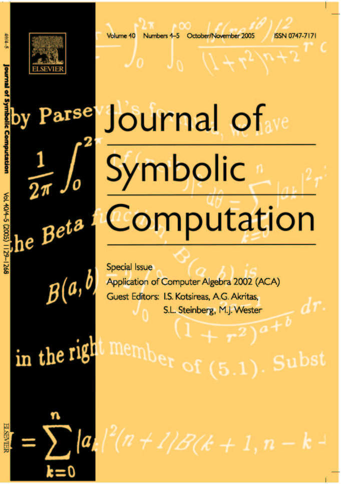








Main computer algebra journal and conference



https://www.journals.elsevier.com/journal-of-symbolic-computation

http://www.issac-conference.org

MOTIVATION

A Philosopher of Science perspective [Sørensen, 2010]

From a philosophical viewpoint, mathematics has traditionally been distinguished from the natural sciences by its formal nature and emphasis on deductive reasoning. Experiments—one of the corner stones of most modern natural science—have had no role to play in mathematics. However, in the past two to three decades, a mathematical subdiscipline has been forming that describes itself as "experimental mathematics", and it is the purpose of this paper to investigate and discuss the ways in which experimental mathematics is *experimental*.

A Philosopher of Science perspective [Sørensen, 2010]

Since the 1990s, many domains of knowledge production have witnessed a "computational turn" during which the wide use of computers has influenced established ways of thinking.[1] In mathematics, computers have been utilized since their first construction, but in the 1990s, their use led to a new subdiscipline of experimental mathematics in which computers were central to most—if not all—the experiments that give the subdiscipline its name. Using high speed computers and software packages such as Maple and Mathematica, mathematicians can now manipulate data and structures of immense complexity through real-time interaction with computers, and these practices are at the heart of experimental mathematics, I will argue. Thus, computers—and the "experiments" that they seem to carry with them—have entered into wide areas of traditional mathematics ranging from combinatorics to partial differential equations.

An Experimental Mathematician perspective [Wilf, 2007]

Albert Einstein once said "You can confirm a theory with experiment, but no path leads from experiment to theory." But that was before computers.

A computer is used by a pure mathematician in much the same way that a telescope is used by a theoretical astronomer. It shows us "what's out there." Neither the computer nor the telescope can provide a theoretical explanation for what it sees, but either of them extends the reach of the mind by providing multitudes of examples that might otherwise be hidden, and from which one has some chance of perceiving, and then demonstrating, the existence of patterns, or universal laws.

An Experimental Mathematician perspective [Wilf, 2007]

When computers first appeared in mathematicians' environments the almost universal reaction was that they would never be useful for proving theorems since a computer can never investigate infinitely many cases, no matter how fast it is. But computers are useful for proving theorems despite that handicap. We have seen several examples of how a mathematician can act in concert with a computer to explore a world within mathematics. From such explorations there can grow understanding, and conjectures, and roads to proofs, and phenomena that would not have been imaginable in the pre-computer era. This role of computation within pure mathematics seems destined only to expand over the near future and to be imbued into our students along with Euclid's axioms and other staples of mathematical education.

Role of computers in mathematics [Borwein, Bailey, 2004]

- heuristics
  - gaining insight and intuition
  - discovering new patterns and relationships
  - using graphical displays to vizualize underlying mathematical principles

- refining and evaluating conjectures
  - testing (and falsifying) conjectures
  - exploring a possible result to see if it is worth formal proof

- aiding in the procedure of proving conjectures
  - suggesting approaches for formal proof
  - replacing lengthy hand derivations with computer-based calculations
  - confirming analytically derived results

Role of computers in Experimental Mathematics

- check huge number of cases (but finitely many)

  ▷ E.g. an algorithm checks 1 936 configurations to complete the proof of the four-color conjecture [Appel, Haken, 1976]

- guess patterns

  ▷ E.g. an algorithm guesses a new (and useful!) formula for π

$$\pi = \sum_{n\geq 0} \frac{1}{16^n} \left(\frac{4}{8n+1} - \frac{2}{8n+4} - \frac{1}{8n+5} - \frac{1}{8n+6} \right)$$

[Bailey, Borwein, Plouffe, 1997]

- prove theorems

  ▷ E.g. an algorithm proves a classification of lattice walks according to properties of generating functions $\longrightarrow$ details on Wednesday!

Some tools and methods from the Exp. Math. toolbox

- Computer Algebra Systems: *Maple*, *Mathematica*, *Sage*, *Magma*, *Pari*, ...
- Databases
  - OEIS (large searchable table of integer sequences),
  - ISC (online service to identify real constants given with good numerical precision)
- Algorithmic tools for guessing patterns
  - *guessing recurrences* satisfied by truncated sequences
  - *guessing equations* (algebraic / differential) from truncated power series
  - *integer relation detection* satisfied by truncated numbers
  - *constant recognition*
- Algorithmic tools for proving theorems
  - *polynomial elimination* (resultants, Gröbner bases)
  - *D-finiteness*
  - *Creative Telescoping*

