Computer Algebra for Lattice Path Combinatorics

Alin Bostan

Inria informatics mathematics

IHP, February 5, 2018

An (innocent looking) combinatorial question

Let $\mathfrak{S} = \{\uparrow, \leftarrow, \searrow\}$. A $\mathfrak{S}$-walk is a path in $\mathbb{Z}^2$ using only steps from $\mathfrak{S}$. Show that, for any integer n, the following quantities are equal:

(i) the number a_n of $\mathfrak{S}$-walks of length n confined to the upper half plane $\mathbb{Z} \times \mathbb{N}$ that start and end at the origin $(0,0)$;

(ii) the number b_n of $\mathfrak{S}$-walks of length n confined to the quarter plane $\mathbb{N}^2$ that start at the origin $(0,0)$ and finish on the diagonal $x = y$.

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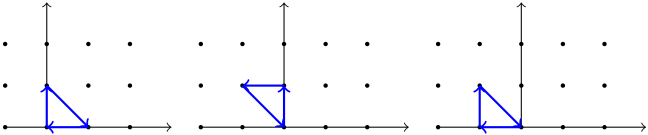
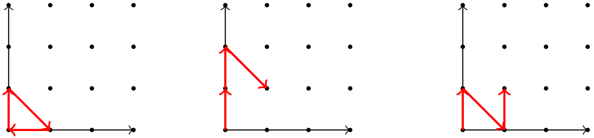
For instance, for $n = 3$, this common value is $a_3 = b_3 = 3$:

(*i*)



(*ii*)



Teasers

Teaser 1: This problem can be solved using computer algebra!

Teaser 2: The answer has a nice closed form!

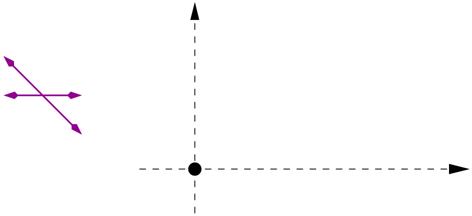
$$a_{3n} = b_{3n} = \frac{(3n)!}{n!^2 \cdot (n+1)!}, \qquad \text{and} \quad a_m = b_m = 0 \quad \text{if 3 does not divide } m.$$

Teaser 3: A certain group attached to the step set $\{\uparrow, \leftarrow, \searrow\}$ is finite!

Combinatorial context: lattice paths confined to cones

Let $\mathfrak{S}$ be a subset of $\mathbb{Z}^d$ (step set, or model) and $p_0 \in \mathbb{Z}^d$ (starting point).

Example: $\mathfrak{S} = \{(1,0),(-1,0),(1,-1),(-1,1)\}$, $p_0 = (0,0)$



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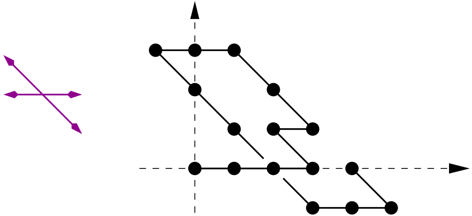

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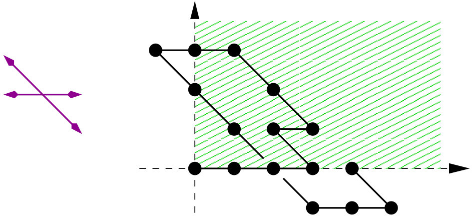
Let $\mathfrak{C}$ be a cone of $\mathbb{R}^d$ (if $x \in \mathfrak{C}$ and $r \geq 0$ then $r \cdot x \in \mathfrak{C}$).

Example: $\mathfrak{S} = \{(1,0), (-1,0), (1,-1), (-1,1)\}$, $p_0 = (0,0)$ and $\mathfrak{C} = \mathbb{R}_+^2$



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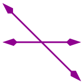

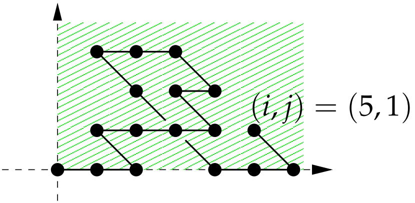

Questions

- What is the number a_n of n-step walks contained in $\mathfrak{C}$?
- For $\boldsymbol{i} \in \mathfrak{C}$, what is the number $a_{n;\boldsymbol{i}}$ of such walks that end at $\boldsymbol{i}$?
- What about their GF's $A(t) = \sum_n a_n t^n$ and $A(t; \boldsymbol{x}) = \sum_{n,\boldsymbol{i}} a_{n;\boldsymbol{i}} \boldsymbol{x}^{\boldsymbol{i}} t^n$?

Why count walks in cones?

Many discrete objects can be encoded in that way:

- discrete mathematics (permutations, trees, words, urns, ...)
- statistical physics (Ising model, ...)
- probability theory (branching processes, games of chance, ...)
- operations research (queueing theory, ...)

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A history and a survey of lattice path enumeration

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ABSTRACT

In celebration of the Sixth International Conference on Lattice Path Counting and Applications, it is befitting to review the history of lattice path enumeration and to survey how the topic has progressed thus far.

We start the history with early games of chance specifically the ruin problem which later appears as the ballot problem. We discuss André's Reflection Principle and its misnomer, its relation with the method of images and possible origins from physics and Brownian motion, and the earliest evidence of lattice path techniques and solutions.

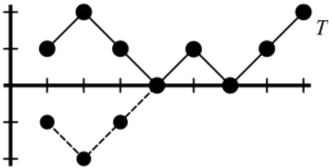
In the survey, we give representative articles on lattice path enumeration found in the literature in the last 35 years by the lattice, step set, boundary, characteristics counted, and solution method. Some of this work appears in the author's 2005 dissertation.

An old topic: ballot problem [Bertrand, 1887]

Suppose that candidates A and B are running in an election. If a votes are cast for A and b votes are cast for B, where $a > b$, then the probability that A stays ahead of B throughout the counting of the ballots is $(a-b)/(a+b)$.

Lattice path reformulation: find the number of paths that start at the origin and never touch the x-axis, consisting of a upsteps $\nearrow$ and b downsteps $\searrow$

Reflection principle [Aebly, 1923]: *paths in* $\mathbb{N}^2$ *from* $(1,1)$ *to* $T(a+b, a-b)$ *that do touch the* x*-axis* are in bijection with *paths in* $\mathbb{Z}^2$ *from* $(1,-1)$ *to* T



Answer: (*paths in* $\mathbb{Z}^2$ *from* $(1,1)$ *to* T) $-$ (*paths in* $\mathbb{Z}^2$ *from* $(1,-1)$ *to* T)

$$\binom{a+b-1}{a-1} - \binom{a+b-1}{b-1} = \frac{a-b}{a+b}\binom{a+b}{a}$$

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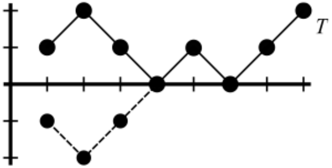


Answer: when $a = n+1$ and $b = n$, this is the Catalan number

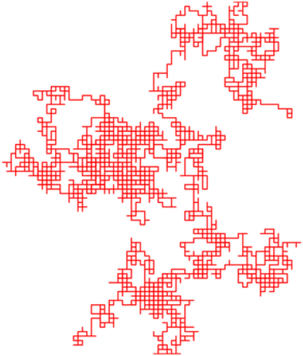
$$C_n = \frac{1}{2n+1}\binom{2n+1}{n+1} = \frac{1}{n+1}\binom{2n}{n}$$

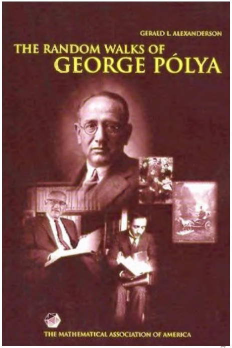
An old topic: Pólya's "promenade au hasard" / "Irrfahrt"

Motto: Drunkard: "Will I ever, ever get home again?"
Polya (1921): "You can't miss; just keep going and stay out of 3D!" (Adam and Delbruck, 1968)

[Pólya, 1921] Simple random walk $\{\pm 1\}^d$ on $\mathbb{Z}^d$ is recurrent in dimensions $d = 1, 2$ ("Alle Wege führen nach Rom"), and transient in dimension $d \geq 3$

Über eine Aufgabe der Wahrscheinlichkeitsrechnung betreffend die Irrfahrt im Straßennetz.





…but still a topical issue

Many recent contributors:

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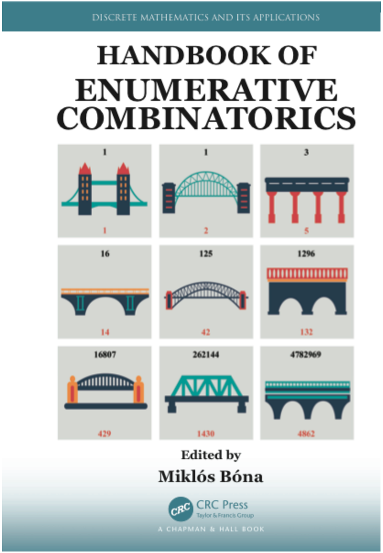
etc.

