Algebraicity and transcendence of power series: combinatorial and computational aspects

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Algorithmic and Enumerative Combinatorics
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Overview

1. Monday: Context and Examples
2. Tuesday: Properties and Criteria (1)
3. Wednesday: Properties and Criteria (2)
4. Thursday: Algorithmic Proofs of Algebraicity
5. Friday: Transcendence in Lattice Path Combinatorics

Part I: Context and Examples

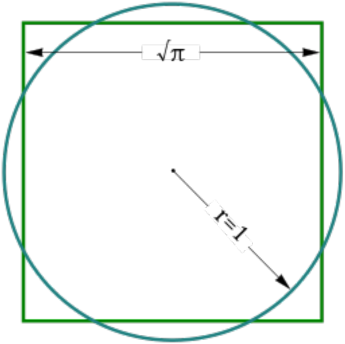
Context

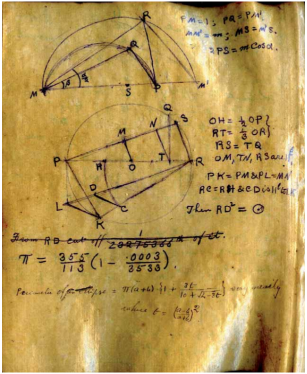
This book will probably be ignored by pure mathematicians. It will appeal only to those applied mathematicians who are willing to share the author's idée fixe. The subject is as quaint and improbable as the title of the book itself, and the author pursues it armed only with the most ordinary of weapons and a relentless preoccupation with detail.

[From a 1967 math book review]

Algebraic and transcendental numbers

▷ Fundamental question in mathematics: arithmetic nature of numbers; motivated by old problems, e.g., squaring a circle (compass & straightedge).





- A complex number α is called *algebraic* if it is a root of some algebraic equation $P(\alpha) = 0$, where $P(x) \in \mathbb{Z}[x] \setminus \{0\}$ Notation: $\overline{\mathbb{Q}}$
- A complex number that is not algebraic is called *transcendental*.

▷ Given some particular constant (e.g., obtained by some limiting process), it is usually very hard to determine whether it is algebraic or transcendental.

A bit of history

- Liouville (1844): transcendental numbers do exist
(algebraic irrationals cannot be approximated "too well" by rationals)
- Eisenstein (1850): the set of algebraic numbers forms a field
- Cantor (1874): "almost all" numbers are transcendental

First explicit examples of transcendental numbers:

- Liouville (1844): $\sum_{n\geq 0} \frac{1}{10^{n!}} = 0.110001000000000000000001000\ldots$
- Hermite (1873): $e = \sum_{n\geq 0} \frac{1}{n!} = 2.718281828459045235\ldots$
- Lindemann (1882): $\pi = 4 \times \sum_{n\geq 0} \frac{(-1)^n}{2n+1} = 3.141592653589793238\ldots$
- Mahler (1937): Champernowne's number $0.12345678910111213\ldots$

Advanced transcendence results

• Hermite-Lindemann (1882): If $\alpha \in \overline{\mathbb{Q}} \setminus \{0\}$, then e^{α} is transcendental.

E.g.: $e, \pi, \log(2), e^{\sqrt{2}}$ are transcendental.

• Lindemann-Weierstrass (1885): If $\alpha_1, \ldots, \alpha_n \in \overline{\mathbb{Q}}$ are distinct, then the exponentials $e^{\alpha_1}, \ldots, e^{\alpha_n}$ are $\overline{\mathbb{Q}}$-linearly independent.

E.g.: $\sin(1), \cos(\sqrt{2}), \tan(\frac{1+\sqrt{5}}{2})$ are transcendental.

• Gel'fond-Schneider (1934): If $a \in \overline{\mathbb{Q}} \setminus \{0, 1\}$ and $b \in \overline{\mathbb{Q}} \setminus \mathbb{Q}$, then $a^b \notin \overline{\mathbb{Q}}$.

E.g.: $2^{\sqrt{2}}, e^{\pi}, i^{i}, \log_2(3)$ are transcendental.

Advanced transcendence results

- Baker (1966): If $\log \alpha_1, \ldots, \log \alpha_n$ are $\mathbb{Q}$-linearly independent, with $\alpha_1, \ldots, \alpha_n \in \overline{\mathbb{Q}}$, then they are also $\overline{\mathbb{Q}}$-linearly independent.