EXAMPLES

Discovering Pascal

Question: Find $B_{i,j}$:= the number of $\{\rightarrow, \uparrow\}$-walks in $\mathbb{N}^2$ from $(0,0)$ to (i,j)

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1. There are 2 ways to get to (i,j), either from $(i-1,j)$, or from $(i,j-1)$:

$$B_{i,j} = B_{i-1,j} + B_{i,j-1}$$

2. There is only one way to get to a point on an axis: $B_{i,0} = B_{0,j} = 1$

▷ These two rules completely determine all the numbers $B_{i,j}$

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(I) Generate data:

$\vdots$

1	7	28	84	210	462	924	
1	6	21	56	126	252	462	
1	5	15	35	70	126	210	
1	4	10	20	35	56	84	
1	3	6	10	15	21	28	
1	2	3	4	5	6	7	
1	1	1	1	1	1	1	$\cdots$

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$$\begin{array}{ccccccc} \vdots \\ 1 & 7 & 28 & 84 & 210 & 462 & 924 \\ 1 & 6 & 21 & 56 & 126 & 252 & 462 \\ 1 & 5 & 15 & 35 & 70 & 126 & 210 \\ 1 & 4 & 10 & 20 & 35 & 56 & 84 \\ 1 & 3 & 6 & 10 & 15 & 21 & 28 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{array}$$

(II) Guess:

$$\begin{array}{l} \longrightarrow \cdots \\ \longrightarrow \frac{(i+1)(i+2)}{2} \\ \longrightarrow i+1 \\ \longrightarrow 1 \end{array}$$

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(II) Guess:

$$B_{i,j} \stackrel{?}{=} \frac{(i+j)!}{i!j!}$$

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1 & 4 & 10 & 20 & 35 & 56 & 84 & \\
1 & 3 & 6 & 10 & 15 & 21 & 28 & \\
1 & 2 & 3 & 4 & 5 & 6 & 7 & \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & \dots
\end{array}$$

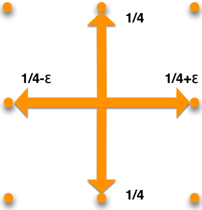
(III) Prove: If $C_{i,j} \stackrel{\text{def}}{=} \frac{(i+j)!}{i!j!}$, then

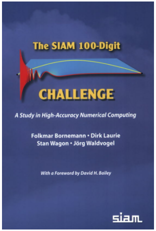
$$\frac{C_{i-1,j}}{C_{i,j}} + \frac{C_{i,j-1}}{C_{i,j}} = \frac{i}{i+j} + \frac{j}{i+j} = 1$$

and $C_{i,0} = C_{0,j} = 1$.

Thus $\boxed{B_{i,j} = C_{i,j}}$

Biasing for a Fair Return [Trefethen, 2002; Bornemann, 2004]





Problem 6

A flea starts at $(0,0)$ *on the infinite two-dimensional integer lattice and executes a biased random walk: At each step it hops north or south with probability* $1/4$*, east with probability* $1/4+\epsilon$*, and west with probability* $1/4-\epsilon$*. The probability that the flea returns to* $(0,0)$ *sometime during its wanderings is* $1/2$*. What is* ϵ*?*

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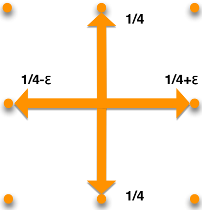

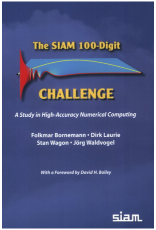

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▷ Computer algebra conjectures and proves

$$p(\epsilon) = 1 - \left(\sqrt{\frac{A}{2}} \cdot {}_2F_1\left(\begin{matrix}\frac{1}{2}, \frac{1}{2}\\ 1\end{matrix} \,\middle|\, \frac{2\sqrt{1-16\epsilon^2}}{A}\right)\right)^{-1}, \quad \text{with } A = 1 + 8\epsilon^2 + \sqrt{1-16\epsilon^2}.$$

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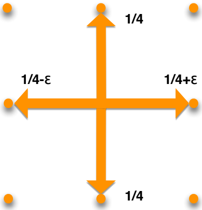

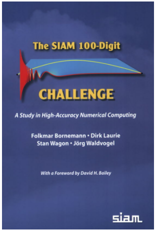

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▷ Computer algebra conjectures and proves

$$\epsilon \approx 0.0619139544739909428481752164732121769996387749983\;6207606146725885993101029759615845907105645752087861\ldots$$

from *SIAM News*, Volume 35, Number 1

A Hundred-dollar, Hundred-digit Challenge

Each October, a few new graduate students arrive in Oxford to begin research for a doctorate in numerical analysis. In their first term, working in pairs, they take an informal course called the "Problem Solving Squad." Each week for six weeks, I give them a problem, stated in a sentence or two, whose answer is a single real number. Their mission is to compute that number to as many digits of precision as they can.

Ten of these problems appear below. I would like to offer them as a challenge to the SIAM community. Can you solve them?

I will give \$100 to the individual or team that delivers to me the most accurate set of numerical answers to these problems before May 20, 2002. With your solutions, send in a few sentences or programs or plots so I can tell how you got them. Scoring will be simple: You get a point for each correct digit, up to ten for each problem, so the maximum score is 100 points.