~~**Specific question**~~

~~**Ad hoc solution**~~ $\longrightarrow$ **Systematic approach**

. . . but still a topical issue



Chapter 10

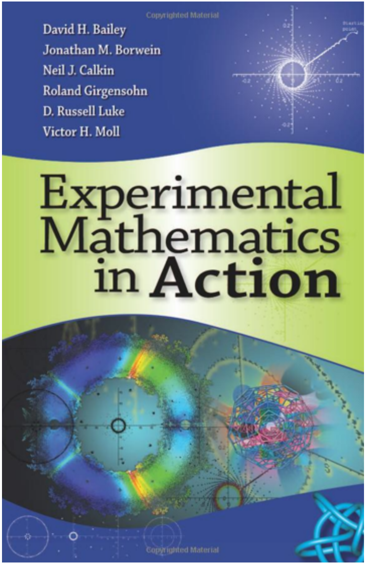
Lattice Path Enumeration

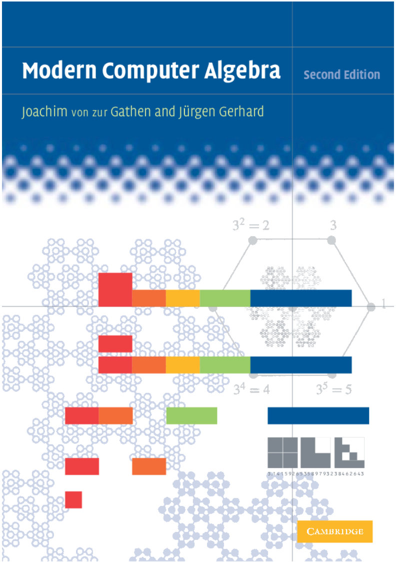
Christian Krattenthaler

Universität Wien

CONTENTS

Approach: Experimental Mathematics using Computer Algebra





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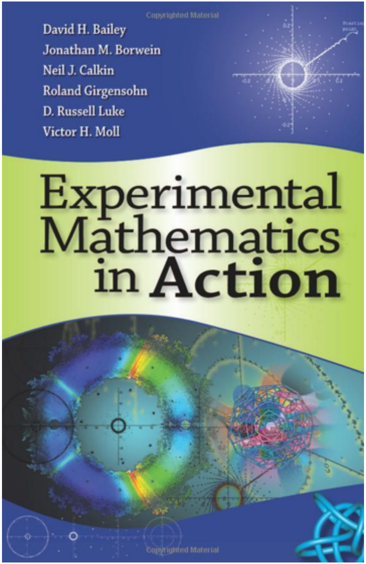
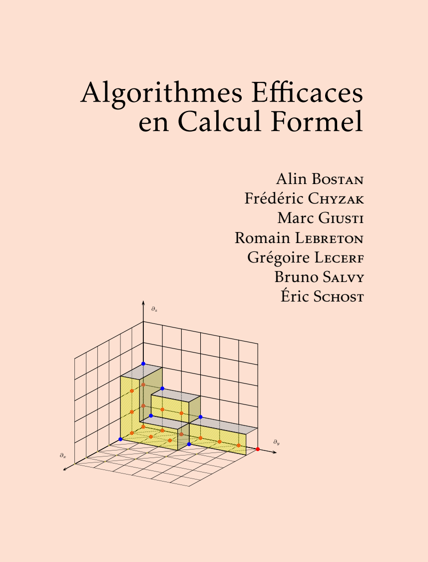
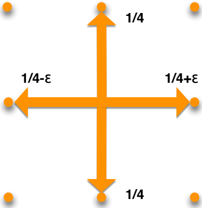
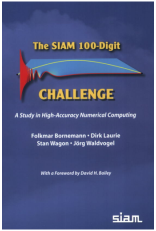




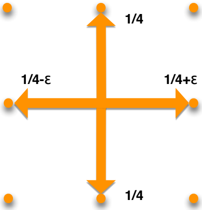


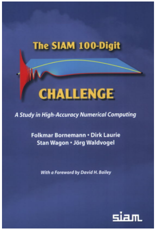


Problem 6

A flea starts at $(0,0)$ *on the infinite two-dimensional integer lattice and executes a biased random walk: At each step it hops north or south with probability* $1/4$*, east with probability* $1/4+\epsilon$*, and west with probability* $1/4-\epsilon$*. The probability that the flea returns to* $(0,0)$ *sometime during its wanderings is* $1/2$*. What is* ϵ*?*

Example: From the SIAM 100-Digit Challenge [Trefethen, 2002]





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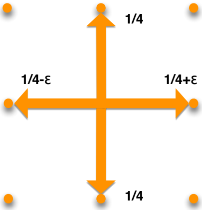
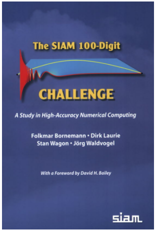
▷ Computer algebra conjectures and proves

$$p(\epsilon) = 1 - \sqrt{\frac{A}{2}} \cdot {}_2F_1\left(\begin{matrix} \frac{1}{2}, \frac{1}{2} \\ 1 \end{matrix} \;\middle|\; \frac{2\sqrt{1-16\epsilon^2}}{A} \right)^{-1}, \quad \text{with } A = 1 + 8\epsilon^2 + \sqrt{1-16\epsilon^2}.$$





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$$\epsilon \approx 0.0619139544739909428481752164732121769996387749983\\ 6207606146725885993101029759615845907105645752087861\ldots$$

A (very) basic cone: the full space

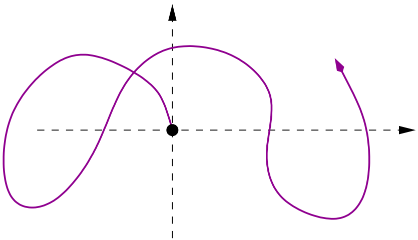
Rational series [folklore]

If $\mathfrak{S} \subset \mathbb{Z}^d$ is finite and $\mathfrak{C} = \mathbb{R}^d$, then

$$a_n = |\mathfrak{S}|^n, \text{ i.e. } \quad A(t) = \sum_{n \geq 0} a_n t^n = \frac{1}{1 - |\mathfrak{S}|\, t}.$$

More generally:

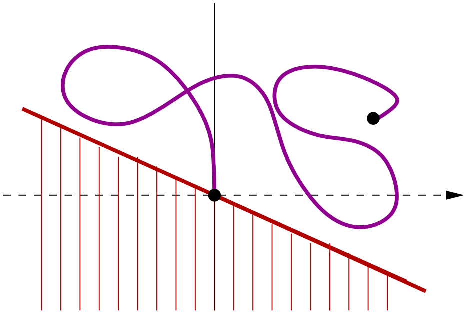
$$A(t;\boldsymbol{x}) = \sum_{n,\boldsymbol{i}} a_{n;\boldsymbol{i}} \boldsymbol{x}^{\boldsymbol{i}} t^n = \frac{1}{1 - t \sum_{\boldsymbol{s} \in \mathfrak{S}} \boldsymbol{x}^{\boldsymbol{s}}}.$$



Also well-known: a (rational) half-space

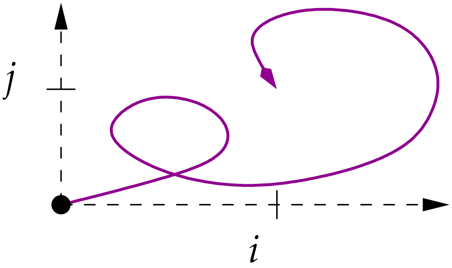
Algebraic series [Bousquet-Mélou, Petkovšek, 2000]

If $\mathfrak{S} \subset \mathbb{Z}^d$ is finite and $\mathfrak{C}$ is a rational half-space, then $A(t; \boldsymbol{x})$ is algebraic, given by an explicit system of polynomial equations.



Example: For Dyck paths (ballot problem), $A(t;1) = \sum_{n \geq 0} C_n t^n = \dfrac{1 - \sqrt{1-4t}}{2t}$

The “next” case: intersection of two half-spaces



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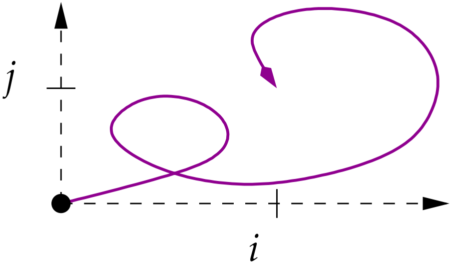


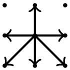
?

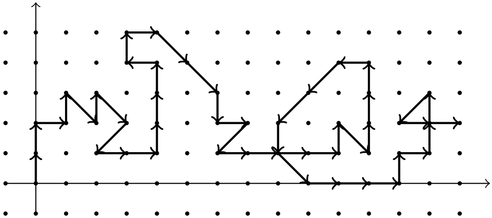
Lattice walks with small steps in the quarter plane

▷ From now on: we focus on nearest-neighbor walks in the quarter plane, i.e. walks in $\mathbb{N}^2$ starting at $(0,0)$ and using steps in a *fixed* subset $\mathfrak{S}$ of

$$\{\swarrow, \leftarrow, \nwarrow, \uparrow, \nearrow, \rightarrow, \searrow, \downarrow\}.$$

▷ Example with $n = 45$, $i = 14$, $j = 2$ for:

$\mathfrak{S} =$ 



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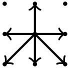
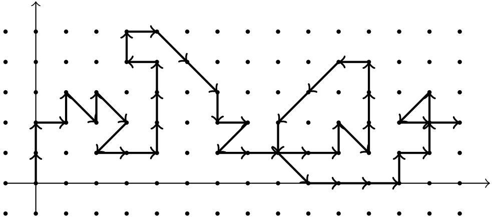
$\mathfrak{S} =$ 



▷ Counting sequence: $f_{n;i,j}$ = number of walks of length n ending at (i, j).

