# Efficient experimental mathematics for combinatorics and number theory

**Alin Bostan**

Inria informatics mathematics

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# Problem Session 2

# Exercises

1. Explain why $\sum_n F_n t^n$ is rational, where $F_{n+2} = F_{n+1} + F_n, F_0 = 0, F_1 = 1$. Find a general statement.
2. Show that the series $\displaystyle\sum_n \binom{2n}{n} t^n$ and $\displaystyle\sum_n \binom{5n}{n} t^n$ are both algebraic.
3. Prove that the series
   - $\sqrt{1-4t} = 1 - 2\,t - 2\,t^2 - 4\,t^3 - 10\,t^4 - 28\,t^5 - \cdots$
   - $\sqrt[3]{1-9t} = 1 - 3\,t - 9\,t^2 - 45\,t^3 - 270\,t^4 - 1782\,t^5 - \cdots$

   have only integer coefficients. Try to generalize.
4. Prove that $\tan(t) = t + \frac{1}{3}t^3 + \frac{2}{15}t^5 + \frac{17}{315}t^7 + \frac{62}{2835}t^9 + \cdots$ is not D-finite.
5. Let $M_{n,k}$ be the number of $\{(1,1),(1,-1)\}$-walks in $\mathbb{N}^2$ of length $n$ that start at $(0,0)$ and end at vertical altitude $k$. Let $M(x,y) = \displaystyle\sum_{n,k} M_{n,k} x^n y^k$.
   (a) Show that $(y - x(1+y^2)) \cdot M(x,y) = y - x \cdot M(x,0)$
   (b) Deduce that $M(x,y) = \dfrac{\sqrt{1-4x^2} + 2xy - 1}{2x(y - x(1+y^2))}$

# Shank's 1974 identity

$$\sqrt{11+2\sqrt{29}}+\sqrt{16-2\sqrt{29}+2\sqrt{55-10\sqrt{29}}} = \sqrt{5}+\sqrt{22+2\sqrt{5}}$$

# Dyson's 1962 problem

Prove that for any $a_1, \dots, a_n \in \mathbb{N}^\star$, the constant term $[x_1^0 \cdots x_n^0]$ in

$$\prod_{1 \le i \ne j \le n} \left(1 - \frac{x_i}{x_j}\right)^{a_j}$$

is equal to

$$\frac{(a_1 + \cdots + a_n)!}{a_1! \cdots a_n!}.$$

Hint. Follow the next steps:

1. Prove that for any value of $x$:

$$\sum_{j=1}^{n} \prod_{\substack{i=1 \\ i \ne j}}^{n} \frac{x - x_i}{x_j - x_i} = 1$$

2. Let $F_n(a_1, \dots, a_n)$ be our product. Prove the equality:

$$F_n(a_1, \dots, a_n) = \sum_{j=1}^{n} F_n(a_1, \dots, a_j - 1, \dots, a_n)$$

3. Show that the constant term of $F_n(a_1, \dots, a_n)$ satisfies the same recurrence and boundary conditions as the multinomial coefficient.

# Solution, ex. 2(b)

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Strategy: First guess a polynomial $P(t, y)$ in $\mathbb{Q}[t, y]$, s.t. $P(t, f(t)) = 0 \bmod t^?$, then prove that $P$ admits the power series $f(t)$ as a root, i.e., $P(t, f(t)) = 0$.

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2. Implicit function theorem: $\exists!$ root $r(t) \in \mathbb{Q}[[t]]$ of $P$.

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2. Implicit function theorem: $\exists!$ root $r(t) \in \mathbb{Q}[[t]]$ of $P$.
3. $r(t) = \sum_{n=0}^{\infty} r_n t^n$ being algebraic, it is D-finite, and so $(r_n)$ is P-recursive. Conclude that $r_n = \binom{5n}{n}$, thus $f(t) = r(t)$ is algebraic.