Fine print? You are free to get ideas and advice from friends and literature far and wide, but any team that enters the contest should have no more than half a dozen core members. Contestants must assure me that they have received no help from students at Oxford or anyone else who has already seen these problems.

Hint: They're hard! If anyone gets 50 digits in total, I will be impressed. The ten magic numbers will be published in the July/August issue of *SIAM News*, together with the names of winners and strong runners-up.—*Nick Trefethen, Oxford University.*

The Hundred-dollar, Hundred-digit Challenge Problems

1. What is $\lim_{\varepsilon \to 0} \int_\varepsilon^1 x^{-1} \cos(x^{-1} \log x)\, dx$?

2. A photon moving at speed 1 in the x–y plane starts at $t = 0$ at $(x,y) = (0.5, 0.1)$ heading due east. Around every integer lattice point (i,j) in the plane, a circular mirror of radius 1/3 has been erected. How far from the origin is the photon at $t = 10$?

3. The infinite matrix A with entries $a_{11} = 1$, $a_{12} = 1/2$, $a_{21} = 1/3$, $a_{13} = 1/4$, $a_{22} = 1/5$, $a_{31} = 1/6$, etc., is a bounded operator on ℓ^2. What is $\|A\|$?

4. What is the global minimum of the function

$$\exp(\sin(50x)) + \sin(60e^y) + \sin(70 \sin(x)) + \sin(\sin(80y)) - \sin(10(x+y)) + \tfrac{1}{4}(x^2+y^2) \ ?$$

5. Let $f(z) = 1/\Gamma(z)$, where $\Gamma(z)$ is the gamma function, and let $p(z)$ be the cubic polynomial that best approximates $f(z)$ on the unit disk in the supremum norm $\|\cdot\|_\infty$. What is $\|f - p\|_\infty$?

6. A flea starts at $(0, 0)$ on the infinite 2D integer lattice and executes a biased random walk: At each step it hops north or south with probability 1/4, east with probability $1/4 + \varepsilon$, and west with probability $1/4 - \varepsilon$. The probability that the flea returns to (0, 0) sometime during its wanderings is 1/2. What is ε?

7. Let A be the $20{,}000 \times 20{,}000$ matrix whose entries are zero everywhere except for the primes 2, 3, 5, 7, . . . , 224737 along the main diagonal and the number 1 in all the positions a_{ij} with $|i - j| = 1,2,4,8,\ldots,16384$. What is the (1, 1) entry of A^{-1}?

8. A square plate $[-1, 1] \times [-1, 1]$ is at temperature $u = 0$. At time $t = 0$ the temperature is increased to $u = 5$ along one of the four sides while being held at $u = 0$ along the other three sides, and heat then flows into the plate according to $u_t = \Delta u$. When does the temperature reach $u = 1$ at the center of the plate?

9. The integral $I(\alpha) = \int_0^2 [2 + \sin(10\alpha)] x^\alpha \sin(\alpha/(2-x))\, dx$ depends on the parameter α. What is the value $\alpha \in [0, 5]$ at which $I(\alpha)$ achieves its maximum?

10. A particle at the center of a 10×1 rectangle undergoes Brownian motion (i.e., 2D random walk with infinitesimal step lengths) till it hits the boundary. What is the probability that it hits at one of the ends rather than at one of the sides?

Solutions should be sent to Nick Trefethen at Oxford University (LNT@comlab.ox.ac.uk), no later than May 20, 2002.

Another (innocent looking) combinatorial question

Let $\mathscr{S} = \{\uparrow, \leftarrow, \searrow\}$. An $\mathscr{S}$-walk is a path in $\mathbb{Z}^2$ using only steps from $\mathscr{S}$. Show that, for any integer n, the following quantities are equal:

(i) number a_n of n-steps $\mathscr{S}$-walks confined to the upper half plane $\mathbb{Z} \times \mathbb{N}$ that start and finish at the origin $(0, 0)$ (*excursions*);

(ii) number b_n of n-steps $\mathscr{S}$-walks confined to the quarter plane $\mathbb{N}^2$ that start at the origin $(0, 0)$ and finish on the diagonal of $\mathbb{N}^2$ (*diagonal walks*).

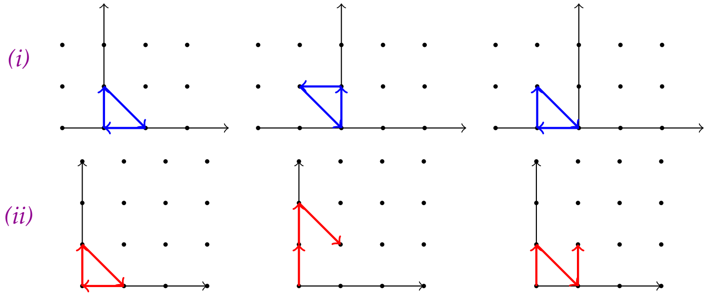
Another (innocent looking) combinatorial question

Let $\mathscr{S} = \{\uparrow, \leftarrow, \searrow\}$. An $\mathscr{S}$-walk is a path in $\mathbb{Z}^2$ using only steps from $\mathscr{S}$. Show that, for any integer n, the following quantities are equal:

(i) number a_n of n-steps $\mathscr{S}$-walks confined to the upper half plane $\mathbb{Z} \times \mathbb{N}$ that start and finish at the origin $(0,0)$ (*excursions*);

(ii) number b_n of n-steps $\mathscr{S}$-walks confined to the quarter plane $\mathbb{N}^2$ that start at the origin $(0,0)$ and finish on the diagonal of $\mathbb{N}^2$ (*diagonal walks*).