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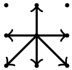
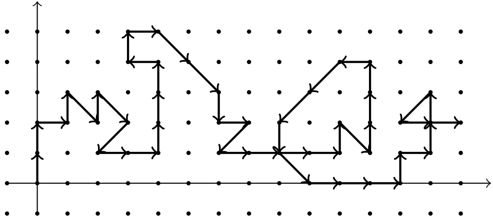
$\mathfrak{S} =$ 



▷ Counting sequence: $f_{n;i,j}$ = number of walks of length n ending at (i,j).

▷ Specializations:

- $f_{n;0,0}$ = number of walks of length n returning to origin ("excursions");
- $f_n = \sum_{i,j\geq 0} f_{n;i,j}$ = number of walks with prescribed length n.

Generating functions and combinatorial problems

▷ Complete generating function:

$$F(t;x,y) = \sum_{n=0}^{\infty} \left(\sum_{i,j=0}^{\infty} f_{n;i,j} x^i y^j \right) t^n \ \in \mathbb{Q}[x,y][[t]].$$

Generating functions and combinatorial problems

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▷ Specializations:

- GF of excursions: $F(t;0,0)$;
- GF of walks: $F(t;1,1) = \sum_{n \geq 0} f_n t^n$;
- GF of horizontal returns: $F(t;1,0)$;
- GF of diagonal returns: "$F(t;0,\infty)$" $:= \left[x^0\right] F(t;x,1/x)$.

▷ Complete generating function:

$$F(t;x,y) = \sum_{n=0}^{\infty} \left(\sum_{i,j=0}^{\infty} f_{n;i,j} x^i y^j \right) t^n \in \mathbb{Q}[x,y][[t]].$$

▷ Specializations:

- GF of excursions: $F(t;0,0)$;
- GF of walks: $F(t;1,1) = \sum_{n \geq 0} f_n t^n$;
- GF of horizontal returns: $F(t;1,0)$;
- GF of diagonal returns: "$F(t;0,\infty)$" $:= \left[x^0\right] F(t;x,1/x)$.

Combinatorial questions:
Given $\mathfrak{S}$, what can be said about $F(t;x,y)$, resp. $f_{n;i,j}$, and their variants?

- Structure of F: algebraic? transcendental? solution of ODE?
- Explicit form: of F? of $f_{n;i,j}$?
- Asymptotics of $f_{n;0,0}$? of f_n?

Generating functions and combinatorial problems

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Our goal: Use computer algebra to give computational answers.

Small-step models of interest

Among the 2^8 step sets $\mathfrak{S} \subseteq \{-1,0,1\}^2 \setminus \{(0,0)\}$, some are:

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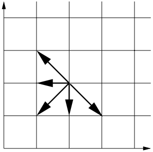

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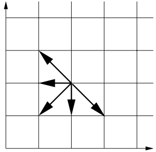

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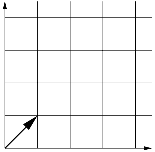

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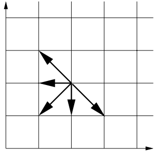

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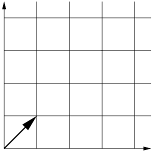

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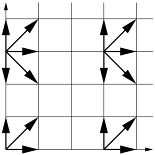

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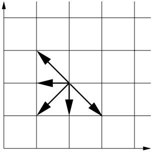

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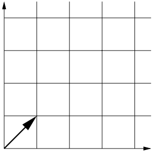

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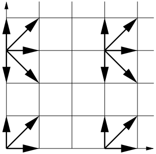

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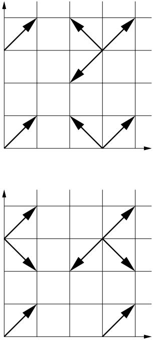

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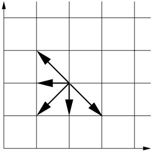

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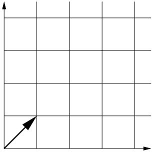

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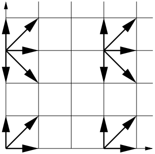

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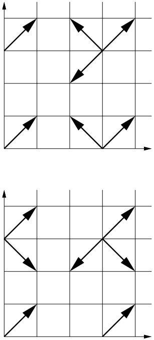

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One is left with 79 interesting distinct models.

Small-step models of interest

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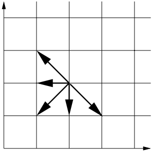

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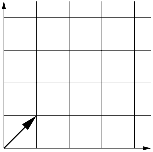

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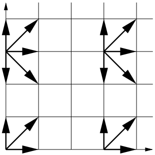

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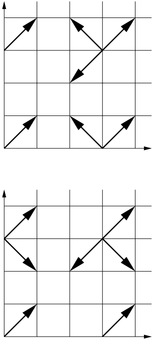
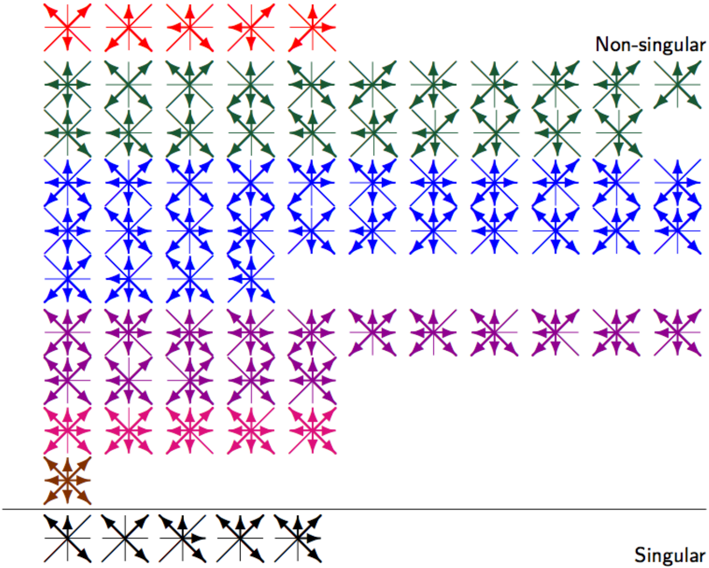


symmetrical.

One is left with 79 interesting distinct models.

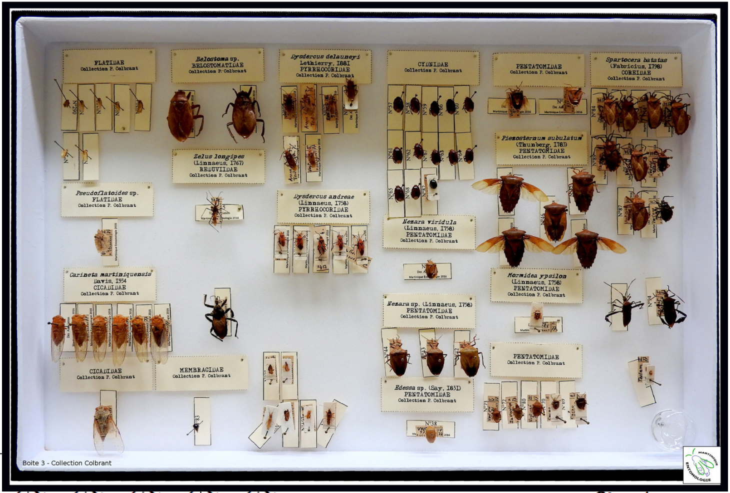
Is any further classification possible?

The 79 models



The 79 models

Non-singular

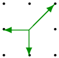


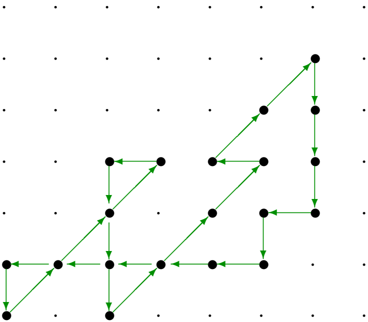
Singular

Two important models: Kreweras and Gessel walks

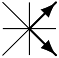
$\mathfrak{S} = \{\downarrow, \leftarrow, \nearrow\}$ $\qquad F_{\mathfrak{S}}(t;x,y) \equiv K(t;x,y)$

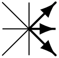
$\mathfrak{S} = \{\nearrow, \swarrow, \leftarrow, \rightarrow\}$ $\qquad F_{\mathfrak{S}}(t;x,y) \equiv G(t;x,y)$





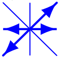
Example: A Kreweras excursion.

