Overview

Lecture 1: Context, Motivation, Examples
Lecture 2: Exp. Math. for Combinatorics
Lecture 3: Inside the Exp. Math. Toolbox
Lecture 2: Exp. Math. for Combinatorics

Efficient experimental mathematics for combinatorics and number theory
Computer Algebra for Enumerative Combinatorics: a showcase of Experimental Mathematics

**Enumerative Combinatorics:** science of counting

Area of mathematics primarily concerned with counting discrete objects.

▷ Main outcome: theorems

**Computer Algebra:** effective mathematics

Area of computer science primarily concerned with the algorithmic manipulation of algebraic objects.

▷ Main outcome: algorithms

**Computer Algebra for Enumerative Combinatorics ⊂ Experimental Math.**

An (innocent looking) combinatorial question

Let $\mathcal{I} = \{\uparrow, \leftarrow, \searrow\}$. An $\mathcal{I}$-walk is a path in $\mathbb{Z}^2$ using only steps from $\mathcal{I}$. Show that, for any integer $n$, the following quantities are equal:

(i) number $a_n$ of $n$-steps $\mathcal{I}$-walks confined to the upper half plane $\mathbb{Z} \times \mathbb{N}$ that start and finish at the origin $(0, 0)$ (excursions);

(ii) number $b_n$ of $n$-steps $\mathcal{I}$-walks confined to the quarter plane $\mathbb{N}^2$ that start at the origin $(0, 0)$ and finish on the diagonal of $\mathbb{N}^2$ (diagonal walks).

For instance, for $n = 3$, this common value is $a_3 = b_3 = 3$: 
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For instance, for $n = 3$, this common value is $a_3 = b_3 = 3$:

(i)

(ii)
Teaser 1: This “exercise” is non-trivial

Teaser 2: It can be solved using Experimental Math and Computer Algebra

Teaser 3: ... by two robust and efficient algorithmic techniques, Guess-and-Prove and Creative Telescoping
Why care about counting walks?

Many objects can be encoded by walks:

- probability theory (voting, games of chance, branching processes, …)
- discrete mathematics (permutations, trees, words, urns, …)
- statistical physics (Ising model, …)
- operations research (queueing theory, …)
Suppose that candidates $A$ and $B$ are running in an election. If $a$ votes are cast for $A$ and $b$ votes are cast for $B$, where $a > b$, then the probability that $A$ stays ahead of $B$ throughout the counting of the ballots is $(a - b)/(a + b)$.

**Lattice path reformulation**: find the number of paths in $\mathbb{Z}^2$ with $a$ upsteps $\uparrow$ and $b$ downsteps $\downarrow$ that start at the origin and never touch the $x$-axis.
Counting walks is an old topic: the ballot problem [Bertrand, 1887]

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Lattice path reformulation: find the number of paths in $\mathbb{Z}^2$ with $a - 1$ upsteps $\uparrow$ and $b$ downsteps $\downarrow$ that start at $(1, 1)$ and never touch the $x$-axis

Reflection principle [Aebly, 1923]: paths in $\mathbb{Z}^2$ from $(1, 1)$ to $T(a + b, a - b)$ that do touch the $x$-axis are in bijection with paths in $\mathbb{Z}^2$ from $(1, -1)$ to $T$

Answer: \[
\binom{a + b - 1}{a - 1} - \binom{a + b - 1}{b - 1}
\]
Counting walks is an old topic: the ballot problem [Bertrand, 1887]

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Lattice path reformulation: find the number of paths in $\mathbb{Z}^2$ with $a - 1$ upsteps $\uparrow$ and $b$ downsteps $\downarrow$ that start at $(1, 1)$ and never touch the $x$-axis.

Reflection principle [Aebly, 1923]: paths in $\mathbb{Z}^2$ from $(1, 1)$ to $T(a + b, a - b)$ that do touch the $x$-axis are in bijection with paths in $\mathbb{Z}^2$ from $(1, -1)$ to $T$.

Answer: \[
\left( \binom{a + b - 1}{a - 1} \right) - \left( \binom{a + b - 1}{b - 1} \right) = \frac{a - b}{a + b} \binom{a + b}{a}
\]
Lot of recent activity; many recent contributors:

Arquès, Bacher, Banderier, Beaton, Bernardi, Bostan, Bousquet-Mélou,
Buchacher, Budd, Chyzak, Cori, Courtiel, Denisov, Dreyfus, Du, Duchon,
Dulucq, Duraj, Fayolle, Fisher, Flajolet, Fusy, Garbit, Gessel,
Gouyou-Beauchamps, Guttmann, Guy, Hardouin, van Hoeij, Hou,
Iasnogorodski, Johnson, Kauers, Kenyon, Koutschan, Krattenthaler,
Kreweras, Kurkova, Lecouvey, Malyshev, Melczer, Miller, Mishna,
Niederhausen, Owczarek, Pech, Petkovšek, Prellberg, Raschel, Rechnitzer,
Roques, Sagan, Salvy, Sheffield, Singer, Tarrago, Viennot, Wachtel, Wallner,
Wang, Wilf, D. Wilson, M. Wilson, Yatchak, Xu, Yeats, Zeilberger, . . .

etc.
...but it is still a very hot topic

Lot of recent activity; many recent contributors:


etc.

Specific question
Ad hoc solution

Systematic approach
Chapter 10

Lattice Path Enumeration

Christian Krattenthaler
Universität Wien

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Our approach: Experimental Mathematics using Computer Algebra
Our approach: Experimental Mathematics using Computer Algebra

Algorithmes Efficaces en Calcul Formel

Alin Bostan
Frédéric Chyzak
Marc Giusti
Romain Lebreton
Grégoire Lecerf
Bruno Salvy
Éric Schost
Lattice walks with small steps in the quarter plane

▶ Nearest-neighbor walks in the quarter plane: 
\( \mathcal{S} \)-walks in \( \mathbb{N}^2 \): starting at \((0,0)\) and using steps in a fixed subset \( \mathcal{S} \) of 
\[
\{ \searrow, \leftarrow, \swarrow, \uparrow, \nearrow, \rightarrow, \nwarrow, \downarrow \}
\]

▶ Counting sequence \( q_{\mathcal{S}}(n) \): number of \( \mathcal{S} \)-walks of length \( n \)

▶ Generating function:
\[
Q_{\mathcal{S}}(t) = \sum_{n=0}^{\infty} q_{\mathcal{S}}(n) t^n \in \mathbb{Z}[[t]]
\]
Lattice walks with small steps in the quarter plane

▷ Nearest-neighbor walks in the quarter plane:
\( \mathcal{S} \)-walks in \( \mathbb{N}^2 \): starting at \((0,0)\) and using steps in a fixed subset \( \mathcal{S} \) of
\[
\{ \searrow, \leftarrow, \swarrow, \uparrow, \nearrow, \rightarrow, \nwarrow, \downarrow \}
\]

▷ Counting sequence \( q_{\mathcal{S}}(i,j;n) \): number of walks of length \( n \) ending at \((i,j)\)

▷ Complete generating function (with “catalytic” variables \( x, y \)):
\[
Q_{\mathcal{S}}(x,y;t) = \sum_{i,j,n=0}^{\infty} q_{\mathcal{S}}(i,j;n) x^i y^j t^n \in \mathbb{Z}[\![x,y,t]\!]
\]
Entire books dedicated to small step walks in the quarter plane!
Small-step models of interest

Among the $2^8$ step sets $\mathcal{S} \subseteq \{-1,0,1\}^2 \setminus \{(0,0)\}$, some are:

- trivial
- simple
- intrinsic to the half plane
- symmetrical

One is left with 79 interesting distinct models.
Small-step models of interest

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Among the $2^8$ step sets $\mathcal{S} \subseteq \{-1, 0, 1\}^2 \setminus \{(0, 0)\}$, some are:

- trivial,
- simple,
- intrinsic to the half plane,
- symmetrical.

One is left with 79 interesting distinct models.
The 79 small steps models of interest
Task: classify their generating functions!