For instance, for $n = 3$, this common value is $a_3 = b_3 = 3$:



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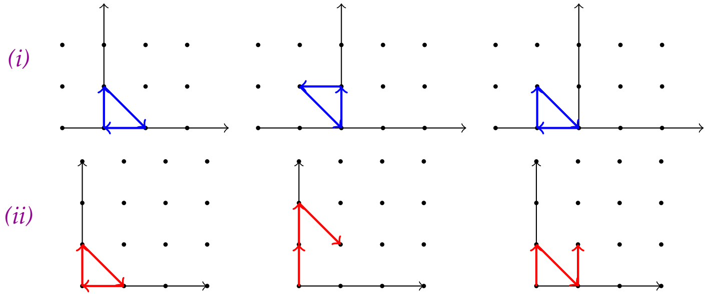

▷ Can be solved using Exp. Math. and Computer Algebra

A money changing problem

Question[†]: The number of ways one can change any amount of banknotes of 10 €, 20 €, … using coins of 50 cents, 1 € and 2 € is always a perfect square.

† Inspired by Pb. 1, Ch. 1, p. 1, vol. 1 of Pólya and Szegö's Problems Book (1925).

▷ Can be solved using Exp. Math. and Computer Algebra

Def. A *partition* of $n \in \mathbb{N}^\star$ is a representation $n = x_1 + x_2 + \cdots + x_k$ as a sum of positive integers $x_1 \geq \cdots \geq x_k$.

▷ Denote by $p(n)$ the number of partitions of n (by convention, $p(0) = 1$)

E.g., there are $p(5) = 7$ partitions of $n = 5$:

$5 = 4+1 = 3+2 = 3+1+1 = 2+2+1 = 2+1+1+1 = 1+1+1+1+1$

Euler's pentagonal theorem – masterpiece of (early) experimental math

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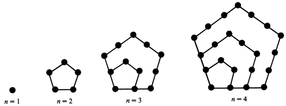
Pentagonal theorem [Euler, 1780] The inverse of the generating function

$$\sum_{n\geq 0} p(n)x^n = 1 + x + 2\,x^2 + 3\,x^3 + 5\,x^4 + 7\,x^5 + 11\,x^6 + 15\,x^7 + \cdots$$

is equal to

$$\prod_{n=1}^{\infty}(1-x^n) = 1 + \sum_{n=1}^{\infty}(-1)^n \left(x^{n(3n+1)/2} + x^{n(3n-1)/2}\right)$$

$$= 1 - x^1 - x^2 + x^5 + x^7 - x^{12} - x^{15} + x^{22} + \cdots$$



Euler's pentagonal theorem – masterpiece of (early) experimental math

One of Euler's most profound discoveries, the *Pentagonal Number Theorem* [7], has been beautifully described by André Weil:

> Playing with series and products, he discovered a number of facts which to him looked quite isolated and very surprising. He looked at this infinite product
>
> $$(1-x)(1-x^2)(1-x^3)\cdots$$
>
> and just formally started expanding it. He had many products and series of that kind; in some cases he got something which showed a definite law, and in other cases things seemed to be rather random. But with this one, he was very successful. He calculated at least fifteen or twenty terms; the formula begins like this:
>
> $$\Pi(1-x^n) = 1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} \cdots$$
>
> where the law, to your untrained eyes, may not be immediately apparent at first sight. In modern notation, it is as follows:
>
> $$\prod_1^{\infty}(1-q^n) = \sum_{-\infty}^{+\infty}(-1)^n q^{n(3n+1)/2} \qquad (1)$$
>
> where I've changed x into q since q has become the standard notation in elliptic function-theory since Jacobi. The exponents make up a progression of a simple nature. This became immediately apparent to Euler after writing down some 20 terms; quite possibly he calculated about a hundred. He very reasonably says, "this is quite certain, although I cannot prove it;" ten years later he does prove it. He could not possibly guess that both series and product are part of the theory of elliptic modular functions. It is another tie-up between number-theory and elliptic functions [**22**, pp. 97–98].

[Andrews, 1983]

Euler's pentagonal theorem – a short digression

Thm [Kolberg, 1959] $p(n)$ takes both even and odd values infinitely often.

Modern proof. Assume the contrary. Then, $\sum_{k=-\infty}^{\infty} (-1)^k x^{k(3k-1)/2}$ is a rational function modulo 2. By Christol's thm. (1978), the sequence of coefficients would be 2-automatic. Contradiction with Ritchie's thm (1963).