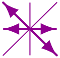
Dyck: 

Motzkin: 

Pólya: 

Kreweras: 

Gessel: 

Gouyou-Beauchamps: 

King walks: 

Tandem walks: 

$$\mathfrak{S} = \{\nearrow, \swarrow, \leftarrow, \rightarrow\}$$

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1,2,11,85 Search Hints
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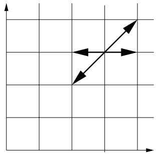
Search: **seq:1,2,11,85**

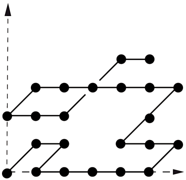
Displaying 1-1 of 1 result found. page 1

Sort: relevance | references | number | modified | created Format: long | short | data

A135404 Gessel sequence: the number of paths of length 2m in the plane, starting and ending at (0,1), with unit steps in the four directions (north, east, south, west) and staying in the region y>0, x>-y. +20 6

1, 2, 11, 85, 782, 8004, 88044, 1020162, 12294260, 152787976, 1946310467, 25302036071, 334560525538, 4488007049900, 60955295750460, 836838395382645, 11597595644244186, 162074575606984788, 2281839419729917410, 32340239369121304038, 461109219391987625316, 6610306991283738684600 (list; graph; refs; listen; history; text; internal format)



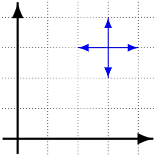


Conjecture 1 The generating function of Gessel excursions is equal to

$$G(t;0,0) = {}_3F_2\left(\begin{matrix}5/6 & 1/2 & 1\\ & 5/3 & 2\end{matrix}\,\middle|\, 16t^2\right)$$
$$= \sum_{n=0}^{\infty} \frac{(5/6)_n (1/2)_n}{(5/3)_n (2)_n} (4t)^{2n}$$
$$= 1 + 2t^2 + 11t^4 + 85t^6 + 782t^8 + \cdots$$

Conjecture 2
The full generating function $G(t;x,y)$ is not D-finite.

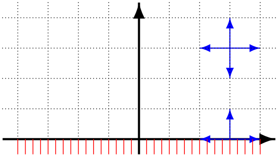
The simple walk in the plane

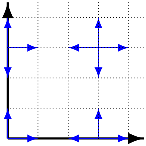


[Pólya, 1921]:

▷ Formula $\binom{2n}{n}^2$ for $2n$-excursions
▷ Rational generating function

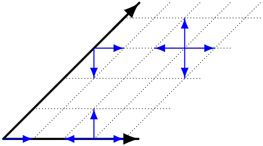
The simple walk in the half-plane and in the quarter-plane

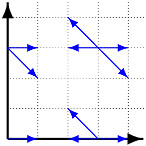




▷ Formulas $\binom{2n+1}{n}C_n$, resp. $C_n C_{n+1}$, for $2n$-excursions [Arquès, 1986]
▷ Full generating functions: algebraic [Bousquet-Mélou, Petkovšek, 2000], resp. D-finite [Bousquet-Mélou, 2002]

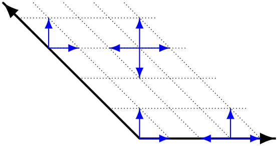
The simple walk in the cone with angle 45°

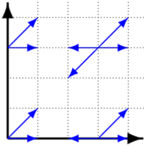




- ▷ Formula $C_n C_{n+2} - C_{n+1}^2$ for $2n$-excursions [Gouyou-Beauchamps, 1986]
- ▷ D-finite generating function [Gessel, Zeilberger, 1992]

What about the simple walk in the cone with angle 135°?

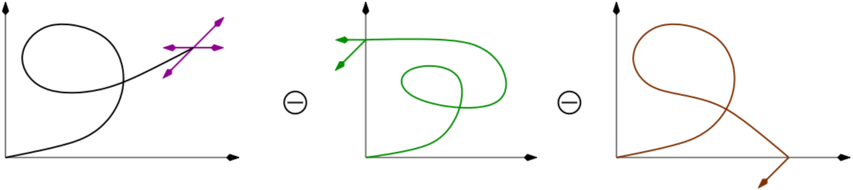




Generating function: $G(t;x,y) = \sum_{n=0}^{\infty}\sum_{i=0}^{n}\sum_{j=0}^{n} g_{n;i,j} t^n x^i y^j \in \mathbb{Q}[x,y][[t]]$

"Kernel equation":

$$\begin{aligned} G(t;x,y) =& 1 + t\left(xy + x + \frac{1}{xy} + \frac{1}{x}\right) G(t;x,y) \\ & - t\left(\frac{1}{x} + \frac{1}{x}\frac{1}{y}\right) G(t;0,y) - t\frac{1}{xy}\left(G(t;x,0) - G(t;0,0)\right) \end{aligned}$$



Algebraic reformulation: solving a functional equation

Generating function: $G(t;x,y) = \sum_{n=0}^{\infty} \sum_{i=0}^{n} \sum_{j=0}^{n} g_{n;i,j} t^n x^i y^j \in \mathbb{Q}[x,y][[t]]$

"Kernel equation":

$$G\,(t;x,y) = 1 + t\left(xy + x + \frac{1}{xy} + \frac{1}{x}\right) G\,(t;x,y)$$
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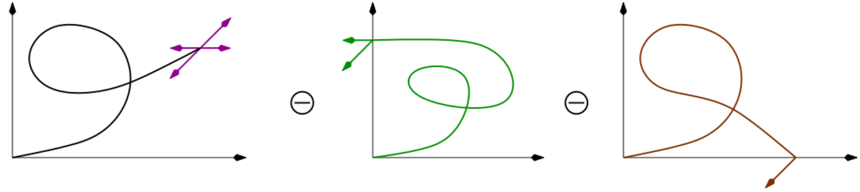

Task: Solve this functional equation!

Algebraic reformulation: solving a functional equation

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"Kernel equation":

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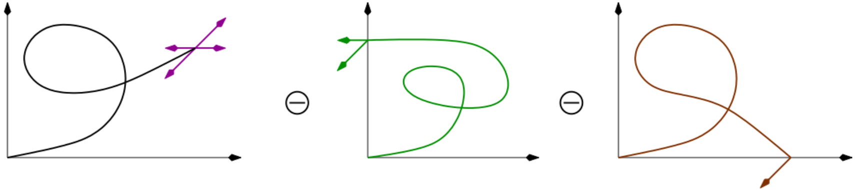
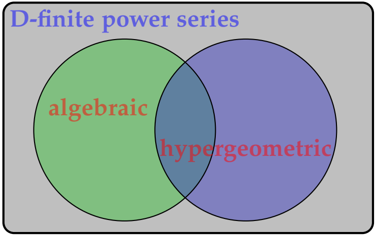
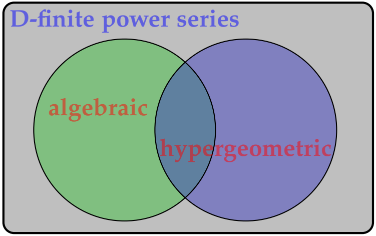


Task: For the other models – solve 78 similar equations!

Classification of univariate power series

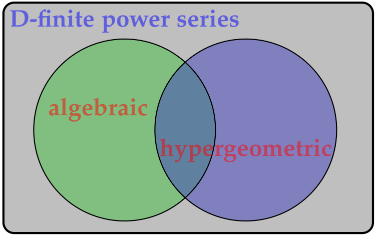




$$S(t) = \sum_{n=0}^{\infty} s_n t^n \in \mathbb{Q}[[t]] \quad \text{is}$$

▷ *algebraic* if $P(t, S(t)) = 0$ for some $P(x, y) \in \mathbb{Z}[x, y] \setminus \{0\}$;

Classification of univariate power series



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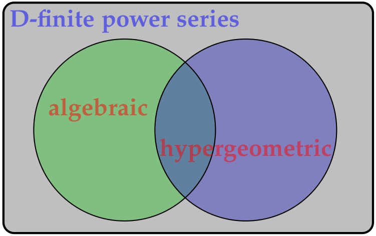
▷ *algebraic* if $P(t, S(t)) = 0$ for some $P(x,y) \in \mathbb{Z}[x,y] \setminus \{0\}$;

▷ *D-finite* if $c_r(t)S^{(r)}(t) + \cdots + c_0(t)S(t) = 0$ for some $c_i \in \mathbb{Z}[t]$, not all zero;

Classification of univariate power series



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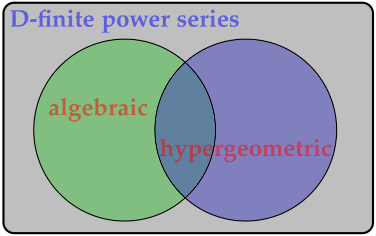

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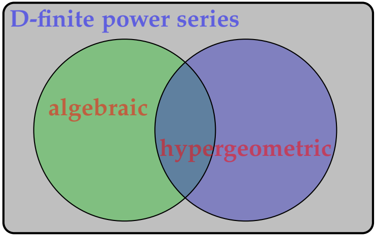
▷ *hypergeometric* if $\frac{s_{n+1}}{s_n} \in \mathbb{Q}(n)$. E.g.,

$$\ln(1-t); \quad \frac{\arcsin(\sqrt{t})}{\sqrt{t}}; \quad (1-t)^{\alpha}, \alpha \in \mathbb{Q}$$