E.g.: $\alpha \log(2) + \beta \log(3) + \gamma \log(5)$ is transcendental for $\alpha, \beta, \gamma \in \overline{\mathbb{Q}} \setminus \{0\}$.

- Schneider (1940): Let $a, b \in \mathbb{Q} \setminus \mathbb{Z}$ be such that $a + b \notin \mathbb{Z}$. Then

$$B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} = \int_0^1 t^{a-1}(1-t)^{b-1}\, dt \quad \text{is transcendental.}$$

- Chudnovsky (1976): $\Gamma(1/3), \Gamma(1/4)$ and $\Gamma(1/6)$ are transcendental.

A few constants whose transcendence is not known yet

- $\pi + e = 5.859874\ldots$, $\pi \times e = 8.539734\ldots$ and $\pi^e = 22.459158\ldots$
- $\log(2) \times \log(3) = 0.761500\ldots$
- Euler's constant $\gamma = \lim_{n\to\infty} \left(\sum_{k=1}^{n} \frac{1}{k} - \log n \right) = 0.577215\ldots$
- Catalan's constant $G = \sum_{n\geq 0} \frac{(-1)^n}{(2n+1)^2} = 0.915966\ldots$
- Apéry's constant $\zeta(3) = \sum_{n\geq 1} \frac{1}{n^3} = 1.202057\ldots$
- Chudnovsky's constant $\Gamma(1/5) = \int_0^\infty t^{-4/5} e^{-t}\, dt = 4.590844\ldots$

More puzzling examples

$$\sum 2^{-n!}$$

is transcendental by Liouville (1844)

$$\sum 2^{-n^2}$$

is transcendental by Nesterenko-Bertrand (1996)

$$\sum 2^{-n^3}$$

is very probably transcendental, but no proof is known yet

In contrast with the "hard" theory of arithmetic transcendence, it is usually "easy" to establish transcendence of functions.

[Flajolet, Sedgewick, 2009]

- A power series f in $\mathbb{Q}[[t]]$ is called *algebraic* if it is a root of some algebraic equation $P(t, f(t)) = 0$, where $P(x, y) \in \mathbb{Z}[x, y] \setminus \{0\}$.
- A power series that is not algebraic is called *transcendental*.

▷ Task: Given a power series, either in explicit or in implicit form, determine whether it is algebraic or transcendental.

Motivations

- Number theory: first step towards proving the transcendence of a complex number is to prove that a power series is transcendental
- Combinatorics: a generating series is algebraic if the counted objects have strong underlying structures
- Computer science: are algebraic power series (intrinsically) easier to manipulate?

Examples

One of the author's most wearisome idiosyncrasies is to work from the special case to the more general, which only serves to emphasize the caprice with which the material was selected.

[From a 1967 math book review]

First examples

- $\sum_n t^n, \quad \sum_n n^{2016} t^n$ $\qquad \sum_n \text{poly}(n)\, t^n$

- $\sum_n \frac{1}{n} t^n, \quad \sum_n \frac{1}{n^2+1} t^n$ $\qquad \sum_n \text{rational}(n)\, t^n$

- $\sum_n 2^n t^n, \quad \sum_n F_n t^n$ $\qquad \sum_n \text{rec. seq. ct. coeffs}(n)\, t^n$

- $\sum_n \frac{1}{2^n} t^n, \quad \sum_n \frac{1}{F_n} t^n$ $\qquad \sum_n \frac{1}{\text{rec. seq. ct. coeffs}(n)}\, t^n$

First examples

- $\sum_n \frac{1}{n!} t^n, \quad \sum_n \frac{(2n)!}{4^n (n!)^2 (2n+1)} t^n$ — exp-trig

- $\sum_n H_n t^n, \quad \sum_n \binom{2n}{n} H_n t^n$ — harmonic sums

- $\sum_n \binom{2n}{n}^{2016} t^n, \quad \sum_n \frac{1}{(2015\,n+1)} \binom{2016\,n}{n} t^n$ — binomial series

- $\sum_n \sum_{k=0}^{n} \binom{n}{k}^3 t^n, \quad \sum_n \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 t^n$ — binomial sums series

- $\sum_n \frac{(2n)!(5n)!^2}{(3n)!^4} t^n, \quad \sum_n \frac{(30n)!n!}{(15n)!(10n)!(6n)!} t^n$ integer ratios of factorials