▷ Conjecture [Subbarao, 1966]: same *for any arithmetic progression* $p(an+b)$.

▷ Proved by [Ono, 1996] (even case) and [Radu, 2012] (odd case)

▷ Transcendence and automata version by [B., Radu, 2019]

Thm [Ramanujan, 1919] The numbers in the arithmetic progression

$$(p(5n+4))_n = (5, 30, 135, 490, 1575, 4565, 12310, 31185, \ldots)$$

are all divisible by 5.

Euler's pentagonal theorem – another short digression

Thm [Ramanujan, 1919] The numbers in the arithmetic progression

$$(p(5n+4))_n = (5, 30, 135, 490, 1575, 4565, 12310, 31185, \ldots)$$

are all divisible by 5.

▷ This is a simple consequence of the following identity

$$\sum_{n=0}^{\infty} p(5n+4)x^n = 5 \prod_{n=1}^{\infty} \frac{1-x^{5n}}{(1-x^n)^6}$$

also conjectured by Ramanujan.

▷ Radu's algorithms can automatically discover this kind of identity!

Two beautiful Monthly (AMM) problems


THE AMERICAN MATHEMATICAL
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An Official Publication of the Mathematical Association of America

Expansion of a Symmetric Determinant

E 2297 [1971, 543]. *Proposed by Richard Stanley, Harvard University*

Let $L(n)$ be the total number of distinct monomials appearing in the expansion of the determinant of an $n \times n$ symmetric matrix $A = (a_{ij})$. For instance, $L(3) = 5$. Show that

$$\sum_{n=0}^{\infty} L(n)x^n/n! = (1-x)^{-1/2} \exp(\tfrac{1}{2}x + \tfrac{1}{4}x^2),$$

where $|x| < 1$, and where we define $L(0) = 1$.

The First Third

6637 [1990, 621]. *Proposed by Herbert S. Wilf, University of Pennsylvania, Philadelphia, PA.*

Let $f(n)$ be the sum of the first one-third of the coefficients in the expansion of $(1+x)^{3n}$, i.e.,

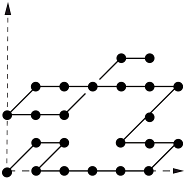
$$f(n) = \sum_{k=0}^{n} \binom{3n}{k} \qquad (n = 0, 1, 2, \ldots).$$

Prove that

$$\sum_{n=0}^{\infty} f(n) \left(\frac{4u^2}{27} \right)^n = \frac{u}{u - 2\sin(\frac{1}{3}\arcsin u)} - \frac{2u}{2u - 3\sin(\frac{1}{3}\arcsin u)}.$$

Gessel walks (2000)

- $g(n)$ = number of n-steps $\{\nearrow, \swarrow, \leftarrow, \rightarrow\}$-walks in $\mathbb{N}^2$
  $1, 2, 7, 21, 78, 260, 988, 3458, 13300, 47880, \ldots$



Question: What is the nature of the generating function

$$G(t) = \sum_{n=0}^{\infty} g(n)\, t^n \ ?$$

Gessel walks (2000)

- $g(i,j;n)$ = number of n-steps $\{\nearrow, \swarrow, \leftarrow, \rightarrow\}$-walks in $\mathbb{N}^2$ from $(0,0)$ to (i,j)

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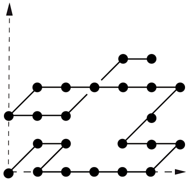
$$G(x,y;t) = \sum_{i,j,n=0}^{\infty} g(i,j;n)\, x^i y^j t^n \ ?$$



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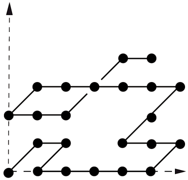

Theorem [B., Kauers, 2010]

$G(x,y;t)$ is an algebraic function†.

▷ computer-driven discovery/proof via *algorithmic Guess-and-Prove*

† Minimal polynomial $P(G(x,y;t);x,y,t) = 0$ has $> 10^{11}$ terms; $\approx 30\,\text{Gb}$ (6 DVDs!)

- $g(n)$ = number of n-steps $\{\nearrow, \swarrow, \leftarrow, \rightarrow\}$-walks in $\mathbb{N}^2$
  1, 2, 7, 21, 78, 260, 988, 3458, 13300, 47880, ...

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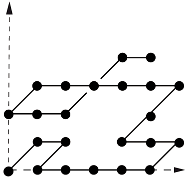

Corollary [B., Kauers, 2010] (former conjecture of Gessel's)

$(3n+1)\, g(2n) = (12n+2)\, g(2n-1)$ and $(n+1)\, g(2n+1) = (4n+2)\, g(2n)$

▷ computer-driven discovery/proof via *algorithmic Guess-and-Prove*

An Integrality Question

Question: Let (a_n) be a sequence with $a_0 = a_1 = 1$ satisfying the recurrence

$$(n+3)a_{n+1} = (2n+3)a_n + 3na_{n-1}.$$

Show that all a_n is an integer for all n.