Classification of univariate power series



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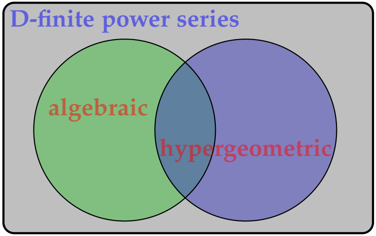
▷ *hypergeometric* if $\frac{s_{n+1}}{s_n} \in \mathbb{Q}(n)$. E.g.,

$${}_2F_1\left(\begin{matrix} a\ b \\ c \end{matrix} \,\middle|\, t\right) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{t^n}{n!}, \quad (a)_n = a(a+1)\cdots(a+n-1).$$



$$S(t) = \sum_{n=0}^{\infty} s_n t^n \in \mathbb{Q}[[t]] \quad \text{is}$$

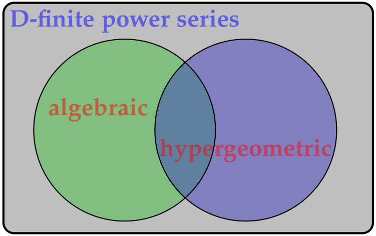
▷ *algebraic* if $P\big(t, S(t)\big) = 0$ for some $P(x,y) \in \mathbb{Z}[x,y] \setminus \{0\}$;

▷ *D-finite* if $c_r(t)S^{(r)}(t) + \cdots + c_0(t)S(t) = 0$ for some $c_i \in \mathbb{Z}[t]$, not all zero;

▷ *hypergeometric* if $\frac{s_{n+1}}{s_n} \in \mathbb{Q}(n)$. E.g.,

$${}_3F_2\left(\begin{matrix} a\ b\ c \\ d\ e \end{matrix} \,\middle|\, t\right) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n (c)_n}{(d)_n (e)_n} \frac{t^n}{n!}, \quad (a)_n = a(a+1)\cdots(a+n-1).$$

Classification of univariate power series



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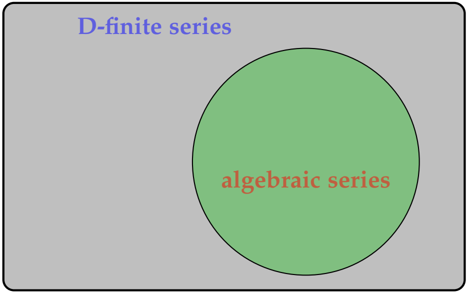
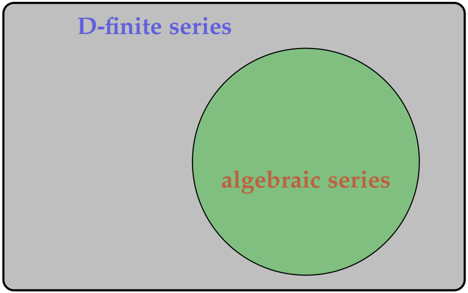
Theorem [Schwarz, 1873; Beukers, Heckman, 1989]

Characterization of { *hypergeometric* } $\cap$ { *algebraic* }.



▷ $S \in \mathbb{Q}[[x,y,t]]$ is *algebraic* if it is the root of a polynomial $P \in \mathbb{Q}[x,y,t,T]$.

Classification of multivariate power series



▷ $S \in \mathbb{Q}[[x,y,t]]$ is *algebraic* if it is the root of a polynomial $P \in \mathbb{Q}[x,y,t,T]$.

▷ $S \in \mathbb{Q}[[x,y,t]]$ is *D-finite* if it satisfies a system of linear partial differential equations with polynomial coefficients

$$\sum_i a_i(t,x,y)\frac{\partial^i S}{\partial x^i} = 0, \quad \sum_i b_i(t,x,y)\frac{\partial^i S}{\partial y^i} = 0, \quad \sum_i c_i(t,x,y)\frac{\partial^i S}{\partial t^i} = 0.$$

Main results (I): algebraicity of Gessel walks

Theorem [Kreweras, 1965; 100 pages long combinatorial proof!]

$$K(t;0,0) = {}_3F_2\left(\begin{matrix}1/3 & 2/3 & 1\\ & 3/2 & 2\end{matrix}\,\middle|\, 27\,t^3\right) = \sum_{n=0}^{\infty} \frac{4^n \binom{3n}{n}}{(n+1)(2n+1)} t^{3n}.$$

Theorem [Kauers, Koutschan, Zeilberger, 2009: former Gessel's conj. 1]

$$G(t;0,0) = {}_3F_2\left(\begin{matrix}5/6 & 1/2 & 1\\ & 5/3 & 2\end{matrix}\,\middle|\, 16t^2\right) = \sum_{n=0}^{\infty} \frac{(5/6)_n (1/2)_n}{(5/3)_n (2)_n} (4t)^{2n}.$$

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▷ Computer-driven discovery and proof.
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† Minimal polynomial $P(x,y,t,G(t;x,y)) = 0$ has $> 10^{11}$ terms; $\approx 30\,$Gb (!)

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Main results (II): Explicit form for $G(t;x,y)$

Theorem [B., Kauers, van Hoeij, 2010]

Let $V = 1 + 4t^2 + 36t^4 + 396t^6 + \cdots$ be a root of

$$(V-1)(1+3/V)^3 = (16t)^2,$$

let $U = 1 + 2t^2 + 16t^4 + 2xt^5 + 2(x^2+83)t^6 + \cdots$ be a root of

$$x(V-1)(V+1)U^3 - 2V(3x+5xV-8Vt)U^2$$
$$-xV(V^2-24V-9)U + 2V^2(xV-9x-8Vt) = 0,$$

let $W = t^2 + (y+8)t^4 + 2(y^2+8y+41)t^6 + \cdots$ be a root of

$$y(1-V)W^3 + y(V+3)W^2 - (V+3)W + V - 1 = 0.$$

Then $G(t;x,y)$ is equal to

$$\frac{\frac{64(U(V+1)-2V)V^{3/2}}{x(U^2-V(U^2-8U+9-V))^2} - \frac{y(W-1)^4(1-Wy)V^{-3/2}}{t(y+1)(1-W)(W^2y+1)^2}}{(1+y+x^2y+x^2y^2)t - xy} - \frac{1}{tx(y+1)}.$$

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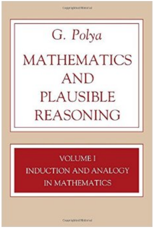

Guessing and Proving

George Pólya

What is "scientific method"? Philosophers and non-philosophers have discussed this question and have not yet finished discussing it. Yet as a first introduction it can be described in three syllables:

Guess and test.

Mathematicians too follow this advice in their research although they sometimes refuse to confess it. They have, however, something which the other scientists cannot really have. For mathematicians the advice is

First guess, then prove.

A typical *guess'n'prove* algorithmic proof

Theorem

$$g(t) := G(\sqrt{t}; 0, 0) = \sum_{n=0}^{\infty} \frac{(5/6)_n (1/2)_n}{(5/3)_n (2)_n} (16t)^n \quad \text{is algebraic.}$$

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Proof: First guess a polynomial $P(t, T)$ in $\mathbb{Q}[t, T]$, then prove that P admits the power series $g(t) = \sum_{n=0}^{\infty} g_n t^n$ as a root.

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1. Find P such that $P(t,g(t)) = 0 \bmod t^{100}$ by (structured) linear algebra.

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1. Find P such that $P(t, g(t)) = 0 \bmod t^{100}$ by (structured) linear algebra.
2. Implicit function theorem: $\exists!$ root $r(t) \in \mathbb{Q}[[t]]$ of P.

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1. Find P such that $P(t, g(t)) = 0 \bmod t^{100}$ by (structured) linear algebra.
2. Implicit function theorem: $\exists!$ root $r(t) \in \mathbb{Q}[[t]]$ of P.
3. $r(t) = \sum_{n=0}^{\infty} r_n t^n$ being algebraic, it is D-finite, and so is (r_n):

$$(n+2)(3n+5)r_{n+1} - 4(6n+5)(2n+1)r_n = 0, \qquad r_0 = 1$$

$\Rightarrow$ solution $r_n = \frac{(5/6)_n (1/2)_n}{(5/3)_n (2)_n} 16^n = g_n$, thus $g(t) = r(t)$ is algebraic.

	OEIS	$\mathfrak{S}$	Pol size	LDE size	Rec size		OEIS	$\mathfrak{S}$	Pol size	LDE size	Rec size
1	A005566		—	(3, 4)	(2, 2)	13	A151275		—	(5, 24)	(9, 18)
2	A018224		—	(3, 5)	(2, 3)	14	A151314		—	(5, 24)	(9, 18)
3	A151312		—	(3, 8)	(4, 5)	15	A151255		—	(4, 16)	(6, 8)
4	A151331		—	(3, 6)	(3, 4)	16	A151287		—	(5, 19)	(7, 11)
5	A151266		—	(5, 16)	(7, 10)	17	A001006		(2, 2)	(2, 3)	(2, 1)
6	A151307		—	(5, 20)	(8, 15)	18	A129400		(2, 2)	(2, 3)	(2, 1)
7	A151291		—	(5, 15)	(6, 10)	19	A005558		—	(3, 5)	(2, 3)
8	A151326		—	(5, 18)	(7, 14)						
9	A151302		—	(5, 24)	(9, 18)	20	A151265		(6, 8)	(4, 9)	(6, 4)
10	A151329		—	(5, 24)	(9, 18)	21	A151278		(6, 8)	(4, 12)	(7, 4)
11	A151261		—	(4, 15)	(5, 8)	22	A151323		(4, 4)	(2, 3)	(2, 1)
12	A151297		—	(5, 18)	(7, 11)	23	A060900		(8, 9)	(3, 5)	(2, 3)

Equation sizes = (order, degree)