Non-singular

Singular
Classification criterion: properties of generating functions

- Algebraic
- Hypergeometric
- Differentially finite (holonomic)
- Differentially algebraic
Classification criterion: properties of generating functions

- Algebraic
- Hypergeometric
- Differentially finite (holonomic)
- Differentially algebraic

\[(1 - t)\alpha\]

Efficient experimental mathematics for combinatorics and number theory
Classification criterion: properties of generating functions

- Algebraic
- Hypergeometric
- Differentially finite (holonomic)
- Differentially algebraic

\[(1-t)^{\alpha} \sqrt{1-t} + \sqrt[3]{1-2t}\]
Classification criterion: properties of generating functions

- algebraic
- hypergeometric
- differentially finite (holonomic)
- differentially algebraic

\[
\sqrt{1 - t} + \sqrt[3]{1 - 2t}
\]

\[
\ln(1 - t)
\]

\[
(1 - t)^{\alpha}
\]
Classification criterion: properties of generating functions

\begin{align*}
2F_1 \left( \begin{array}{c} a \\ b \\ c \end{array} \right | t \right) := \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} t^n, \quad \text{where} \quad (a)_n = a(a+1) \cdots (a+n-1).
\end{align*}
Classification criterion: properties of generating functions

- **algebraic**
- **hypergeometric**
- **differentially finite (holonomic)**
- **differentially algebraic**

E.g., \((1-t)^\alpha = 2F_1\left(\begin{array}{c} -\alpha, 1 \\ 1 \end{array} \bigg| t\right)\), \(\ln(1-t) = -t \cdot 2F_1\left(\begin{array}{c} 1, 1 \\ 2 \end{array} \bigg| t\right) = - \sum_{n=1}^{\infty} \frac{t^n}{n}\)
Classification criterion: properties of generating functions

\[ \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{t^n}{n!}, \text{ where } (a)_n = a(a+1) \cdots (a+n-1). \]
Classification criterion: properties of generating functions

\[ 2F_1 \left( \begin{array}{c} a \\ c \end{array} \left| \frac{b}{t} \right. \right) := \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n} \frac{t^n}{n!}, \text{ where } (a)_n = a(a+1) \cdots (a+n-1). \]
Algebraic reformulation of main task: solving a functional equation

Generating function: \( Q(x, y) \equiv Q(x, y; t) = \sum_{i,j,n=0}^{\infty} q(i, j; n)x^iy^jt^n \in \mathbb{Z}[[x, y, t]] \)

Recursive construction yields the kernel equation

\[
Q(x, y) = 1 + t \left( y + \frac{1}{x} + x\frac{1}{y} \right)Q(x, y) - t\frac{1}{x}Q(0, y) - tx\frac{1}{y}Q(x, 0)
\]
Algebraic reformulation of main task: solving a functional equation

Generating function: \( Q(x,y) \equiv Q(x,y;t) = \sum_{i,j,n=0}^{\infty} q(i,j;n) x^i y^j t^n \in \mathbb{Z}[[x,y,t]] \)

Recursive construction yields the kernel equation

\[
\left( 1 - t \left( y + \frac{1}{x} + x \frac{1}{y} \right) \right) xyQ(x,y) = xy - tyQ(0,y) - tx^2 Q(x,0)
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\]

New task: Solve this functional equation!
Algebraic reformulation of main task: solving a functional equation

Generating function: \( Q(x, y) \equiv Q(x, y; t) = \sum_{i,j,n=0}^{\infty} q(i, j; n) x^i y^j t^n \in \mathbb{Z}[[x, y, t]] \)

Recursive construction yields the kernel equation

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\]

New task: For the other models – solve 78 similar equations!
“Special” models of walks in the quarter plane

Dyck: 

Motzkin: 

Pólya: 

Kreweras: 

Gessel: 

Gouyou-Beauchamps: 

King walks: 

Tandem walks: 

Alin Bostan
Efficient experimental mathematics for combinatorics and number theory
Gessel walks (2000)

- $g(n) =$ number of $n$-steps \{↗,↙,←,→\}-walks in $\mathbb{N}^2$
  
  1, 2, 7, 21, 78, 260, 988, 3458, 13300, 47880, …

**Question:** What is the nature of the generating function

$$G(t) = \sum_{n=0}^{\infty} g(n) t^n$$
• $g(i,j;n) =$ number of $n$-steps $\{\nearrow, \searrow, \leftarrow, \rightarrow\}$-walks in $\mathbb{N}^2$ from $(0,0)$ to $(i,j)$

**Question:** What is the nature of the generating function

$$G(x,y;t) = \sum_{i,j,n=0}^{\infty} g(i,j;n) x^i y^j t^n$$
Gessel walks (2000)

- $g(i, j; n) =$ number of $n$-steps $\{\uparrow, \downarrow, \leftarrow, \rightarrow\}$-walks in $\mathbb{N}^2$ from $(0, 0)$ to $(i, j)$

**Question:** What is the nature of the generating function

$$G(x, y; t) = \sum_{i,j,n=0}^{\infty} g(i, j; n) x^i y^j t^n$$

**Theorem [B., Kauers, 2010]**

$G(x, y; t)$ is an algebraic function$^\dagger$.

▶ computer-driven discovery/proof via *algorithmic Guess-and-Prove

$^\dagger$ Minimal polynomial $P(G(x, y; t); x, y, t) = 0$ has $> 10^{11}$ terms; $\approx 30$ Gb (6 DVDs!)
Gessel walks (2000)

- \( g(n) \) = number of \( n \)-steps \( \{↗,↙,←,→\} \)-walks in \( \mathbb{N}^2 \)

**Question**: What is the nature of the generating function

\[
G(t) = \sum_{n=0}^{\infty} g(n) t^n
\]

**Corollary [B., Kauers, 2010]** (former conjecture of Gessel’s)

- \((3n + 1) g(2n) = (12n + 2) g(2n - 1)\) and \((n + 1) g(2n + 1) = (4n + 2) g(2n)\)

▷ computer-driven discovery/proof via *algorithmic Guess-and-Prove*
What is “scientific method”? Philosophers and non-philosophers have discussed this question and have not yet finished discussing it. Yet as a first introduction it can be described in three syllables:

**Guess and test.**

Mathematicians too follow this advice in their research although they sometimes refuse to confess it. They have, however, something which the other scientists cannot really have. For mathematicians the advice is

**First guess, then prove.**
Guessing and Proving

George Pólya

What is “scientific method”? Philosophers and non-philosophers have discussed this question and have not yet finished discussing it. Yet as a first introduction it can be described in three syllables:

Guess and test.

Mathematicians too follow this advice in their research although they sometimes refuse to confess it. They have, however, something which the other scientists cannot really have. For mathematicians the advice is

First guess, then prove.

generate data → make conjectures → prove them
Question: Find $B_{i,j} :=$ the number of $\{\to, \uparrow\}$-walks in $\mathbb{N}^2$ from $(0,0)$ to $(i,j)$
**Guess-and-Prove: a toy example**

**Question:** Find $B_{i,j} :=$ the number of $\{\rightarrow, \uparrow\}$-walks in $\mathbb{N}^2$ from $(0,0)$ to $(i,j)$

1. There are 2 ways to get to $(i,j)$, either from $(i-1,j)$, or from $(i,j-1)$:
   \[ B_{i,j} = B_{i-1,j} + B_{i,j-1} \]

2. There is only one way to get to a point on an axis: $B_{i,0} = B_{0,j} = 1$

▷ These two rules completely determine all the numbers $B_{i,j}$
Guess-and-Prove: a toy example

Question: Find $B_{i,j} :=$ the number of $\{\rightarrow, \uparrow\}$-walks in $\mathbb{N}^2$ from $(0,0)$ to $(i,j)$

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$$B_{i,j} = B_{i-1,j} + B_{i,j-1}$$

2. There is only one way to get to a point on an axis: $B_{i,0} = B_{0,j} = 1$

These two rules completely determine all the numbers $B_{i,j}$

\[
\begin{array}{cccccccc}
1 & 7 & 28 & 84 & 210 & 462 & 924 \\
1 & 6 & 21 & 56 & 126 & 252 & 462 \\
1 & 5 & 15 & 35 & 70 & 126 & 210 \\
1 & 4 & 10 & 20 & 35 & 56 & 84 \\
1 & 3 & 6 & 10 & 15 & 21 & 28 \\
1 & 2 & 3 & 4 & 5 & 6 & 7 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & \ldots
\end{array}
\]

(l) Generate data:
Question: Find $B_{i,j} :=$ the number of $\{\rightarrow, \uparrow\}$-walks in $\mathbb{N}^2$ from $(0,0)$ to $(i,j)$

1. There are 2 ways to get to $(i,j)$, either from $(i-1,j)$, or from $(i,j-1)$:
   
   $$B_{i,j} = B_{i-1,j} + B_{i,j-1}$$

2. There is only one way to get to a point on an axis: $B_{i,0} = B_{0,j} = 1$

$\Rightarrow$ These two rules completely determine all the numbers $B_{i,j}$

(I) Generate data:

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(II) Guess:

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Question: Find $B_{i,j} :=$ the number of $\{\rightarrow, \uparrow\}$-walks in $\mathbb{N}^2$ from $(0, 0)$ to $(i, j)$

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(II) Guess:

$$B_{i,j} = \frac{(i+j)!}{i!j!}$$
Guess-and-Prove: a toy example

Question: Find $B_{i,j} := \text{the number of } \{\rightarrow, \uparrow\}\text{-walks in } \mathbb{N}^2 \text{ from } (0,0) \text{ to } (i,j)$

1. There are 2 ways to get to $(i,j)$, either from $(i-1,j)$, or from $(i,j-1)$:

   $$B_{i,j} = B_{i-1,j} + B_{i,j-1}$$

2. There is only one way to get to a point on an axis: $B_{i,0} = B_{0,j} = 1$

▷ These two rules completely determine all the numbers $B_{i,j}$

(1) Generate data:

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</table>

(III) Prove: If $C_{i,j} \overset{\text{def}}{=} \frac{(i+j)!}{i!j!}$, then

$$\frac{C_{i-1,j}}{C_{i,j}} + \frac{C_{i,j-1}}{C_{i,j}} = \frac{i}{i+j} + \frac{j}{i+j} = 1$$

and $C_{i,0} = C_{0,j} = 1$.