▷ Computer-aided solution: Let's compute the first 10 terms of the sequence:

```
> rec:=(n+3)*a(n+1)-(2*n+3)*a(n)-3*n*a(n-1): ini:=a(0)=1,a(1)=1:
> pro:=gfun:-rectoproc({rec,ini}, a(n), list);
> pro(10);
```

$$[1, 1, 2, 4, 9, 21, 51, 127, 323, 835, 2188]$$

▷ gfun's `seriestoalgeq` command allows to guess that $\sum_n a_n x^n$ is algebraic:

```
> pol:=gfun:-listtoalgeq(%,y(x))[1];
```

$$1 + (x-1)y(x) + x^2 y(x)^2$$

Question: Let (a_n) be a sequence with $a_0 = a_1 = 1$ satisfying the recurrence

$$(n+3)a_{n+1} = (2n+3)a_n + 3na_{n-1}.$$

Show that all a_n is an integer for all n.

▷ Thus it is very likely that $y = \sum_{n\geq 0} a_n x^n$ verifies $1 + (x-1)y + x^2y^2 = 0$.
▷ By coefficient extraction, (a_n) *conjecturally* verifies the nonlinear recurrence

$$a_{n+2} = a_{n+1} + \sum_{k=0}^{n} a_k \cdot a_{n-k}. \tag{1}$$

▷ Clearly (1) implies $a_n \in \mathbb{N}$. *To prove* (1), we proceed the other way around: we start with $P(x,y) = 1 + (x-1)y + x^2y^2$, and show that it admits a power series solution whose coefficients satisfy the same linear recurrence as (a_n):

```
> deq:=gfun:-algeqtodiffeq(pol,y(x)):
> recb:=gfun:-diffeqtorec(deq,y(x),b(n));
```

$$\{(3\,n+3)\,b(n) + (2\,n+5)\,b\,(n+1) - (n+4)\,b\,(n+2)\,,\; b\,(0) = 1,\, b\,(1) = 1\}$$

Other Integrality Questions

Let m and n be nonnegative integers. Prove that the following are integers:

- $\dfrac{(2m)!(2n)!}{m!n!(m+n)!}$ [Catalan, 1874]; [von Szily, 1894], [Feemster, 1910], [IMO 1972/3]

- $\dfrac{m!(2m+2n)!}{(2m)!n!(m+n)!}$ [Gessel, 1985]

- $\dfrac{(3m+3n)!(3n)!(2m)!(2n)!}{(2m+3n)!(m+2n)!(m+n)!m!n!^2}$ [Askey, 1986]

- $\dfrac{(5m)!(5n)!}{m!n!(3m+n)!(m+3n)!}$ [USAMO 1975]

▷ [Putnam, 1999/6]
The sequence $(a_n)_{n\geq 1}$ is defined by $a_1 = 1, a_2 = 2, a_3 = 24$, and, for $n \geq 4$,

$$a_n = \frac{6a_{n-1}^2 a_{n-3} - 8a_{n-1}a_{n-2}^2}{a_{n-2}a_{n-3}}.$$

Show that a_n is an integer multiple of n, for all n.

▷ [Romanian TST, 2004/10]
Prove that if $n, m \in \mathbb{N}^\star$ and m odd, the following number is an integer

$$\frac{1}{3^m\, n} \sum_{k=0}^{m} \binom{3m}{3k} (3n-1)^k.$$

A non-trivial non-vanishing

Thm. [Conca, Krattenthaler, Watanabe, 2009] For for any $h \geq 1$, the rational

$$a_h = \sum_{b=0}^{\lfloor h/3 \rfloor} \frac{(-1)^{h-b}}{h-b} \binom{h-b}{2\,b} \left(\frac{2}{3}\right)^b$$

is non-zero, except for $h = 3$.

▷ Exp. Math. proof [B., 2018]:

1. GF $\sum_{n \geq 3} a_h z^{h-3} = -\frac{5}{12} z + \frac{4}{5} z^2 - \frac{19}{18} z^3 + \cdots$ of $(a_h)_{h \geq 3}$ is equal to

$$-\frac{1}{z^3} \int \frac{(2\,z-5)\,z^3}{2\,z^3 - 3\,z^2 - 6\,z - 3}\, \mathrm{d}z.$$

2. The (integer) coefficients of $G(z) = \dfrac{6\,z-5}{18\,z^3 - 9\,z^2 - 6\,z - 1}$ are all $\neq 0$:
   1. Coefficients of $G_{\mathsf{even}}(z) = 5 + 171\,z + 1485\,z^2 + \cdots$ satisfy

   $$u_{n+3} = 324\,u_n - 297\,u_{n+1} + 18\,u_{n+2},$$

   thus $u_{n+3} = u_{n+1} \bmod 2$ for all n, so all coeffs of G_{even} are odd.
   2. Similarly, all coefficients of $G_{\mathsf{odd}}(z)/36 = 1 + 17\,z + 9\,z^2 + \cdots$ are all odd.