- ${}_2F_1\left(\begin{matrix} a & b \\ & c \end{matrix} \middle| t\right) = \sum_n \frac{t^n}{n!} \prod_{i=0}^{n-1} \frac{(a+i)(b+i)}{c+i}$ Gaussian hypergeometric series

- ${}_2F_1\left(\begin{matrix} \frac{1}{12} & \frac{5}{12} \\ & 1 \end{matrix} \middle| 1728\, t\right) = 1 + 60\, t + 39780\, t^2 + 38454000\, t^3 + \cdots$

- ${}_3F_2\left(\begin{matrix} \frac{1}{3} & \frac{2}{3} & 1 \\ & \frac{3}{2} & 2 \end{matrix} \middle| 27\, t\right) = \sum_{n=0}^{\infty} \frac{4^n \binom{3n}{n}}{(n+1)(2n+1)} t^n$ hypergeometric series

Power series from the hypergeometric world

- $(1-t)^{-a} = {}_2F_1\left(\begin{matrix} a & 1 \\ & 1 \end{matrix} \middle| \, t\right)$

- $\dfrac{1}{2t} \log \dfrac{1+t}{1-t} = {}_2F_1\left(\begin{matrix} \frac{1}{2} & 1 \\ & \frac{3}{2} \end{matrix} \middle| \, t^2\right)$

- $\dfrac{\arcsin(t)}{t} = {}_2F_1\left(\begin{matrix} \frac{1}{2} & \frac{1}{2} \\ & \frac{3}{2} \end{matrix} \middle| \, t^2\right)$

- $P_n(t) = 2^n {}_2F_1\left(\begin{matrix} -n & n+1 \\ & 1 \end{matrix} \middle| \, \dfrac{1+t}{2}\right)$

Legendre polynomials

$P_n(t) = \frac{1}{n!}(d/dt)^n (1-t^2)^n$

- $T_n(t) = (-1)^n {}_2F_1\left(\begin{matrix} -n & n \\ & \frac{1}{2} \end{matrix} \middle| \, \dfrac{1+t}{2}\right)$

Chebyshev polynomials

$T_n(cosa) = cos(na)$

Lacunary series

- $\sum t^{n!} = 2t + t^2 + t^6 + t^{24} + t^{120} + \cdots$ Liouville
- $\sum t^{2^n} = t + t^2 + t^4 + t^8 + t^{16} + t^{32} + t^{64} + t^{128} + \cdots$ Mahler
- $\Theta = \sum_n t^{n^2} = 1 + t + t^4 + t^9 + t^{16} + t^{25} + t^{36} + t^{49} + \cdots$ Jacobi
- $\sum t^{F_n} = 1 + 2t + t^2 + t^3 + t^5 + t^8 + t^{13} + t^{21} + t^{34} + t^{55} + \cdots$
- $\sum t^{p_n} = t^2 + t^3 + t^5 + t^7 + t^{11} + t^{13} + t^{17} + \cdots$
  where p_n is the nth prime number

Exotic series

- $\sum p_n t^n = 2t + 3t^2 + 5t^3 + 7t^4 + 11t^5 + 13t^6 + 17t^7 + 19t^8 + 23t^{10} + \cdots$
- $\sum a_n t^n = 4 + t + 4t^2 + 2t^3 + t^4 + 3t^5 + 5t^6 + 6t^7 + 2t^8 + \cdots,$
  where a_n is the nth decimal digit of $\sqrt{2}$
- $\sum \lfloor \tan(n) \rfloor t^n = t - 3t^2 - t^3 + t^4 - 4t^5 - t^6 - 7t^8 - t^9 - 226t^{11} - t^{12} + \cdots$
- $$\sum \lfloor n \tanh(\pi) \rfloor t^n = t^2 + 2t^3 + 3t^4 + \cdots + 267t^{268} + 267t^{269} + 268t^{270} + \cdots = \frac{t^2}{(t-1)^2} - t^{269} - t^{270} - \cdots$$
- $\sum_n \lfloor n\sqrt{2} \rfloor t^n = t + 2t^2 + 4t^3 + 5t^4 + 7t^5 + 8t^6 + 9t^7 + 11t^8 + 12t^9 + \cdots$

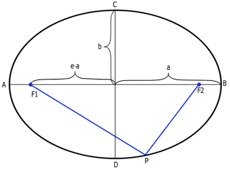
Number theoretic sequences, and their power series