Thus $B_{i,j} = C_{i,j}$
Guess-and-Prove for Gessel walks

• $g(i, j; n) = \text{number of } n\text{-steps } \{\uparrow, \downarrow, \leftarrow, \rightarrow\}\text{-walks in } \mathbb{N}^2 \text{ from } (0, 0) \text{ to } (i, j)$

**Question:** What is the nature of the generating function

$$G(x, y; t) = \sum_{i,j,n=0}^{\infty} g(i, j; n) x^i y^j t^n$$

**Answer:** [B., Kauers, 2010] $G(x, y; t)$ is an algebraic function\(^\dagger\).

**Approach:**

1. **Generate data:** compute $G$ to precision $t^{1200}$ ($\approx 1.5$ billion coeffs!)
2. **Guess:** conjecture polynomial equations for $G(x, 0; t)$ and $G(0, y; t)$ (degree 24 each, coeffs. of degree $(46, 56)$, with 80-bits digits coeffs.)
3. **Prove:** multivariate resultants of (very big) polynomials (30 pages each)

\(^\dagger\) Minimal polynomial $P(G(x, y; t); x, y, t) = 0$ has $> 10^{11}$ terms; $\approx 30$ Gb (6 DVDs!)
A typical Guess-and-Prove algorithmic proof

Theorem [“Gessel excursions are algebraic”]

\[ g(t) := G(0, 0; \sqrt{t}) = \sum_{n=0}^{\infty} \frac{(5/6)_n (1/2)_n}{(5/3)_n (2)_n} (16t)^n \text{ is algebraic.} \]
A typical Guess-and-Prove algorithmic proof

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\]

**Proof**: First guess a polynomial \( P(t, T) \) in \( \mathbb{Q}[t, T] \), then prove that \( P \) admits the power series \( g(t) = \sum_{n=0}^{\infty} g_n t^n \) as a root.
A typical Guess-and-Prove algorithmic proof

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1. Find \( P \) such that \( P(t, g(t)) = 0 \mod t^{100} \) by (structured) linear algebra.
A typical Guess-and-Prove algorithmic proof

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3. \( r(t) = \sum_{n=0}^{\infty} r_n t^n \) being algebraic, it is D-finite, and so \( (r_n) \) is P-recursive:

\[
(n + 2)(3n + 5)r_{n+1} - 4(6n + 5)(2n + 1)r_n = 0, \quad r_0 = 1
\]

\[ \Rightarrow \text{solution } r_n = \frac{(5/6)^n(1/2)^n}{(5/3)^n(2)^n} 16^n = g_n, \text{ thus } g(t) = r(t) \text{ is algebraic.} \]
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\[
> \text{P:}=\text{gfun:-listtoalgeq([seq(pochhammer(5/6,n)*pochhammer(1/2,n)/pochhammer(5/3,n)/pochhammer(2,n)*}\text{16^n}, \text{n=0..100}], g(t))}: \\
> \text{gfun:-diffeqtorec(gfun:-algeqtodiffeq(P[1], g(t)), g(t), r(n))};
\]
Algorithmic classification of models with D-Finite $Q_\mathcal{F}(t) := Q_\mathcal{F}(1,1; t)$

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Equation sizes = (order, degree)

- Computerized discovery: enumeration + guessing [B., Kauers, 2009]
### Algorithmic classification of models with D-Finite $Q_\mathcal{F}(t) := Q_\mathcal{F}(1,1;t)$

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Equation sizes = (order, degree)

▷ Computerized discovery: enumeration + guessing [B., Kauers, 2009]
▷ 23: DF confirmed by a human proof in [B., Kurkova, Raschel, 2017]
▷ All: explicit eqs. proved via CA [B., Chyzak, van Hoeij, Kauers, Pech, 2017]
Algorithmic classification of models with D-Finite $Q_S(t) := Q_S(1,1;t)$

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$A = 1 + \sqrt{2}$, $B = 1 + \sqrt{3}$, $C = 1 + \sqrt{6}$, $\lambda = 7 + 3\sqrt{6}$, $\mu = \sqrt{\frac{4\sqrt{6}-1}{19}}$

▷ Computerized discovery: convergence acceleration + LLL [B., Kauers, ’09]
Algorithmic classification of models with D-Finite $Q_S(t) := Q_S(1, 1; t)$

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<td>A151266</td>
<td>N</td>
<td>$\frac{1}{2} \sqrt{\frac{3}{\pi}} \frac{3^n}{n^{1/2}}$</td>
<td>17</td>
<td>A001006</td>
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<td>$\frac{1}{2} \sqrt{\frac{5}{2\pi}} \frac{5^n}{n^{1/2}}$</td>
<td>18</td>
<td>A129400</td>
<td>Y</td>
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<tr>
<td>A151291</td>
<td>N</td>
<td>$\frac{4}{3\sqrt{\pi}} \frac{4^n}{n^{1/2}}$</td>
<td>19</td>
<td>A005558</td>
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<tr>
<td>A151326</td>
<td>N</td>
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<td>20</td>
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<td>Y</td>
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<tr>
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<td>21</td>
<td>A151278</td>
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<tr>
<td>A151329</td>
<td>N</td>
<td>$\frac{1}{3} \sqrt{\frac{7}{3\pi}} \frac{7^n}{n^{1/2}}$</td>
<td>22</td>
<td>A151323</td>
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<tr>
<td>A151261</td>
<td>N</td>
<td>$\frac{12\sqrt{3}}{\pi} \frac{(2\sqrt{3})^n}{n^2}$</td>
<td>23</td>
<td>A060900</td>
<td>Y</td>
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<tr>
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<td></td>
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</tr>
</tbody>
</table>

- $A = 1 + \sqrt{2}$, $B = 1 + \sqrt{3}$, $C = 1 + \sqrt{6}$, $\lambda = 7 + 3\sqrt{6}$, $\mu = \sqrt{\frac{4\sqrt{6} - 1}{19}}$

- Computerized discovery: convergence acceleration + LLL [B., Kauers, ’09]
- Asympt. confirmed by human proofs via ACSV in [Melczer, Wilson, 2016]
- Transcendence proofs via CA [B., Chyzak, van Hoeij, Kauers, Pech, 2017]
Theorem [B., Chyzak, van Hoeij, Kauers, Pech, 2017]

Let $\mathcal{I}$ be one of the models 1–19. Then

- $Q_{\mathcal{I}}(x, y; t)$ is expressible using iterated integrals of $\text{2F1}$ expressions.
- $Q_{\mathcal{I}}(x, y; t)$ is transcendental.
Theorem [B., Chyzak, van Hoeij, Kauers, Pech, 2017]

Let $\mathcal{S}$ be one of the models 1–19. Then

- $Q_{\mathcal{S}}(t)$ is expressible using iterated integrals of $\binom{2}{1}$ expressions.
- $Q_{\mathcal{S}}(t)$ is transcendental, except for $\mathcal{S} = \uparrow$ and $\mathcal{S} = \downarrow$.

Example (King walks in the quarter plane, A151331)

$$Q_{\downarrow\uparrow}(t) = \frac{1}{t} \int_0^t \frac{1}{(1+4x)^3} \cdot \binom{3}{2} \left( \frac{3}{2} \right) \binom{3}{2} \left( \frac{16x(1+x)}{(1+4x)^2} \right) dx$$

\[= 1 + 3t + 18t^2 + 105t^3 + 684t^4 + 4550t^5 + 31340t^6 + 219555t^7 + \cdots\]
Theorem [B., Chyzak, van Hoeij, Kauers, Pech, 2017]

Let $\mathcal{I}$ be one of the models 1–19. Then

- $Q_\mathcal{I}(t)$ is expressible using iterated integrals of $\,_2F_1$ expressions.
- $Q_\mathcal{I}(t)$ is transcendental, except for $\mathcal{I} = \begin{huge} \text{\ding{52}} \end{huge}$ and $\mathcal{I} = \begin{huge} \text{\ding{53}} \end{huge}$.