# Solution, ex. 2(b)

```
> f5:=sum(binomial(5*n,n)*t^n, n=0..infinity):
> simplify(f5) assuming t>0 and t<1/100;
```

$$ {}_4F_3\left(\left[\frac{1}{5},\frac{2}{5},\frac{3}{5},\frac{4}{5}\right];\left[\frac{1}{4},\frac{1}{2},\frac{3}{4}\right];\frac{3125\,t}{256}\right) $$

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```
> P5:=subs(y(t) = y, seriestoalgeq(series(f5,t,100), y(t))[1]);
```

$$ 1+15\,y+80\,y^2+160\,y^3+(3125\,t-256)\,y^5 $$

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$$

```
> subs({t=0, y=1}, P5), subs({t=0, y=1}, diff(P5,y));
```

$$
0, \quad -625
$$

# Solution, ex. 2(b)

```
> deq5:=algeqtodiffeq(P5, y(t));
```

$$\left\{120\,y(t) + (15000\,t - 24)\,\frac{d}{dt}y(t) + \left(45000\,t^2 - 816\,t\right)\frac{d^2}{dt^2}y(t) + \left(25000\,t^3 - 1152\,t^2\right)\frac{d^3}{dt^3}y(t) + \left(3125\,t^4 - 256\,t^3\right)\frac{d^4}{dt^4}y(t),\ y(0) = RootOf\left(4\_Z^2 - 3\_Z - 1\right)\right\}$$

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```
> rec5:=map(factor, diffeqtorec(deq5, y(t), r(n))[1]);
```

$$5\,(5\,n+1)\,(5\,n+2)\,(5\,n+3)\,(5\,n+4)\,r(n) - 8\,(4\,n+1)\,(2\,n+1)\,(4\,n+3)\,(n+1)\,r(n+1) = 0$$

# Solution, ex. 2(b)

```
> r5n := rsolve({rec5, r(0)=1}, r(n));
```

$$\frac{\sqrt{5}}{4\,\pi}\left(\frac{3125}{64}\right)^{n}\frac{\Gamma\left(n+\frac{1}{5}\right)\Gamma\left(n+\frac{2}{5}\right)\Gamma\left(n+\frac{3}{5}\right)\Gamma\left(n+\frac{4}{5}\right)}{\Gamma\left(2\,n+\frac{1}{2}\right)\Gamma\left(n+\frac{1}{2}\right)\Gamma\left(n+1\right)}$$

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```
> b5n:=convert(binomial(5*n,n), GAMMA);
> simplify(convert(r5n/b5n, GAMMA));
```

$$\frac{\Gamma\left(5\,n+1\right)}{\Gamma\left(n+1\right)\Gamma\left(4\,n+1\right)},\quad 1$$

$$\mathfrak{S} = \{(1,1),(1,-1)\}$$

$$M_{n+1,k} = M_{n,k-1} + M_{n,k+1}, \quad M_{0,0} = 1,\ M_{-1,k} = M_{n,-1} = 0 \text{ for } k,n \geq 0$$

Multiply by $x^{n+1}y^{k+1}$, and sum over $n,k \in \mathbb{N}$

$$\Longrightarrow \qquad y \cdot \Big( M(x,y) - \underbrace{\sum_{k\geq 0} M_{0,k} y^k}_{M(0,y)=1} \Big) = y^2 x \cdot M(x,y) + x \cdot \Big( M - \underbrace{\sum_{n\geq 0} M_{n,0} x^n}_{M(x,0)} \Big)$$

$$\Longrightarrow \qquad (y - x(1+y^2)) \cdot M(x,y) = y - x \cdot M(x,0) \qquad \text{(kernel equation)}$$

# Solution, ex. 5(b)

$$\mathfrak{S} = \{(1,1),(1,-1)\}$$

$$(y - x(1+y^2)) \cdot M(x,y) = y - x \cdot M(x,0) \qquad \text{(kernel equation)}$$

Kernel method: let $y_0 \in \mathbb{Q}[[x]]$ the power series root of $K = y - x(1+y^2)$

$$y_0 = \frac{1-\sqrt{1-4x^2}}{2x} = x + x^3 + 2x^5 + \cdots \in \mathbb{Q}[[x]]$$

Plugging $y = y_0$ in the (kernel equation) $\implies$ $E(x) = M(x,0) = \dfrac{y_0}{x}$

$$\implies \qquad M(x,y) = \frac{y-y_0}{K(x,y)} = \frac{\sqrt{1-4x^2}+2xy-1}{2x(y-x(1+y^2))}$$