- Euler's totient function $\varphi(n) = \#\{1 \leq k \leq n : \gcd(n,k) = 1\}$
- the Möbius function $\mu(n)$ = parity of the number of prime factors of n, if n is square-free; 0 if not
- the divisor function $\sigma_k(n) = \sum_{d|n} d^k$
  - $\sigma_0(n) = d(n)$ = number of positive divisors of n
  - $\sigma_1(n) = \sigma(n)$ = sum of positive divisors of n
- $\omega(n)$ = number of distinct prime factors of n
- $\Omega(n)$ = number of distinct prime factors of n, counted with multiplicity
- the Liouville function $\lambda(n) = (-1)^{\Omega(n)}$
- $\rho(n) = 2^{\omega(n)}$ = number of squarefree positive divisors of n
- $r_2(n) = \#\{(a,b) \in \mathbb{Z}^2 : a^2 + b^2 = n\}$ $\qquad \sum r_2(n) t^n = \Theta(t)^2$

Power series from the elliptic world

- Perimeter of an ellipse of eccentricity e, semi-major axis 1 [Euler, 1733]

$$p(e) = 4\int_0^1 \sqrt{\frac{1-e^2x^2}{1-x^2}}\,\mathrm{d}x = 2\pi - \frac{\pi}{2}e^2 - \frac{3\pi}{32}e^4 - \cdots$$



- Complete elliptic integrals
  - of the first kind $\quad K(e) = \displaystyle\int_0^1 \sqrt{\frac{1}{(1-x^2)(1-e^2x^2)}}\,\mathrm{d}x = \frac{\pi}{2}\,{}_2F_1\left(\left.\begin{matrix}\frac{1}{2} & \frac{1}{2}\\ & 1\end{matrix}\,\right|\, e^2\right)$
  - of the second kind $\quad E(e) = \displaystyle\frac{\pi}{2}\sum_{n=0}^{\infty}\left[\frac{(2n)!}{2^{2n}(n!)^2}\right]^2\frac{k^{2n}}{1-2e} = \frac{\pi}{2}\,{}_2F_1\left(\left.\begin{matrix}\frac{1}{2} & -\frac{1}{2}\\ & 1\end{matrix}\,\right|\, e^2\right)$
- Elliptic integrals $f(t) = \int R\left(t, \sqrt{P(t)}\right)\mathrm{d}t$, where R is a bivariate rational function, P a squarefree polynomial of degree 3 or 4
- Weierstrass elliptic function: inverse $y = \wp(t)$ of the elliptic integral

$$t = \int_y^\infty \frac{\mathrm{d}s}{\sqrt{4s^3 - g_2 s - g_3}}$$

$$\wp(t) = \frac{1}{t^2} + \frac{g_2}{20}t^2 + \frac{g_3}{28}t^4 + \frac{g_2^2}{1200}t^6 + \frac{3g_2g_3}{6160}t^8 + \frac{49g_2^3 + 750g_3^2}{7644000}t^{10}\cdots$$

Power series from the modular world

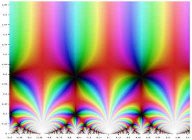
- Eisenstein modular series

$$E_4 = \quad 1 + 240 \sum_n \sigma_3(n) q^n = 1 + 240\, q + \;\; 2160\, q^2 + \quad 6720\, q^3 + \cdots$$

$$E_6 = \quad 1 - 504 \sum_n \sigma_5(n) q^n = 1 - 504\, q - 16632\, q^2 - 122976\, q^3 + \cdots$$

- Ramanujan's modular discriminant

$$\Delta = \frac{E_4^3 - E_6^2}{1728} = q \prod_{n \geq 1} (1 - q^n)^{24} = q - 24\, q^2 + 252\, q^3 + \cdots$$



- Klein's modular invariant

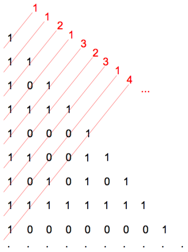
$$J = \frac{E_4^3}{\Delta} = \frac{1}{q} + 744 + 196884\, q + 21493760\, q^2 + 864299970\, q^3 + \cdots$$