Example (King walks in the quarter plane, A151331)

$$Q_{\begin{huge} \text{\ding{53}} \end{huge}}(t) = \frac{1}{t} \int_0^t \frac{1}{(1+4x)^3} \cdot \,_2F_1\left(\frac{3}{2}, \frac{3}{2}; \frac{16x(1+x)}{(1+4x)^2}\right) dx$$

$$= 1 + 3t + 18t^2 + 105t^3 + 684t^4 + 4550t^5 + 31340t^6 + 219555t^7 + \cdots$$

- Computer-driven discovery and proof; no human proof yet.
- Proof uses: (1) kernel method + (2) creative telescoping.
The kernel $K(x, y; t) := 1 - t \cdot \sum_{i,j} x^i y^j = 1 - t \left( x + \frac{1}{x} + y + \frac{1}{y} \right)$ is left invariant under the change of $(x, y)$ into the elements of

$G_S := \left\{ (x, y), \left( \frac{1}{x}, y \right), \left( \frac{1}{x}, \frac{1}{y} \right), (x, \frac{1}{y}) \right\}$
The kernel $K(x, y; t) := 1 - t \cdot \sum_{(i,j) \in \mathcal{S}} x^i y^j = 1 - t \left( x + \frac{1}{x} + y + \frac{1}{y} \right)$ is left invariant under the change of $(x, y)$ into the elements of

$$
\mathcal{G}_\mathcal{S} := \left\{ (x, y), \left( \frac{1}{x}, y \right), \left( \frac{1}{x}, \frac{1}{y} \right), (x, \frac{1}{y}) \right\}
$$

Kernel equation:

$$
K(x, y; t)xyQ(x, y; t) = xy - txQ(x, 0; t) - tyQ(0, y; t)
$$
The kernel $K(x, y; t) := 1 - t \cdot \sum_{(i,j) \in S} x^i y^j = 1 - t \left( x + \frac{1}{x} + y + \frac{1}{y} \right)$ is left invariant under the change of $(x, y)$ into the elements of

$$G_S := \left\{ (x, y), \left( \frac{1}{x}, y \right), \left( x, \frac{1}{y} \right), (x, \frac{1}{y}) \right\}$$

Kernel equation:

$$K(x, y; t) xy Q(x, y; t) = xy - tx Q(x, 0; t) - ty Q(0, y; t) - K(x, y; t) \frac{1}{x} y Q\left( \frac{1}{x}, y; t \right) = - \frac{1}{x} y + t \frac{1}{x} Q\left( \frac{1}{x}, 0; t \right) + ty Q(0, y; t)$$
The kernel $K(x, y; t) := 1 - t \cdot \sum_{(i,j) \in \mathcal{S}} x^i y^j = 1 - t \left( x + \frac{1}{x} + y + \frac{1}{y} \right)$ is left invariant under the change of $(x, y)$ into the elements of

$$G_{\mathcal{S}} := \left\{ (x, y), \left( \frac{1}{x}, y \right), \left( \frac{1}{x}, \frac{1}{y} \right), \left( x, \frac{1}{y} \right) \right\}$$

Kernel equation:

$$K(x, y; t) xy Q(x, y; t) = xy - tx Q(x, 0; t) - ty Q(0, y; t)$$

$$- K(x, y; t) \frac{1}{x} y Q\left( \frac{1}{x}, y; t \right) = - \frac{1}{x} y + t \frac{1}{x} Q\left( \frac{1}{x}, 0; t \right) + ty Q(0, y; t)$$

$$K(x, y; t) \frac{1}{x} \frac{1}{y} Q\left( \frac{1}{x}, \frac{1}{y}; t \right) = \frac{1}{x} \frac{1}{y} - t \frac{1}{x} Q\left( \frac{1}{x}, 0; t \right) - t \frac{1}{y} Q(0, \frac{1}{y}; t)$$
The kernel $K(x, y; t) := 1 - t \cdot \sum_{(i,j) \in \mathcal{S}} x^i y^j = 1 - t \left( x + \frac{1}{x} + y + \frac{1}{y} \right)$ is left invariant under the change of $(x, y)$ into the elements of

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Kernel equation:

$$K(x, y; t) xy Q(x, y; t) = xy - t x Q(x, 0; t) - t y Q(0, y; t)$$

$$- K(x, y; t) \frac{1}{x} y Q \left( \frac{1}{x}, y; t \right) = - \frac{1}{x} y + t \frac{1}{x} Q \left( \frac{1}{x}, 0; t \right) + t y Q(0, y; t)$$

$$K(x, y; t) \frac{1}{x} \frac{1}{y} Q \left( \frac{1}{x}, \frac{1}{y}; t \right) = \frac{1}{x} \frac{1}{y} - t \frac{1}{x} Q \left( \frac{1}{x}, 0; t \right) - t \frac{1}{y} Q(0, \frac{1}{y}; t)$$

$$- K(x, y; t) x \frac{1}{y} Q(x, \frac{1}{y}; t) = - x \frac{1}{y} + t x Q(x, 0; t) + t \frac{1}{y} Q(0, \frac{1}{y}; t)$$
The kernel $K(x, y; t) := 1 - t \cdot \sum_{(i,j) \in \mathcal{S}} x^i y^j = 1 - t \left( x + \frac{1}{x} + y + \frac{1}{y} \right)$ is left invariant under the change of $(x, y)$ into the elements of

$$G_{\mathcal{S}} := \left\{ (x, y), \left( \frac{1}{x}, y \right), \left( \frac{1}{x}, \frac{1}{y} \right), (x, \frac{1}{y}) \right\}$$

Kernel equation:

$$K(x, y; t)xyQ(x, y; t) = xy - txQ(x, 0; t) - tyQ(0, y; t)$$

$$- K(x, y; t)\frac{1}{x}yQ\left( \frac{1}{x}, y; t \right) = -\frac{1}{x}y + t\frac{1}{x}Q\left( \frac{1}{x}, 0; t \right) + tyQ(0, y; t)$$

$$K(x, y; t)\frac{1}{x}\frac{1}{y}Q\left( \frac{1}{x}, \frac{1}{y}; t \right) = \frac{1}{x}\frac{1}{y} - t\frac{1}{x}Q\left( \frac{1}{x}, 0; t \right) - t\frac{1}{y}Q(0, \frac{1}{y}; t)$$

$$- K(x, y; t)x\frac{1}{y}Q(x, \frac{1}{y}; t) = -\frac{1}{y}x + txQ(x, 0; t) + t\frac{1}{y}Q(0, \frac{1}{y}; t)$$

Summing up yields the orbit equation:

$$\sum_{\theta \in G} (-1)^\theta \theta(x y \, Q(x, y; t)) = \frac{xy - \frac{1}{x}y + \frac{1}{x}\frac{1}{y} - x\frac{1}{y}}{K(x, y; t)}$$
The kernel $K(x, y; t) := 1 - t \cdot \sum_{(i,j) \in \mathcal{S}} x^i y^j = 1 - t \left( x + \frac{1}{x} + y + \frac{1}{y} \right)$ is left invariant under the change of $(x, y)$ into the elements of $G_{\mathcal{S}} := \{(x, y), \left( \frac{1}{x}, y \right), \left( \frac{1}{x}, \frac{1}{y} \right), (x, \frac{1}{y}) \}$.