- ▷ Computerized discovery: enumeration + guessing [B., Kauers, 2009]
- ▷ 1–22: Confirmed by human proofs in [Bousquet-Mélou, Mishna, 2010]
- ▷ 23: Confirmed by a human proof in [B., Kurkova, Raschel, 2013]

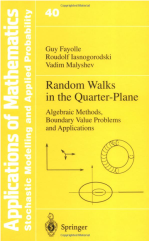
Main results (III): Models with D-Finite $F(t;1,1)$

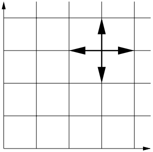
	OEIS	$\mathfrak{S}$	algebraic?	asymptotics		OEIS	$\mathfrak{S}$	algebraic?	asymptotics
1	A005566		N	$\frac{4}{\pi}\frac{4^n}{n}$	13	A151275		N	$\frac{12\sqrt{30}}{\pi}\frac{(2\sqrt{6})^n}{n^2}$
2	A018224		N	$\frac{2}{\pi}\frac{4^n}{n}$	14	A151314		N	$\frac{\sqrt{6}\lambda\mu C^{5/2}}{5\pi}\frac{(2C)^n}{n^2}$
3	A151312		N	$\frac{\sqrt{6}}{\pi}\frac{6^n}{n}$	15	A151255		N	$\frac{24\sqrt{2}}{\pi}\frac{(2\sqrt{2})^n}{n^2}$
4	A151331		N	$\frac{8}{3\pi}\frac{8^n}{n}$	16	A151287		N	$\frac{2\sqrt{2}A^{7/2}}{\pi}\frac{(2A)^n}{n^2}$
5	A151266		N	$\frac{1}{2}\sqrt{\frac{3}{\pi}}\frac{3^n}{n^{1/2}}$	17	A001006		Y	$\frac{3}{2}\sqrt{\frac{3}{\pi}}\frac{3^n}{n^{3/2}}$
6	A151307		N	$\frac{1}{2}\sqrt{\frac{5}{2\pi}}\frac{5^n}{n^{1/2}}$	18	A129400		Y	$\frac{3}{2}\sqrt{\frac{3}{\pi}}\frac{6^n}{n^{3/2}}$
7	A151291		N	$\frac{4}{3\sqrt{\pi}}\frac{4^n}{n^{1/2}}$	19	A005558		N	$\frac{8}{\pi}\frac{4^n}{n^2}$
8	A151326		N	$\frac{2}{\sqrt{3\pi}}\frac{6^n}{n^{1/2}}$					
9	A151302		N	$\frac{1}{3}\sqrt{\frac{5}{2\pi}}\frac{5^n}{n^{1/2}}$	20	A151265		Y	$\frac{2\sqrt{2}}{\Gamma(1/4)}\frac{3^n}{n^{3/4}}$
10	A151329		N	$\frac{1}{3}\sqrt{\frac{7}{3\pi}}\frac{7^n}{n^{1/2}}$	21	A151278		Y	$\frac{3\sqrt{3}}{\sqrt{2}\Gamma(1/4)}\frac{3^n}{n^{3/4}}$
11	A151261		N	$\frac{12\sqrt{3}}{\pi}\frac{(2\sqrt{3})^n}{n^2}$	22	A151323		Y	$\frac{\sqrt{2}3^{3/4}}{\Gamma(1/4)}\frac{6^n}{n^{3/4}}$
12	A151297		N	$\frac{\sqrt{3}B^{7/2}}{2\pi}\frac{(2B)^n}{n^2}$	23	A060900		Y	$\frac{4\sqrt{3}}{3\Gamma(1/3)}\frac{4^n}{n^{2/3}}$

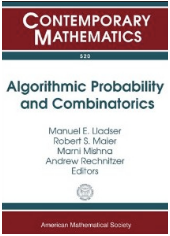
$$A = 1+\sqrt{2},\ B = 1+\sqrt{3},\ C = 1+\sqrt{6},\ \lambda = 7+3\sqrt{6},\ \mu = \sqrt{\tfrac{4\sqrt{6}-1}{19}}$$

▷ Computerized discovery: conv. acc. + LLL/PSLQ [B., Kauers, 2009]

▷ Confirmed by human proofs using ACSV in [Melczer, Wilson, 2015]

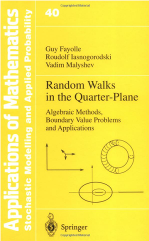


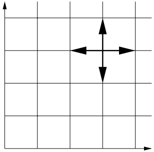


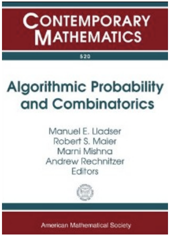


The characteristic polynomial $\chi_{\mathfrak{S}} := x + \frac{1}{x} + y + \frac{1}{y}$

The group of a model: the simple walk case



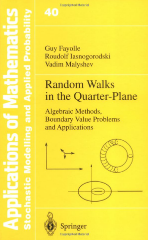


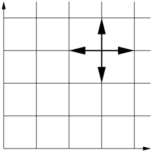


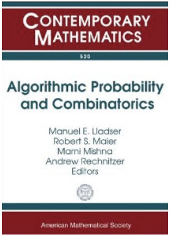
The characteristic polynomial $\chi_{\mathfrak{S}} := x + \frac{1}{x} + y + \frac{1}{y}$ is left invariant under

$$\psi(x,y) = \left(x, \frac{1}{y}\right), \quad \phi(x,y) = \left(\frac{1}{x}, y\right),$$

The group of a model: the simple walk case







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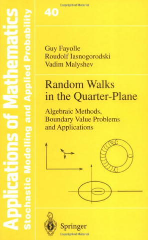
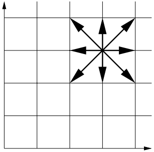
$$\psi(x,y) = \left(x, \frac{1}{y}\right), \qquad \phi(x,y) = \left(\frac{1}{x}, y\right),$$

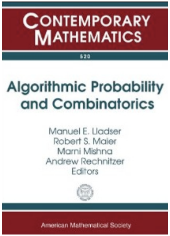
and thus under any element of the group

$$\langle \psi, \phi \rangle = \left\{ (x,y), \left(x, \frac{1}{y}\right), \left(\frac{1}{x}, \frac{1}{y}\right), \left(\frac{1}{x}, y\right) \right\}.$$

The group of a model: the general case







The polynomial $\chi_{\mathfrak{S}} := \sum_{(i,j)\in\mathfrak{S}} x^i y^j = \sum_{i=-1}^{1} B_i(y) x^i = \sum_{j=-1}^{1} A_j(x) y^j$

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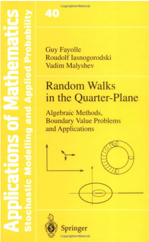

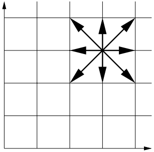

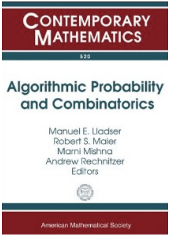

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$$\psi(x,y) = \left(x, \frac{A_{-1}(x)}{A_{+1}(x)}\frac{1}{y}\right), \qquad \phi(x,y) = \left(\frac{B_{-1}(y)}{B_{+1}(y)}\frac{1}{x}, y\right),$$

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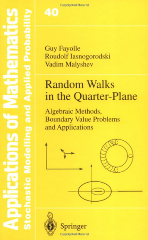

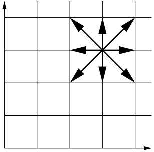

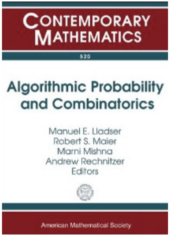

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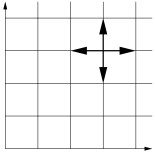
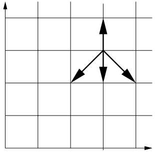
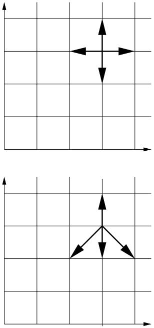
Examples of groups





Order 4,



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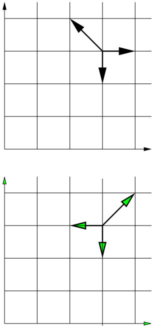
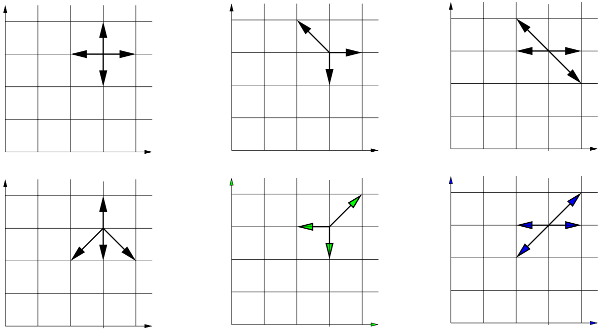


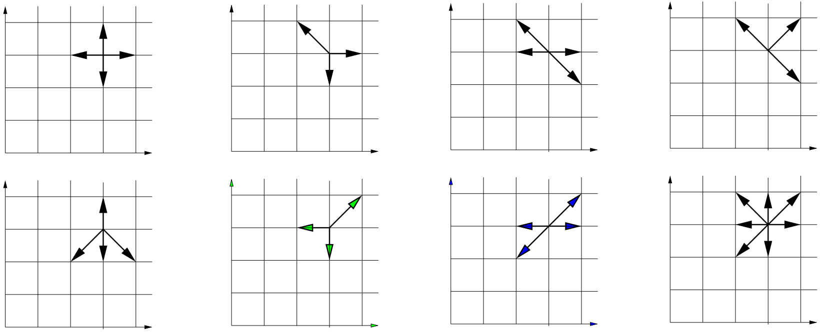
order 6,

Examples of groups



Order 4, order 6, order 8,

Examples of groups



Order 4, order 6, order 8, order ∞.