▷ [Fricke, Klein, 1897] $\quad \Delta = \frac{1}{J} \cdot {}_2F_1\left(\left. \begin{matrix} \frac{1}{12} & \frac{5}{12} \\ & 1 \end{matrix} \right| \frac{1728}{J} \right)^{12},$

$$E_4 = {}_2F_1\left(\left. \begin{matrix} \frac{1}{12} & \frac{5}{12} \\ & 1 \end{matrix} \right| \frac{1728}{J} \right)^{4}, \qquad E_6 = \sqrt{1 - \frac{1728}{J}} \cdot {}_2F_1\left(\left. \begin{matrix} \frac{1}{12} & \frac{5}{12} \\ & 1 \end{matrix} \right| \frac{1728}{J} \right)^{6}.$$

Power series from the automatic world

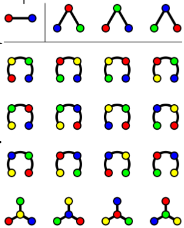
- Thue-Morse: $\sum s(n)t^n = t + t^2 + t^4 + t^7 + t^8 + t^{11} + t^{13} + t^{14} + \cdots$, where $s(n)$ is the parity of number of 1s in the base-2 expansion of n
- Baum–Sweet: $\sum b_n t^n = 1 + t + t^3 + t^4 + t^7 + t^9 + t^{12} + t^{15} + \cdots$, where $b_n = 1$ if the base-2 expansion of n contains no block of consecutive 0s of odd length, and $b_n = 0$ otherwise
- Rudin-Shapiro: $\sum (-1)^{a_n} t^n = 1 + t + t^2 - t^3 + t^4 + t^5 - t^6 + t^7 + \cdots$
  $a_n =$ the number of pairs of consecutive 1's in the base-2 expansion of n
- Stern:
  $\sum f_n t^n = t + t^2 + 2t^3 + t^4 + 3t^5 + 2t^6 + 3t^7 + t^8 + 4t^9 + \cdots$
  with $f_{2n+1} = f_n + f_{n+1}$, $f_{2n} = f_n$,
  $f_0 = 0, f_1 = 1$



- Partitions
$$\sum_{n=0}^{\infty} p(n) t^n = \prod_{k=1}^{\infty} \left(\frac{1}{1-t^k} \right) = 1 + t + 2t^2 + 3t^3 + 5t^4 + 7t^5 + 11t^6 + 15t^7 + \cdots$$
$$5 = 4+1 = 3+2 = 3+1+1 = 2+2+1 = 2+1+1+1 = 1+1+1+1+1$$

- Permutations in $\mathcal{S}_n$ containing 3 subsequences of type 132 [Bóna, 1997]
$\sum_n s_3(n) t^n = 1t^4 + 14t^5 + 82t^5 + 410t^6 + 1918t^7 + \cdots \quad (s_3(4) = 1\text{: } 1432)$

- Alternating permutations [André, 1881]
$$\sum_n \frac{a_n}{n!} t^n = \tan(t) + \sec(t) = 1 + t + \frac{1}{2!}t^2 + \frac{2}{3!}t^3 + \frac{5}{4!}t^4 + \frac{16}{5!}t^5 + \cdots$$
$a_4 = 5 : \{1,3,2,4\}, \{1,4,2,3\}, \{2,3,1,4\}, \{2,4,1,3\}, \{3,4,1,2\}$



- Labeled trees [Borchardt, 1860], [Cayley, 1889]
$$\sum_{n \geq 2} n^{n-2} t^n = 1t^2 + 3t^3 + 16t^4 + 125t^5 + 1296t^6 + 6807t^7 + \cdots$$

- Planar maps with n edges $\displaystyle\sum_n \frac{2 \cdot 3^n}{(n+1)(n+2)} \binom{2n}{n} t^n$ [Tutte, 1968]

Syracuse problem [Collatz, 1937]

$T : \mathbb{N}^+ \to \mathbb{N}^+$

$$T(n) = \begin{cases} n/2 & \text{if } n \equiv 0 \pmod 2 \\ 3n+1 & \text{if } n \equiv 1 \pmod 2. \end{cases}$$

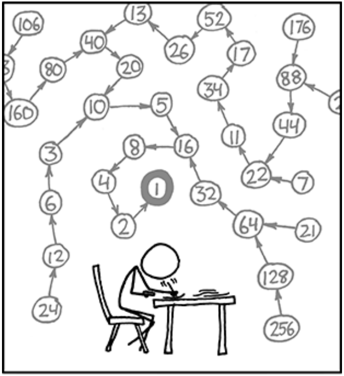
$\mathcal{O}(m) = \{n : T^{\circ j}(n) = m \text{ for some } j\}$