Kernel equation:

\[
K(x, y; t) x y Q(x, y; t) = xy - t x Q(x, 0; t) - t y Q(0, y; t) \\
- K(x, y; t) \frac{1}{x} y Q\left( \frac{1}{x}, y; t \right) = -\frac{1}{x} y + t \frac{1}{x} Q\left( \frac{1}{x}, 0; t \right) + t y Q(0, y; t) \\
K(x, y; t) \frac{1}{x} \frac{1}{y} Q\left( \frac{1}{x}, \frac{1}{y}; t \right) = \frac{1}{x y} - t \frac{1}{x} Q\left( \frac{1}{x}, 0; t \right) - t \frac{1}{y} Q(0, \frac{1}{y}; t) \\
- K(x, y; t) x \frac{1}{y} Q(x, \frac{1}{y}; t) = -x \frac{1}{y} + t x Q(x, 0; t) + t \frac{1}{y} Q(0, \frac{1}{y}; t)
\]

Taking positive parts yields:

\[
[x^y] \sum_{\theta \in G} (-1)^\theta \theta(x y Q(x, y; t)) = [x^y] \frac{xy - \frac{1}{x} y + \frac{1}{x y} - x \frac{1}{y}}{K(x, y; t)}
\]
The kernel $K(x, y; t) := 1 - t \cdot \sum_{(i, j) \in \mathcal{S}} x^i y^j = 1 - t \left( x + \frac{1}{x} + y + \frac{1}{y} \right)$
is left invariant under the change of $(x, y)$ into the elements of

$$G_S := \left\{ (x, y), \left( \frac{1}{x}, y \right), \left( \frac{1}{x}, \frac{1}{y} \right), (x, \frac{1}{y}) \right\}$$

Kernel equation:

$$K(x, y; t)xyQ(x, y; t) = xy - txQ(x, 0; t) - tyQ(0, y; t)$$

$$- K(x, y; t)\frac{1}{x}yQ\left( \frac{1}{x}, y; t \right) = - \frac{1}{x}y + t\frac{1}{x}Q\left( \frac{1}{x}, 0; t \right) + tyQ(0, y; t)$$

$$K(x, y; t)\frac{1}{x}\frac{1}{y}Q\left( \frac{1}{x}, \frac{1}{y}; t \right) = \frac{1}{x \cdot y} - t\frac{1}{x}Q\left( \frac{1}{x}, 0; t \right) - t\frac{1}{y}Q(0, \frac{1}{y}; t)$$

$$- K(x, y; t)x\frac{1}{y}Q(x, \frac{1}{y}; t) = - x\frac{1}{y} + txQ(x, 0; t) + t\frac{1}{y}Q(0, \frac{1}{y}; t)$$

Summing up and taking positive parts yields:

$$xy Q(x, y; t) = [x^y] \frac{xy - \frac{1}{x}y + \frac{1}{x} + \frac{1}{y} - x\frac{1}{y}}{K(x, y; t)}$$
The kernel $K(x, y; t) := 1 - t \cdot \sum_{(i,j) \in \mathcal{S}} x^i y^j = 1 - t \left( x + \frac{1}{x} + y + \frac{1}{y} \right)$ is left invariant under the change of $(x, y)$ into the elements of $\mathcal{G}_\mathcal{S} := \{(x, y), (\frac{1}{x}, y), (\frac{1}{x}, \frac{1}{y}), (x, \frac{1}{y})\}$.

Kernel equation:

$$K(x, y; t)xyQ(x, y; t) = xy - txQ(x, 0; t) - tyQ(0, y; t)$$
$$- K(x, y; t)\frac{1}{x} y Q(\frac{1}{x}, y; t) = -\frac{1}{x} y + t\frac{1}{x} Q(\frac{1}{x}, 0; t) + tyQ(0, y; t)$$
$$K(x, y; t)\frac{1}{x} \frac{1}{y} Q(\frac{1}{x}, \frac{1}{y}; t) = \frac{1}{x} \frac{1}{y} - t\frac{1}{x} Q(\frac{1}{x}, 0; t) - t\frac{1}{y} Q(0, \frac{1}{y}; t)$$
$$- K(x, y; t)x \frac{1}{y} Q(x, \frac{1}{y}; t) = -x \frac{1}{y} + txQ(x, 0; t) + t\frac{1}{y} Q(0, \frac{1}{y}; t)$$

$$\text{GF} = \text{PosPart} \left( \frac{\text{OS}}{\text{kernel}} \right) = \int \int \text{RatFrac}$$
(1) Kernel method [Bousquet-Mélou, Mishna, 2010]

The kernel \( K(x, y; t) := 1 - t \cdot \sum_{(i,j) \in \mathcal{S}} x^i y^j = 1 - t \left( x + \frac{1}{x} + y + \frac{1}{y} \right) \)
is left invariant under the change of \((x, y)\) into the elements of

\[ G_{\mathcal{S}} := \{ (x, y), (\frac{1}{x}, y), (\frac{1}{x}, \frac{1}{y}), (x, \frac{1}{y}) \} \]

Kernel equation:

\[
\begin{align*}
K(x, y; t)xyQ(x, y; t) &= xy - txQ(x, 0; t) - tyQ(0, y; t) \\
- K(x, y; t)\frac{1}{x}yQ(\frac{1}{x}, y; t) &= -\frac{1}{x}y + t\frac{1}{x}Q(\frac{1}{x}, 0; t) + tyQ(0, y; t) \\
K(x, y; t)\frac{1}{x}yQ(\frac{1}{x}, \frac{1}{y}; t) &= \frac{1}{x}y - t\frac{1}{x}Q(\frac{1}{x}, 0; t) - t\frac{1}{y}Q(0, \frac{1}{y}; t) \\
- K(x, y; t)x\frac{1}{y}Q(x, \frac{1}{y}; t) &= -x\frac{1}{y} + txQ(x, 0; t) + t\frac{1}{y}Q(0, \frac{1}{y}; t)
\end{align*}
\]

\[ GF = \text{PosPart} \left( \frac{\text{OS}}{\ker} \right) \text{ is D-finite [Lipshitz, 1988]} \]
The kernel $K(x, y; t) := 1 - t \cdot \sum_{(i,j) \in \mathcal{S}} x^i y^j = 1 - t \left( x + \frac{1}{x} + y + \frac{1}{y} \right)$ is left invariant under the change of $(x, y)$ into the elements of $G_\mathcal{S} := \{(x, y), (\frac{1}{x}, y), (\frac{1}{x}, \frac{1}{y}), (x, \frac{1}{y})\}$

Kernel equation:

\[
K(x, y; t)xyQ(x, y; t) = xy - txQ(x, 0; t) - tyQ(0, y; t)
\]

\[
- K(x, y; t) \frac{1}{x} yQ(\frac{1}{x}, y; t) = - \frac{1}{x} y + t \frac{1}{x} Q(\frac{1}{x}, 0; t) + tyQ(0, y; t)
\]

\[
K(x, y; t) \frac{1}{x} \frac{1}{y} Q(\frac{1}{x}, \frac{1}{y}; t) = \frac{1}{x} \frac{1}{y} - t \frac{1}{x} Q(\frac{1}{x}, 0; t) - t \frac{1}{y} Q(0, \frac{1}{y}; t)
\]

\[
- K(x, y; t) x \frac{1}{y} Q(x, \frac{1}{y}; t) = - x \frac{1}{y} + txQ(x, 0; t) + t \frac{1}{y} Q(0, \frac{1}{y}; t)
\]

\[
GF = \text{PosPart} \left( \frac{OS}{\text{ker}} \right) \text{ is D-finite [Lipshitz, 1988]}
\]

▷ Argument works if $OS \neq 0$: algebraic version of the reflection principle
The kernel $K(x, y; t) := 1 - t \cdot \sum_{(i,j) \in \mathcal{S}} x^i y^j = 1 - t \left( x + \frac{1}{x} + y + \frac{1}{y} \right)$
is left invariant under the change of $(x, y)$ into the elements of

$$\mathcal{G}_\mathcal{S} := \{(x, y), \left( \frac{1}{x}, y \right), \left( \frac{1}{x}, \frac{1}{y} \right), (x, \frac{1}{y})\}$$

Kernel equation:

$$K(x, y; t)xyQ(x, y; t) = xy - txQ(x, 0; t) - tyQ(0, y; t)$$

$$- K(x, y; t)\frac{1}{x} y Q\left( \frac{1}{x}, y \right; t) = - \frac{1}{x} y + t \frac{1}{x} Q\left( \frac{1}{x}, 0; t \right) + tyQ(0, y; t)$$

$$K(x, y; t)\frac{1}{x} y Q\left( \frac{1}{x}, \frac{1}{y} \right; t) = \frac{1}{x} \frac{1}{y} - t \frac{1}{x} Q\left( \frac{1}{x}, 0; t \right) - t \frac{1}{y} Q(0, \frac{1}{y}; t)$$

$$- K(x, y; t)x\frac{1}{y} Q(x, \frac{1}{y}; t) = - \frac{1}{y} + txQ(x, 0; t) + t \frac{1}{y} Q(0, \frac{1}{y}; t)$$

$\text{GF} = \text{PosPart} \left( \frac{\text{OS}}{\text{ker}} \right)$ is D-finite [Lipshitz, 1988]

▷ Creative Telescoping finds a differential equation for $\text{GF} = \int \int \int \text{RatFrac}$
(2) Creative Telescoping

“An algorithmic toolbox for multiple sums and integrals with parameters”

Example [Apéry 1978]: \( A_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 \) satisfies the recurrence

\[
(n + 1)^3 A_{n+1} + n^3 A_{n-1} = (2n + 1) (17n^2 + 17n + 5) A_n.
\]

▷ Key fact used to prove that \( \zeta(3) := \sum_{n \geq 1} \frac{1}{n^3} \approx 1.202056903 \ldots \) is irrational.