All roots on the unit circle

Conjecture [Furter]. For all n, the polynomial

$$P_n(x) = \sum_{i+j=n} \binom{n+i}{n} \binom{n+j}{n} x^i$$

has only roots of modulus 1.

$$2\,x + 2$$

$$6\,x^2 + 9\,x + 6$$

$$20\,x^3 + 40\,x^2 + 40\,x + 20$$

$$70\,x^4 + 175\,x^3 + 225\,x^2 + 175\,x + 70$$

$$252\,x^5 + 756\,x^4 + 1176\,x^3 + 1176\,x^2 + 756\,x + 252$$

$$924\,x^6 + 3234\,x^5 + 5880\,x^4 + 7056\,x^3 + 5880\,x^2 + 3234\,x + 924$$

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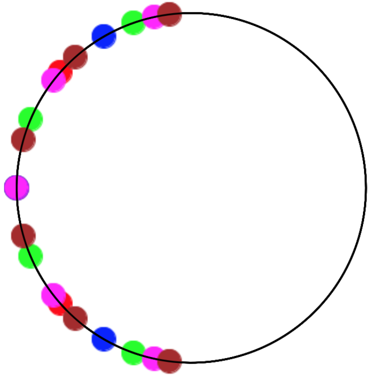
▷ Proved [B., 2019] using Experimental Mathematics.



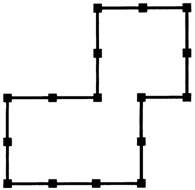
Pólya polygons

Def. Pairs (π, σ) of $\{\uparrow, \rightarrow\}$-paths in $\mathbb{N}^2$ of the same length n, such that:

(i) Both π and σ start at $(0,0)$ and end at the same point;

(ii) π begins with a $\uparrow$ step and σ with a $\rightarrow$ step;

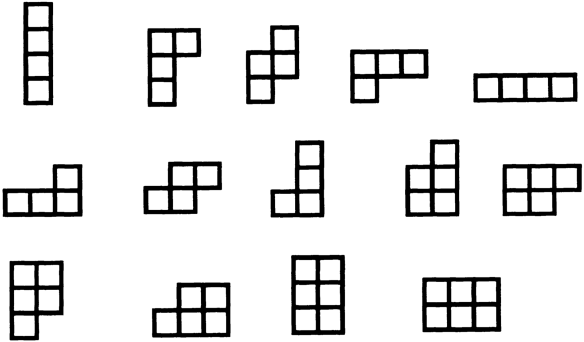
(iii) π and σ do not meet between the origin and their common endpoint.



Thm. [Levine, 1959; Pólya, 1969; Fürlinger, Hofbauer, 1985]

(i) The number of n-Pólya polygons is $C_{n-1} = \frac{1}{n}\binom{2n-2}{n-1}$

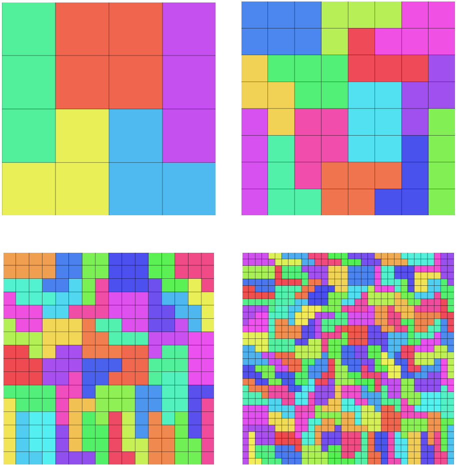
(ii) The total area of all n-Pólya polygons is 4^{n-1}.



Pólya polygons: a crazy conjecture

Conjecture [Schwärzler, 1985] One may tile a square of side 2^{n-1} with the n-Pólya polygons.

▷ Partial answer [Doligez, Sibut-Pinote, Varloot, 2016] This is true for $n \leq 7$.



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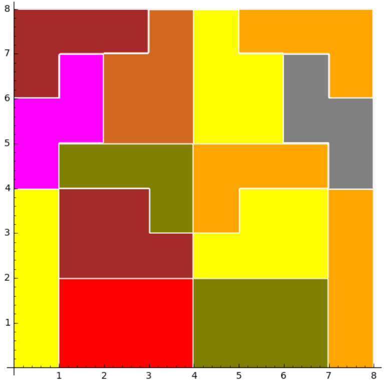
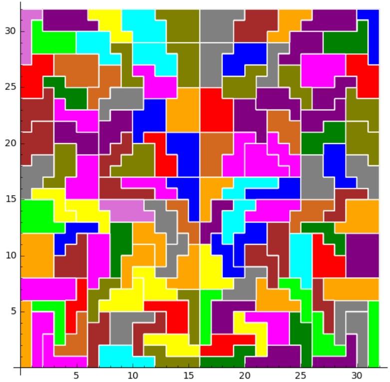
▷ Partial answer [Lemoine, Zimmermann, 2017] Even more is true for $n \leq 7$: one can find symmetric tilings!



$n = 5$



$n = 7$

Catalan's last digit: an amusing conjecture

Let

$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

be the nth Catalan number. Then:

- The last digit (in base 10) of C_n is never 3;
- For $n \gg 0$, the last digit of any odd C_n is always 5.