▷ Open: $\mathcal{O}(1) = \mathbb{N}^+$

$$f_m(t) = \sum_{n \in \mathcal{O}(m)} t^n$$

Theorem [Bell, Lagarias, 2015]

- f_m is transcendental for $m \notin \{1, 2, 4, 8, 16\}$
- f_m is rational for $m \in \{1, 2, 4, 8, 16\}$ iff the Collatz conjecture is true



THE COLLATZ CONJECTURE STATES THAT IF YOU PICK A NUMBER, AND IF IT'S EVEN DIVIDE IT BY TWO AND IF IT'S ODD MULTIPLY IT BY THREE AND ADD ONE, AND YOU REPEAT THIS PROCEDURE LONG ENOUGH, EVENTUALLY YOUR FRIENDS WILL STOP CALLING TO SEE IF YOU WANT TO HANG OUT.

The problem whether a given power series
is algebraic or transcendental at a given algebraic point
may be very deep and involved

[Mahler, 1976]

▷ Easy: Algebraic series take algebraic values at algebraic points and transcendental values at transcendental points

▷ Intuition: Transcendental series tend to take transcendental values at algebraic points $\longrightarrow$ finitely many exceptions?

Theorem [Stäckel 1895, 1902]
There exists a transcendental $f \in \mathbb{Q}[[t]]$ such that $f(\overline{\mathbb{Q}}) \subset \overline{\mathbb{Q}}$.

Theorem [Mahler 1965]
There exists a transcendental $f \in \mathbb{Q}[[t]]$ with $f(\sqrt{2}) \in \overline{\mathbb{Q}}$ and $f(-\sqrt{2}) \notin \overline{\mathbb{Q}}$.

Power series, and their values

The problem whether a given power series
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[Mahler, 1976]

▷ Easy: Algebraic series take algebraic values at algebraic points and transcendental values at transcendental points

▷ Intuition: Transcendental series tend to take transcendental values at algebraic points $\longrightarrow$ false in general, more structure needed

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There exists a transcendental $f \in \mathbb{Q}[[t]]$ such that $f(\overline{\mathbb{Q}}) \subset \overline{\mathbb{Q}}$.

Theorem [Mahler 1965]
There exists a transcendental $f \in \mathbb{Q}[[t]]$ with $f(\sqrt{2}) \in \overline{\mathbb{Q}}$ and $f(-\sqrt{2}) \notin \overline{\mathbb{Q}}$.

Power series, and their values

▷ [Siegel, 1949], [Shidlovski, 1962] For special classes of power series (e.g., E-functions), transcendental power series can take algebraic values at only finitely many algebraic points.

▷ Includes e^t and the Bessel function $J_0 = \sum_n \frac{(-t^2/4)^n}{n!^2}$

▷ False for G-functions, e.g. for many ${}_2F_1\left(\begin{matrix} a & b \\ & c \end{matrix} \middle| \, t\right)$'s.

Theorem [Beukers, Wolfart, 1988]
Let $z \in \mathbb{C}$, $|z| < 1$. Then:
${}_2F_1\left(\begin{matrix} \frac{1}{12} & \frac{5}{12} \\ & \frac{1}{2} \end{matrix} \middle| \, z\right) \in \overline{\mathbb{Q}}$ if and only if $z = 1 - \frac{1728}{J(\tau)}$ for some $\tau \in \mathbb{Q}(i)$, $\operatorname{Im}\tau > 0$.

▷ Example:

$$ {}_2F_1\left(\begin{matrix} \frac{1}{12} & \frac{5}{12} \\ & \frac{1}{2} \end{matrix} \middle| \, \frac{1323}{1331}\right) = \frac{3}{4}\sqrt[4]{11} $$

Two High Precision Frauds

[Borwein, Borwein, 1992]:

- If $e(n)$ and $o(n)$ are the number of even and odd decimal digits of n, then

$$\sum_{n=1}^{\infty} \frac{o(2^n)}{2^n} = \frac{1}{9}, \qquad \text{but } \sum_{n=1}^{\infty} \frac{e(2^n)}{2^n} \approx \frac{3166}{3069} \pm 10^{-30} \text{ is transcendental}$$

- If $\alpha = e^{\pi\sqrt{163/9}}$, then

$$\sum_{n=1}^{\infty} \frac{\lfloor n\alpha \rfloor}{2^n} \approx 1280640 \text{ (to half a billion digits!) is transcendental}$$

Thanks for your attention!