1. Journées Arithmétiques de Marseille-Luminy, June 1978

The board of programme changes informed us that R. Apéry (Caen) would speak Thursday, 14.00 “Sur l’irrationalité de \( \zeta(3) \).” Though there had been earlier rumours of his claiming a proof, scepticism was general. The lecture tended to strengthen this view to rank disbelief. Those who listened casually, or who were afflicted with being non-Francophone, appeared to hear only a sequence of unlikely assertions.

[Van der Poorten, 1979: “A proof that Euler missed”]

7. ICM ’78, Helsinki, August 1978

Neither Cohen nor I had been able to prove \( \text{S} \) or \( \text{S}^* \) in the intervening 2 months. After a few days of fruitless effort the specific problem was mentioned to Don Zagier (Bonn), and with irritating speed he showed that indeed the sequence \( \{b'_n\} \) satisfies the recurrence (4). This more or less broke the dam and \( \text{S} \) and \( \text{S}^* \) were quickly conquered. Henri Cohen addressed a very well-attended meeting at 17.00 on Friday, August 18 in the language of the majority, proving \( \text{S} \) and explaining how this implied the
(2) Creative Telescoping

“An algorithmic toolbox for multiple sums and integrals with parameters”

Example [Apéry 1978]: \( A_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 \) satisfies the recurrence

\[
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\]

Key fact used to prove that \( \zeta(3) \) := \( \sum_{n \geq 1} \frac{1}{n^3} \approx 1.202056903 \ldots \) is irrational.

(2) Creative Telescoping

“An algorithmic toolbox for multiple sums and integrals with parameters”

Example [Euler, 1733]: Perimeter of an ellipse of eccentricity \( e \), semi-major axis 1

\[
p(e) = 4 \int_0^1 \sqrt{\frac{1 - e^2 u^2}{1 - u^2}} \, du = 4 \iint \frac{dudv}{1 - \frac{1-e^2 u^2}{(1-u^2)v^2}}
\]

**Principle:** Find algorithmically
(2) Creative Telescoping

“An algorithmic toolbox for multiple sums and integrals with parameters”

Example [Euler, 1733]: Perimeter of an ellipse of eccentricity $e$, semi-major axis 1

$$p(e) = 4 \int_0^1 \sqrt{\frac{1 - e^2 u^2}{1 - u^2}} \, du = 4 \iint \frac{dudv}{1 - \frac{1 - e^2 u^2}{(1 - u^2)v^2}}$$

Principle: Find algorithmically

$$\left( (e - e^3) \partial_e^2 + (1 - e^2) \partial_e + e \right) \cdot \left( \frac{1}{1 - \frac{1 - e^2 u^2}{(1 - u^2)v^2}} \right) =$$

$$\partial_u \left( \frac{-e(-1-u^2+u^3)v^2(-3+2u+v^2+u^2(-2+3e^2-v^2))}{(-1+v^2+u^2(e^2-v^2))^2} \right) + \partial_v \left( \frac{2e(-1+e^2)u(1+u^3)v^3}{(-1+v^2+u^2(e^2-v^2))^2} \right)$$

▶ Conclusion: $$(e - e^3) \cdot p''(e) + (1 - e^2) \cdot p'(e) + e \cdot p(e) = 0.$$
(2) Creative Telescoping

“An algorithmic toolbox for multiple sums and integrals with parameters”

Example [Euler, 1733]: Perimeter of an ellipse of eccentricity \(e\), semi-major axis 1

\[
p(e) = 4 \int_0^1 \sqrt{\frac{1 - e^2 u^2}{1 - u^2}} \, du = 4 \iint \frac{dudv}{1 - \frac{1-e^2u^2}{(1-u^2)v^2}}
\]

Principle: Find algorithmically

\[
\left( (e - e^3) \partial_e^2 + (1 - e^2) \partial_e + e \right) \cdot \left( \frac{1}{1 - \frac{1-e^2u^2}{(1-u^2)v^2}} \right) = \\
\partial_u \left( \frac{-e(-1+u^2+u^3)v^2(-3+2u+v^2+u^2(-2+3e^2-v^2))}{(-1+v^2+u^2(e^2-v^2))^2} \right) + \partial_v \left( \frac{2e(-1+e^2)u(1+u^3)v^3}{(-1+v^2+u^2(e^2-v^2))^2} \right)
\]

▷ Conclusion: \(p(e) = \frac{\pi}{2} \cdot {}_2F_1 \left( \begin{array}{c} \frac{1}{2} \\ \frac{1}{2} \end{array} \left| e^2 \right. \right) = 2\pi - \frac{\pi}{2} e^2 - \frac{3\pi}{32} e^4 - \cdots.\)
(2) Creative Telescoping

“An algorithmic toolbox for multiple sums and integrals with parameters”

Example [Euler, 1733]: Perimeter of an ellipse of eccentricity $e$, semi-major axis $1$

$$p(e) = 4 \int_0^1 \sqrt{\frac{1 - e^2 u^2}{1 - u^2}} \, du = 4 \iint \frac{dudv}{1 - \frac{1-e^2u^2}{(1-u^2)v^2}}$$

Principle: Find algorithmically

$$\left( (e - e^3) \partial_e^2 + (1 - e^2) \partial_e + e \right) \cdot \left( \frac{1}{1 - \frac{1-e^2u^2}{(1-u^2)v^2}} \right) =$$

$$\partial_u \left( \frac{-e(-1-u+u^2+u^3)v^2(-3+2u+v^2+u^2(-2+3e^2-v^2))}{(-1+v^2+u^2(e^2-v^2))^2} \right) + \partial_v \left( \frac{2e(-1+e^2)u(1+u^3)v^3}{(-1+v^2+u^2(e^2-v^2))^2} \right)$$

▶ Drawback: Size(certificate) $\gg$ Size(telescoper).
Algorithm for the integration of rational functions [B., Lairez, Salvy, 2013]

- **Input:** \( R(e, x) \) a rational function in \( e \) and \( x = x_1, \ldots, x_n \).
- **Output:** A linear ODE \( T(e, \partial_e)y = 0 \) satisfied by \( y(e) = \int R(e, x)dx \).
- **Complexity:** \( O(D^{8n+2}) \), where \( D = \deg R \).
- **Output size:** \( T \) has order \( \leq D^n \) in \( \partial_e \) and degree \( \leq D^{3n+2} \) in \( e \).

- Avoids the (costly) computation of certificates, of size \( \Omega(D^{n^2/2}) \).
- Previous algorithms: complexity (at least) doubly exponential in \( n \).
- Very efficient in practice.
Theorem (Apéry’s power series is transcendental)

\[ f(t) = \sum_{n} A_n t^n, \quad \text{where} \quad A_n = \sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k}^2, \quad \text{is transcendental.} \]
A toy *transcendence proof*: blending *Guess-and-Prove* and CT

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Proof:

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(n+1)^3 A_{n+1} + n^3 A_{n-1} = (2n+1) (17n^2 + 17n + 5) A_n, \quad A_0 = 1, \ A_1 = 5
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3. Guess-and-Prove:
   compute least-order \(L_f^{\text{min}}\) in \(Q(t)\langle\partial_t\rangle\) such that \(L_f^{\text{min}}(f) = 0\)
A toy \textit{transcendence proof}: blending \textbf{Guess-and-Prove} and CT

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\text{compute least-order } L_f^{\min} \text{ in } \mathbb{Q}(t)\langle \partial_t \rangle \text{ such that } L_f^{\min}(f) = 0
\]

4. Basis of formal solutions of \( L_f^{\min} \) at \( t = 0 \):
\[
\left\{ 1 + 5t + O(t^2), \; \ln(t) + (5\ln(t) + 12)t + O(t^2), \; \ln(t)^2 + (5\ln(t)^2 + 24\ln(t))t + O(t^2) \right\}
\]
A toy \textit{transcendence proof}: blending \textbf{Guess-and-Prove} and \textbf{CT}

\textbf{Theorem (Apéry’s power series is transcendental)}

\[ f(t) = \sum_n A_n t^n, \quad \text{where} \quad A_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2, \quad \text{is transcendental}. \]

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5. \textbf{Conclusion:} \( f \) is transcendental\(^\dagger\)

\(^\dagger\) \( f \) algebraic would imply a full basis of algebraic solutions for \( L_{f}^{\text{min}} \) [Tannery, 1875].
Summary: classification of walks with small steps in $\mathbb{N}^2$

$Q, \mathcal{G}$ is D-finite $\iff$ a certain group $\mathcal{G}, \mathcal{G}$ is finite (!)

quadrant models $\mathcal{I}$: 79

$|\mathcal{G}, \mathcal{G}| < \infty$: 23
orbit sum $= 0$: 4
Guess-and-Prove algebraic

$|\mathcal{G}, \mathcal{G}| = \infty$: 56
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Creative Telescoping D-finite

asymptotics + GB
non-D-finite
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    - kernel + CT: transcendental
- $|G\mathcal{I}| = \infty$: 56
  - $\exists$ decoupling*: 9
    - Tutte's invariants: D-algebraic
  - $\nexists$ decoupling: 47
    - diff. Galois: D-transcendental

$\exists U \in \mathbb{Q}(x,t), V \in \mathbb{Q}(y,t)$ s.t. $U(x) + V(y) = xy$ on the kernel $K(x,y;t) = 0$. 