▷ Behavior of C_n modulo 2 and modulo 5 is well understood

The Bank of Bath and Dyck paths

IMO 2019
60TH INTERNATIONAL
MATHEMATICAL OLYMPIAD

English (eng), day 2

Wednesday, July 17, 2019

Problem 4. Find all pairs (k, n) of positive integers such that

$$k! = (2^n - 1)(2^n - 2)(2^n - 4) \cdots (2^n - 2^{n-1}).$$

Problem 5. The Bank of Bath issues coins with an H on one side and a T on the other. Harry has n of these coins arranged in a line from left to right. He repeatedly performs the following operation: if there are exactly $k > 0$ coins showing H, then he turns over the k^{th} coin from the left; otherwise, all coins show T and he stops. For example, if $n = 3$ the process starting with the configuration THT would be $THT \to HHT \to HTT \to TTT$, which stops after three operations.

(a) Show that, for each initial configuration, Harry stops after a finite number of operations.

(b) For each initial configuration C, let $L(C)$ be the number of operations before Harry stops. For example, $L(THT) = 3$ and $L(TTT) = 0$. Determine the average value of $L(C)$ over all 2^n possible initial configurations C.

https://www.imo-official.org/year_info.aspx?year=2019

The Mykonos reformulation

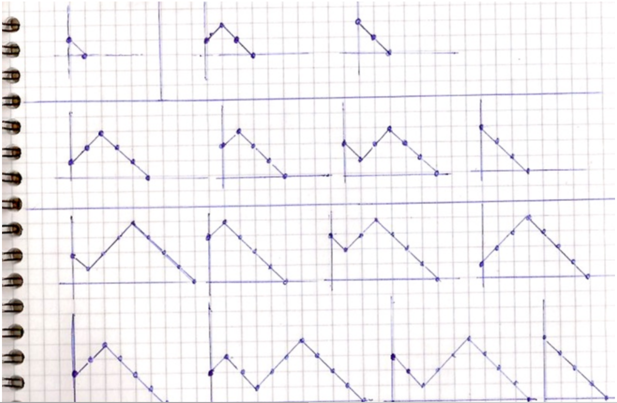
The Mykonos reformulation

Let D_N be the set of $\{\nearrow, \searrow\}$-paths in $\mathbb{N}^2$, such that:

(i) they start from the vertical axis and end at the horizontal axis;
(ii) their maximum height is N;
(iii) each "turn" is bigger than the preceding one.

Show that

(i) there are exactly 2^N paths in D_N;
(ii) the sum of the lengths of walks in D_N is $2^{N-1}\binom{N+1}{2}$;



Two beautiful identities of Ramanujan's

Question: What is the value of

$$\frac{\sin \frac{2\pi}{7}}{\sin^2 \frac{3\pi}{7}} - \frac{\sin \frac{\pi}{7}}{\sin^2 \frac{2\pi}{7}} + \frac{\sin \frac{3\pi}{7}}{\sin^2 \frac{\pi}{7}} \ ?$$

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Answer:

$$2\sqrt{7}$$

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Answer:

$$2\sqrt{7}$$

Question: What is the value of

$$\sqrt[3]{\cos\frac{2\pi}{7}} + \sqrt[3]{\cos\frac{4\pi}{7}} + \sqrt[3]{\cos\frac{8\pi}{7}} \text{ ?}$$

Two beautiful identities of Ramanujan's

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Answer:

$$\sqrt[3]{\frac{5-3\sqrt[3]{7}}{2}}.$$

1. What is the value of

$$\cos\left(\frac{\pi}{7}\right) - \cos\left(\frac{2\,\pi}{7}\right) + \cos\left(\frac{3\,\pi}{7}\right)\ ?$$

2. Show that number of ways one can change any amount of banknotes of 10 €, 20 €, … using coins of 50 cents, 1 € and 2 € is always a perfect square.

$3^{\star\star\star}$. Show that if a, b, q are positive integers with

$$q = \frac{a^2 + b^2}{ab + 1}$$

then q is a perfect square.

4. Let m, n be nonnegative integers. Prove that the following is an integer:

$$\frac{(2m)!(2n)!}{m!n!(m+n)!}$$

5*. Let $f(n)$ be the sum of the first one-third of the coefficients in the expansion of $(1+x)^{3n}$, i.e., $f(n) = \sum_{k=0}^{n} \binom{3n}{k}$, for $n = 0, 1, 2, \ldots$ Prove that

$$\sum_{n=0}^{\infty} f(n) \left(\frac{4u^2}{27}\right)^n = \frac{u}{u - 2\sin\left(\frac{1}{3}\arcsin u\right)} - \frac{2u}{2u - 3\sin\left(\frac{1}{3}\arcsin u\right)}$$

Hint: Prove that $\displaystyle\sum_{n=0}^{\infty} f(n) \frac{a^n}{(1+a)^{3n+1}} = \frac{1}{(1-a)(1-2a)}$ for $|a| < \frac{1}{2}$

6**. Prove that if $n, m \in \mathbb{N}^\star$ and m odd, the following number is an integer

$$\frac{1}{3^m\, n} \sum_{k=0}^{m} \binom{3m}{3k} (3n-1)^k.$$