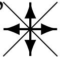
An important concept: the orbit sum (OS)

When $\mathcal{G}_{\mathfrak{S}}$ is finite, the orbit sum of $\mathfrak{S}$ is the polynomial in $\mathbb{Q}[x, x^{-1}, y, y^{-1}]$:

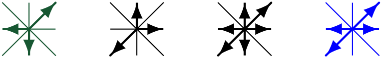
$$\mathrm{OS}_{\mathfrak{S}} := \sum_{\theta \in \mathcal{G}_{\mathfrak{S}}} (-1)^{\theta} \theta(xy)$$

▷ E.g., for the simple walk, with $\mathcal{G}_{\mathfrak{S}} = \left\{ (x,y), \left(x, \frac{1}{y}\right), \left(\frac{1}{x}, \frac{1}{y}\right), \left(\frac{1}{x}, y\right) \right\}$:

$$\mathrm{OS} = x \cdot y - \frac{1}{x} \cdot y + \frac{1}{x} \cdot \frac{1}{y} - x \cdot \frac{1}{y}$$



▷ For 4 models, the orbit sum is zero:



E.g., for the Kreweras model:

$$\mathrm{OS} = x \cdot y - \frac{1}{xy} \cdot y + \frac{1}{xy} \cdot x - y \cdot x + y \cdot \frac{1}{xy} - x \cdot \frac{1}{xy} \quad = 0$$

The 79 models: finite and infinite groups

79 models

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23 admit a finite group

[Mishna'07]

79 models

56 have an infinite group

[Bousquet-Mélou, Mishna'10]

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all $F(t;x,y)$ D-finite

19 transcendental
(OS $\neq 0$)
[Gessel, Zeilberger'92]
[Bousquet-Mélou'02]

4 algebraic (OS $= 0$)
(3 Kreweras-type + Gessel)
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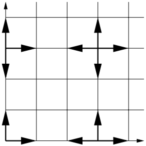
[Bousquet-Mélou, Mishna'10]

$\longrightarrow$ all non-D-finite

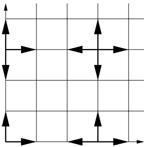
- [Mishna, Rechnitzer'07] and [Melczer, Mishna'13] for 5 singular models
- [Kurkova, Raschel'13] and [B., Raschel, Salvy'13] for all others

D-Finiteness via the finite group [Bousquet-Mélou, Mishna, 2010]



The kernel $J = 1 - t \cdot \sum_{(i,j)\in\mathfrak{S}} x^i y^j = 1 - t\left(x + \frac{1}{x} + y + \frac{1}{y}\right)$ is invariant under the change of (x, y) into, respectively:

$$\left(\tfrac{1}{x}, y\right), \left(\tfrac{1}{x}, \tfrac{1}{y}\right), \left(x, \tfrac{1}{y}\right).$$



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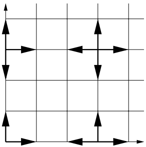
$$J(t; x, y)xyF(t; x, y) = xy - txF(t; x, 0) - tyF(t; 0, y)$$



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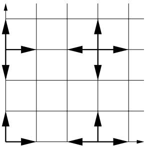

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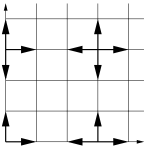

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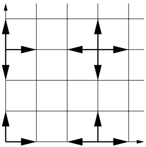
\end{aligned}$$



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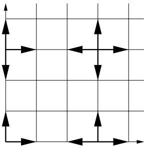
Summing up yields the orbit equation:

$$\sum_{\theta\in\mathcal{G}} (-1)^{\theta}\theta\left(xy\,F(t;x,y)\right) = \frac{xy - \frac{1}{x}y + \frac{1}{x}\frac{1}{y} - x\frac{1}{y}}{J(t;x,y)}$$



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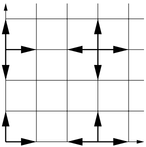
\end{aligned}$$

Taking positive parts yields:

$$[x^{>}y^{>}] \sum_{\theta\in\mathcal{G}} (-1)^{\theta}\theta\big(xy\,F(t;x,y)\big) = [x^{>}y^{>}]\,\frac{xy - \frac{1}{x}y + \frac{1}{x}\frac{1}{y} - x\frac{1}{y}}{J(t;x,y)}$$

D-Finiteness via the finite group [Bousquet-Mélou, Mishna, 2010]



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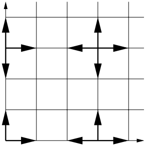

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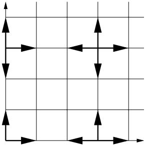
$$\mathsf{GF} = \mathsf{PosPart}\left(\frac{\mathsf{OS}}{\mathsf{kernel}}\right)$$



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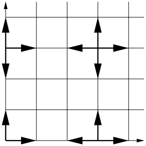
$$\mathsf{GF} = \mathsf{PosPart}\left(\frac{\mathsf{OS}}{\mathsf{ker}}\right) = \text{D-finite [Lipshitz, 1988]}$$



The kernel $J = 1 - t \cdot \sum_{(i,j)\in\mathfrak{S}} x^i y^j = 1 - t\left(x + \frac{1}{x} + y + \frac{1}{y}\right)$ is invariant under the change of (x,y) into, respectively:

$$\left(\tfrac{1}{x}, y\right), \left(\tfrac{1}{x}, \tfrac{1}{y}\right), \left(x, \tfrac{1}{y}\right).$$

Kernel equation:

$$\begin{aligned}
J(t;x,y)xyF(t;x,y) &= xy - txF(t;x,0) - tyF(t;0,y)\\
-J(t;x,y)\tfrac{1}{x}yF(t;\tfrac{1}{x},y) &= -\tfrac{1}{x}y + t\tfrac{1}{x}F(t;\tfrac{1}{x},0) + tyF(t;0,y)\\
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-J(t;x,y)x\tfrac{1}{y}F(t;x,\tfrac{1}{y}) &= -x\tfrac{1}{y} + txF(t;x,0) + t\tfrac{1}{y}F(t;0,\tfrac{1}{y})
\end{aligned}$$

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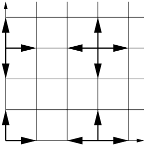
▷ Argument works if $\mathsf{OS} \neq 0$: algebraic version of the reflection principle



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▷ Creative Telescoping finds a differential equation for PosPart(OS/ker)

Main results (IV): explicit expressions for models 1–19

Theorem [B., Chyzak, van Hoeij, Kauers, Pech, 2016]

Let $\mathfrak{S}$ be one of the 19 models with finite group $\mathcal{G}_{\mathfrak{S}}$, and non-zero orbit sum. Then

- $F_{\mathfrak{S}}$ is expressible using iterated integrals of ${}_2F_1$ expressions.
- Among the 19×4 specializations of $F_{\mathfrak{S}}(t; x, y)$ at $(x, y) \in \{0, 1\}^2$, only 4 are algebraic: for $\mathfrak{S} =$  at $(1, 1)$, and $\mathfrak{S} =$  at $(1, 0), (0, 1), (1, 1)$

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Example (King walks in the quarter plane, A025595)

F 

$$(t;1,1) = \frac{1}{t}\int_0^t \frac{1}{(1+4x)^3} \cdot {}_2F_1\left(\begin{matrix}\frac{3}{2} & \frac{3}{2}\\ & 2\end{matrix}\,\middle|\, \frac{16x(1+x)}{(1+4x)^2}\right) dx$$

$$= 1 + 3t + 18t^2 + 105t^3 + 684t^4 + 4550t^5 + 31340t^6 + 219555t^7 + \cdots$$

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▷ Computer-driven discovery and proof; no human proof yet.

▷ Proof uses creative telescoping, ODE factorization, ODE solving.