Alin Bostan  
Efficient experimental mathematics for combinatorics and number theory
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algebraic
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Summary: classification of walks with small steps in \( \mathbb{N}^2 \)

\[ Q_S \text{ is D-finite} \iff \text{a certain group } G_S \text{ is finite (!)} \]

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- Proofs use various tools: algebra, complex analysis, probability theory, differential Galois theory, computer algebra, etc.
Enumerative Combinatorics and Computer Algebra enrich one another

Classification of $Q(x,y;t)$ fully completed for 2D small step walks

Robust algorithmic methods, based on efficient algorithms:
- Guess-and-Prove
- Creative Telescoping

Brute-force and/or use of naive algorithms = hopeless.
E.g. size of algebraic equations for $G(x,y;t) \approx 30$Gb.
Conclusion

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Robust algorithmic methods, based on efficient algorithms:
- Guess-and-Prove
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Brute-force and/or use of naive algorithms = hopeless.
E.g. size of algebraic equations for $G(x,y;t) \approx 30Gb.$

Lack of “purely human” proofs for some results.

Many beautiful open questions for 2D models with repeated or large steps, and in dimension $> 2.$
Exercises

1. Explain why $\sum_n F_n t^n$ is rational, where $F_{n+2} = F_{n+1} + F_n, F_0 = 0, F_1 = 1$. Find a general statement.

2. Show that the series $\sum_n \binom{2n}{n} t^n$ and $\sum_n \binom{5n}{n} t^n$ are both algebraic.

3. Prove that the series
   
   $\sqrt{1 - 4t} = 1 - 2t - 2t^2 - 4t^3 - 10t^4 - 28t^5 - \cdots$

   $\sqrt[3]{1 - 9t} = 1 - 3t - 9t^2 - 45t^3 - 270t^4 - 1782t^5 - \cdots$

   have only integer coefficients. Try to generalize.

4. Prove that $\tan(t) = t + \frac{1}{3} t^3 + \frac{2}{15} t^5 + \frac{17}{315} t^7 + \frac{62}{2835} t^9 + \cdots$ is not D-finite.

5. Let $M_{n,k}$ be the number of $\{(1,1), (1,-1)\}$-walks in $\mathbb{N}^2$ of length $n$ that start at $(0,0)$ and end at vertical altitude $k$. Let $M(x,y) = \sum_{n,k} M_{n,k} x^n y^k$.

   (a) Show that $(y - x(1 + y^2)) \cdot M(x,y) = y - x \cdot M(x,0)$

   (b) Deduce that $M(x,y) = \frac{\sqrt{1 - 4x^2 + 2xy - 1}}{2x(y - x(1 + y^2))}$
Bonus
Beyond dimension 2: walks with small steps in $\mathbb{N}^3$

$2^{3^3-1} \approx 67$ million models, of which $\approx 11$ million inherently 3D

3D octant models $\mathcal{S}$ with $\leq 6$ steps: 20804

- $|G_{\mathcal{S}}| < \infty$: 170
  - orbit sum $\neq 0$: 108
    - Creative Telescoping
      - D-finite
    - 2D-reducible: 43
      - D-finite
    - not 2D-reducible: 19
      - non-D-finite?
- $|G_{\mathcal{S}}| = \infty$: 20634
  - orbit sum $= 0$: 62
    - non-D-finite?

[Alin Bostan] Efficient experimental mathematics for combinatorics and number theory

[B., Bousquet-Mélou, Kauers, Melczer, 2016] + [Du, Hou, Wang, 2017]; completed by [Bacher, Kauers, Yatchak, 2016]
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Question: differential finiteness $\iff$ finiteness of the group?

Answer: probably no
19 mysterious 3D-models: finite $G_\mathcal{F}$ and possibly non-D-finite $Q_\mathcal{F}$
Open question: 3D Kreweras excursions

Two different computations suggest:

\[ k_{4n} = C \cdot 256^n / n^\alpha, \text{ for } \alpha = 3.3257570041744 \ldots, \]

so excursions are very probably non-D-finite
Beyond small steps: Walks in $\mathbb{N}^2$ with large steps

quadrant models with steps in $\{-2, -1, 0, 1\}^2$: 13 110

$|G_S| < \infty$: 240

$|G_S| = \infty$: 12 870

OS $\neq 0$: 431

OS $= 0$: 9

$\alpha$ rational: 16

$\alpha$ irrational: 12 854

D-finite

D-finite?

non-D-finite?

non-D-finite

[B., Bousquet-Mélou, Melczer, 2018]

Question: differential finiteness $\iff$ finiteness of the group?

Answer: ?
Two challenging models with large steps

**Conjecture 1** [B., Bousquet-Mélou, Melczer, 2018]

For the model the excursions generating function $Q(0, 0; t^{1/2})$ equals

\[
\frac{1}{3t} - \frac{1}{6t} \cdot \left( \frac{1 - 12t}{(1 + 36t)^{1/3}} \right) \cdot 2F_1 \left( \frac{1}{6}, \frac{2}{3} \bigg| \frac{108t(1 + 4t)^2}{(1 + 36t)^2} \right) + \\
\sqrt{1 - 12t} \cdot 2F_1 \left( -\frac{1}{6}, \frac{2}{3} \bigg| \frac{108t(1 + 4t)^2}{(1 - 12t)^2} \right).
\]

**Conjecture 2** [B., Bousquet-Mélou, Melczer, 2018]

For the model the excursions generating function $Q(0, 0; t)$ equals

\[
\frac{(1 - 24U + 120U^2 - 144U^3)(1 - 4U)}{(1 - 3U)(1 - 2U)^{3/2}(1 - 6U)^{9/2}},
\]

where $U = t^4 + 53t^8 + 4363t^{12} + \cdots$ is the unique series in $\mathbb{Q}[[t]]$ satisfying

\[
U \cdot (1 - 2U)^3 \cdot (1 - 3U)^3 \cdot (1 - 6U)^9 = t^4 \cdot (1 - 4U)^4.
\]


Solution of the “exercise”
• The kernel equation reads (with $K(x,y) = 1 - t(y + \bar{x} + xy)$):

$$K(x,y)yH(x,y) = y - txH(x,0)$$
Excursions in $\mathbb{Z} \times \mathbb{N}$

- The kernel equation reads (with $K(x,y) = 1 - t(y + \bar{x} + x\bar{y})$):
  \[ K(x,y) yH(x,y) = y - txH(x,0) \]

- Let
  \[ y_0 = \frac{x - t - \sqrt{(t-x)^2 - 4t^2x^3}}{2tx} = xt + t^2 + (x^2 + \bar{x})t^3 + (3x + \bar{x}^2)t^4 + \cdots \]

be the (unique) root in $\mathbb{Q}[x, \bar{x}][[t]]$ of $K(x,y_0) = 0$. 
• The kernel equation reads (with $K(x,y) = 1 - t(y + \bar{x} + x\bar{y})$):

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be the (unique) root in $\mathbb{Q}[x, \bar{x}][[t]]$ of $K(x,y_0) = 0$.

• Then

$$0 = K(x,y_0)yH(x,y_0) = y_0 - txH(x,0),$$

thus

$$H(x,0) = \frac{y_0}{tx} \quad \text{and} \quad A(t) = \begin{bmatrix} x^0 \end{bmatrix} \frac{y_0}{tx}. $$
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be the (unique) root in \( \mathbb{Q}[x, \bar{x}][[t]] \) of \( K(x, y_0) = 0 \).

• Then

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\]

thus

\[
H(x, 0) = \frac{y_0}{tx} \quad \text{and} \quad A(t) = \left[ x^0 \right] \frac{y_0}{tx}.
\]

• **Creative telescoping** then proves:

\[
(27t^4 - t)A''(t) + (108t^3 - 4)A'(t) + 54t^2A(t) = 0.
\]

> Zeilberger(1/x * sqrt((t-x)^2 - 4*t^2*x^3)/(2*t^2*x^2), t, x, Dt);
The group of the model \( \{ \uparrow, \leftarrow, \downarrow \} \)

Step set \( \mathcal{S} = \{(-1, 0), (0, 1), (1, -1)\} \), with characteristic polynomial

\[
\chi(x, y) = \frac{1}{x} + y + x \cdot \frac{1}{y} = \bar{x} + y + x\bar{y}
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The group of the model \{↑, ←, \downarrow\}

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\(\chi(x, y)\) is left unchanged by the rational transformations

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\Phi : (x, y) \mapsto (\bar{x}y, y) \quad \text{and} \quad \Psi : (x, y) \mapsto (x, x\bar{y}).
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\]

\( \Phi \) and \( \Psi \) are involutions, and generate a finite dihedral group \( D_3 \) of order 6:
Diagonal walks in $\mathbb{N}^2$

- Orbit equation:

$$xyQ(x, y) - \bar{x}y^2 Q(\bar{x}y, y) + \bar{x}^2 yQ(\bar{x}y, \bar{x})$$

$$- \bar{x}yQ(\bar{y}, \bar{x}) + x\bar{y}^2 Q(\bar{y}, x\bar{y}) - x^2 \bar{y}Q(x, x\bar{y}) =$$

$$\frac{xy - \bar{x}y^2 + \bar{x}^2 y - \bar{x}\bar{y} + x\bar{y}^2 - x^2 \bar{y}}{1 - t(y + \bar{x} + x\bar{y})}$$
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- Corollary [Bousquet-Mélou & Mishna, 2010]:

$$xyQ(x, y) = [x>0y>0] \frac{xy - \bar{x}y^2 + \bar{x}^2 y - \bar{x}y + x\bar{y}^2 - x^2 \bar{y}}{1 - t(y + \bar{x} + x\bar{y})}$$
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$$xyQ(x, y) - \bar{x}y^2Q(\bar{x}y, y) + \bar{x}^2yQ(\bar{x}y, \bar{x})$$

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- Corollary [B.-Chyzak-van Hoeij-Kauers-Pech, 2015]:

$$B(t) = [z^0]Q(z, \bar{z}) = [u^{-1} v^{-1} z^{-1}] \frac{\bar{u}\bar{v} - u\bar{v}^2 + u^2\bar{v} - uv + \bar{u}v^2 - \bar{u}^2v}{z(1 - zu)(1 - v\bar{z})(1 - t(\bar{v} + u + \bar{u}v))}$$
Diagonal walks in $\mathbb{N}^2$

- Orbit equation:

\[
xyQ(x, y) - \bar{x}y^2Q(\bar{x}y, y) + \bar{x}^2yQ(\bar{x}y, \bar{x}) - \bar{x}\bar{y}Q(\bar{y}, \bar{x}) + x\bar{y}^2Q(y, x\bar{y}) - x^2\bar{y}Q(x, x\bar{y}) = \\
\frac{xy - \bar{x}y^2 + \bar{x}^2y - \bar{x}\bar{y} + x\bar{y}^2 - x^2\bar{y}}{1 - t(y + \bar{x} + x\bar{y})}
\]

- Corollary [Bousquet-Mélou & Mishna, 2010]:

\[
xyQ(x, y) = [x>0y>0] \frac{xy - \bar{x}y^2 + \bar{x}^2y - \bar{x}\bar{y} + x\bar{y}^2 - x^2\bar{y}}{1 - t(y + \bar{x} + x\bar{y})}
\]

- Corollary [B.-Chyzak-van Hoeij-Kauers-Pech, 2015]:

\[
B(t) = [z^0]Q(z, \bar{z}) = [u^{-1}v^{-1}z^{-1}] \frac{\bar{u}\bar{v} - u\bar{v}^2 + u^2\bar{v} - uv + \bar{u}v^2 - \bar{u}^2v}{z(1 - zu)(1 - v\bar{z})(1 - t(\bar{v} + u + \bar{u}v))}
\]

- Creative Telescoping gives a differential equation for $B(t)$:

\[
(27t^4 - t)B''(t) + (108t^3 - 4)B'(t) + 54t^2B(t) = 0.
\]
We have proved that $A(t)$ and $B(t)$ are both solutions of

$$(27t^4 - t)y''(t) + (108t^3 - 4)y'(t) + 54t^2y(t) = 0.$$ 

Solving this equation proves:

$$A(t) = B(t) = 2F_1\left(\begin{array}{c}1/3 \\ 2\end{array} ; \frac{2}{3} \bigg| 27t^3 \right) = \sum_{n=0}^{\infty} \frac{(3n)!}{n!^3} \frac{t^{3n}}{n + 1}.$$ 

Thus the two sequences are equal to

$$a_{3n} = b_{3n} = \frac{(3n)!}{n!^2 \cdot (n + 1)!},$$

and

$$a_m = b_m = 0 \quad \text{if 3 does not divide } m.$$
Example with infinite group: the scarecrows

[B., Raschel, Salvy, 2014]: $Q_\scr G(0, 0; t)$ is not D-finite for the models

\[ Q_\scr G(0, 0; t) \]

For the 1st and the 3rd, the excursions sequence $[t^n] Q_\scr G(0, 0; t)$

\[ 1, 0, 0, 2, 4, 8, 28, 108, 372, \ldots \]

is $\sim K \cdot 5^n \cdot n^{-\alpha}$, with $\alpha = 1 + \pi / \arccos(1/4) = 3.383396 \ldots$

[Denisov, Wachtel, 2015]

- For the 1st and the 3rd, the excursions sequence $[t^n] Q_\scr G(0, 0; t)$

- The irrationality of $\alpha$ prevents $Q_\scr G(0, 0; t)$ from being D-finite.

[Katz, 1970; Chudnovsky, 1985; André, 1989]
The group of a model: the simple walk case

The characteristic polynomial \( \chi_S := x + \frac{1}{x} + y + \frac{1}{y} \)
The characteristic polynomial $\chi_S := x + \frac{1}{x} + y + \frac{1}{y}$ is left invariant under

$$\psi(x, y) = \left(x, \frac{1}{y}\right), \quad \phi(x, y) = \left(\frac{1}{x}, y\right),$$
The group of a model: the simple walk case

The characteristic polynomial \( \chi_S := x + \frac{1}{x} + y + \frac{1}{y} \) is left invariant under

\[ \psi(x, y) = \left( x, \frac{1}{y} \right), \quad \phi(x, y) = \left( \frac{1}{x}, y \right), \]

and thus under any element of the group

\[ \langle \psi, \phi \rangle = \left\{ (x, y), \left( x, \frac{1}{y} \right), \left( \frac{1}{x}, \frac{1}{y} \right), \left( \frac{1}{x}, y \right) \right\}. \]
The group of a model

The generating polynomial \( \chi_{\mathcal{S}} := \sum_{(i,j) \in \mathcal{S}} x^i y^j = \sum_{i=-1}^{1} B_i(y)x^i = \sum_{j=-1}^{1} A_j(x)y^j \)
The group of a model

The generating polynomial

\[ \chi_S := \sum_{(i,j) \in S} x^i y^j = \sum_{i=-1}^{1} B_i(y) x^i = \sum_{j=-1}^{1} A_j(x) y^j \]

is left invariant under the birational involutions

\[ \psi(x, y) = \left( x, \frac{A_{-1}(x)}{A_{+1}(x)} \frac{1}{y} \right), \quad \phi(x, y) = \left( \frac{B_{-1}(y)}{B_{+1}(y)} \frac{1}{x}, y \right), \]

and thus under any element of the (dihedral) group

\[ G_S := \langle \psi, \phi \rangle. \]
Examples of groups

Order 4,
Examples of groups

Order 4,

order 6,
Examples of groups

Order 4, order 6, order 8,
Examples of groups

Order 4, order 6, order 8, order $\infty$. 

\[ \Phi \] 
\[ \Psi \] 
\[ \Phi \Psi \] 
\[ \Phi \Psi \] 
\[ \Psi \Phi \] 
\[ \Psi \Phi \] 
\[ \Phi \Psi \] 
\[ \Phi \Psi \] 

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Examples of groups

Order 4,

order 6,

order 8,

order ∞.

\[
\begin{align*}
\Phi & \quad (x, y) \quad \Psi \\
\Phi & \quad (x, \frac{x}{y}) \quad \Psi \\
\Psi & \quad (x, \frac{y}{x}) \quad \Phi \\
\Psi & \quad (\frac{y}{x}, \frac{1}{x}) \quad \Phi \\
\Phi & \quad (\frac{1}{y}, \frac{1}{x})
\end{align*}
\]