Hypergeometric Series Occurring in Explicit Expressions for $F(t;x,y)$

	$\mathfrak{S}$	occurring ${}_2F_1$	w		$\mathfrak{S}$	occurring ${}_2F_1$	w
1		${}_2F_1\left(\begin{matrix}\frac{1}{2}\ \frac{1}{2}\\ 1\end{matrix}\,\middle\vert\, w\right)$	$16t^2$	11		${}_2F_1\left(\begin{matrix}\frac{1}{2}\ \frac{1}{2}\\ 1\end{matrix}\,\middle\vert\, w\right)$	$\frac{16t^2}{4t^2+1}$
2		${}_2F_1\left(\begin{matrix}\frac{1}{2}\ \frac{1}{2}\\ 1\end{matrix}\,\middle\vert\, w\right)$	$16t^2$	12		${}_2F_1\left(\begin{matrix}\frac{1}{4}\ \frac{3}{4}\\ 1\end{matrix}\,\middle\vert\, w\right)$	$\frac{64t^3(2t+1)}{(8t^2-1)^2}$
3		${}_2F_1\left(\begin{matrix}\frac{1}{4}\ \frac{3}{4}\\ 1\end{matrix}\,\middle\vert\, w\right)$	$\frac{64t^2}{(12t^2+1)^2}$	13		${}_2F_1\left(\begin{matrix}\frac{1}{4}\ \frac{3}{4}\\ 1\end{matrix}\,\middle\vert\, w\right)$	$\frac{64t^2(t^2+1)}{(16t^2+1)^2}$
4		${}_2F_1\left(\begin{matrix}\frac{1}{2}\ \frac{1}{2}\\ 1\end{matrix}\,\middle\vert\, w\right)$	$\frac{16t(t+1)}{(4t+1)^2}$	14		${}_2F_1\left(\begin{matrix}\frac{1}{4}\ \frac{3}{4}\\ 1\end{matrix}\,\middle\vert\, w\right)$	$\frac{64t^2(t^2+t+1)}{(12t^2+1)^2}$
5		${}_2F_1\left(\begin{matrix}\frac{1}{4}\ \frac{3}{4}\\ 1\end{matrix}\,\middle\vert\, w\right)$	$64t^4$	15		${}_2F_1\left(\begin{matrix}\frac{1}{4}\ \frac{3}{4}\\ 1\end{matrix}\,\middle\vert\, w\right)$	$64t^4$
6		${}_2F_1\left(\begin{matrix}\frac{1}{4}\ \frac{3}{4}\\ 1\end{matrix}\,\middle\vert\, w\right)$	$\frac{64t^3(t+1)}{(1-4t^2)^2}$	16		${}_2F_1\left(\begin{matrix}\frac{1}{4}\ \frac{3}{4}\\ 1\end{matrix}\,\middle\vert\, w\right)$	$\frac{64t^3(t+1)}{(1-4t^2)^2}$
7		${}_2F_1\left(\begin{matrix}\frac{1}{2}\ \frac{1}{2}\\ 1\end{matrix}\,\middle\vert\, w\right)$	$\frac{16t^2}{4t^2+1}$	17		${}_2F_1\left(\begin{matrix}\frac{1}{3}\ \frac{2}{3}\\ 1\end{matrix}\,\middle\vert\, w\right)$	$27t^3$
8		${}_2F_1\left(\begin{matrix}\frac{1}{4}\ \frac{3}{4}\\ 1\end{matrix}\,\middle\vert\, w\right)$	$\frac{64t^3(2t+1)}{(8t^2-1)^2}$	18		${}_2F_1\left(\begin{matrix}\frac{1}{3}\ \frac{2}{3}\\ 1\end{matrix}\,\middle\vert\, w\right)$	$27t^2(2t+1)$
9		${}_2F_1\left(\begin{matrix}\frac{1}{4}\ \frac{3}{4}\\ 1\end{matrix}\,\middle\vert\, w\right)$	$\frac{64t^2(t^2+1)}{(16t^2+1)^2}$	19		${}_2F_1\left(\begin{matrix}\frac{1}{2}\ \frac{1}{2}\\ 1\end{matrix}\,\middle\vert\, w\right)$	$16t^2$
10		${}_2F_1\left(\begin{matrix}\frac{1}{4}\ \frac{3}{4}\\ 1\end{matrix}\,\middle\vert\, w\right)$	$\frac{64t^2(t^2+t+1)}{(12t^2+1)^2}$				

▷ All related to the complete elliptic integrals $\int_0^{\pi/2}(1-k^2\sin^2\theta)^{\pm\frac{1}{2}}\,d\theta$

Main results (V): non-D-finiteness in models with an infinite group

Theorem [B., Raschel, Salvy, 2013]

Let $\mathfrak{S}$ be one of the 51 non-singular models with infinite group $\mathcal{G}_{\mathfrak{S}}$.
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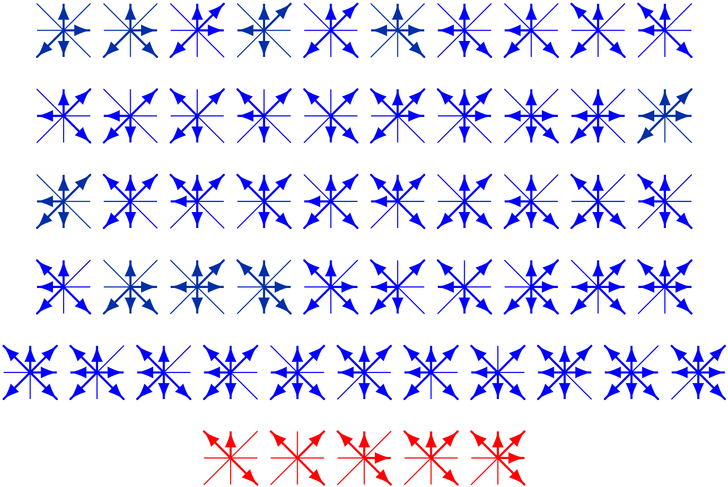
▷ [Bernardi, Bousquet-Mélou, Raschel, 2016] For 9 of these 51 models, $F_{\mathfrak{S}}(t;x,y)$ is nevertheless D-algebraic!
▷ [Dreyfus, Hardouin, Roques, Singer, 2017]: hypertranscendence of the remaining 42 models.

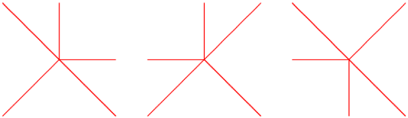
The 56 models with infinite group



In blue, non-singular models, solved by [B., Raschel, Salvy, 2013]
In red, singular models, solved by [Melczer, Mishna, 2013]

Example: the scarecrows

[B., Raschel, Salvy, 2013]: $F_{\mathfrak{S}}(t;0,0)$ is not D-finite for the models



For the 1st and the 3rd, the excursions sequence $[t^n]\, F_{\mathfrak{S}}(t;0,0)$

$$1, 0, 0, 2, 4, 8, 28, 108, 372, \ldots$$

is $\sim K \cdot 5^n \cdot n^{-\alpha}$, with $\alpha = 1 + \pi / \arccos(1/4) = 3.383396\ldots$
[Denisov, Wachtel, 2013]

The irrationality of α prevents $F_{\mathfrak{S}}(t;0,0)$ from being D-finite.
[Katz, 1970; Chudnovsky, 1985; André, 1989]

The Main Theorem Let $\mathfrak{S}$ be one of the 74 non-singular models. The following assertions are equivalent:

(1) The full generating function $F_{\mathfrak{S}}(t;x,y)$ is D-finite

(2) the excursions generating function $F_{\mathfrak{S}}(t;0,0)$ is D-finite

(3) the excursions sequence $[t^n]\,F_{\mathfrak{S}}(t;0,0)$ is $\sim K \cdot \rho^n \cdot n^\alpha$, with $\alpha \in \mathbb{Q}$

(4) the group $\mathcal{G}_{\mathfrak{S}}$ is finite (and $|\mathcal{G}_{\mathfrak{S}}| = 2 \cdot \min\{\ell \in \mathbb{N}^\star \mid \frac{\ell}{\alpha+1} \in \mathbb{Z}\}$)

(5) the step set $\mathfrak{S}$ has either an axial symmetry, or zero drift and cardinality different from 5.

Summary: Classification of 2D non-singular walks

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In this case, $F_{\mathfrak{S}}(t;x,y)$ is expressible using nested radicals.
If not, $F_{\mathfrak{S}}(t;x,y)$ is expressible using iterated integrals of ${}_2F_1$ expressions.

Summary: Walks with small steps in $\mathbb{N}^2$

quadrant models $\mathfrak{S}$: 79

$|\mathcal{G}_{\mathfrak{S}}| < \infty$: 23

$|\mathcal{G}_{\mathfrak{S}}| = \infty$: 56

orbit sum $\neq 0$: 19

orbit sum $= 0$: 4

asymptotics + GB

Kernel method + CT

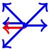
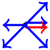
Guess'n'Prove

non-D-finite

D-finite

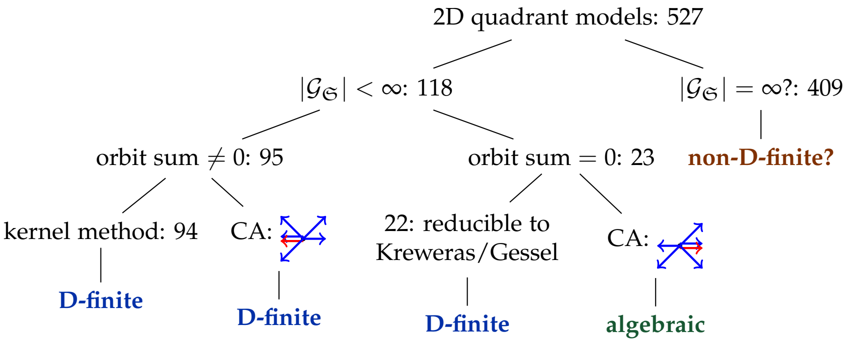
algebraic

Extensions: Walks in $\mathbb{N}^2$ with small repeated steps

- 2D quadrant models: 527
  - $|\mathcal{G}_{\mathfrak{S}}| < \infty$: 118
    - orbit sum $\neq 0$: 95
      - kernel method: 94
        - **D-finite**
      - CA: 
        - **D-finite**
    - orbit sum $= 0$: 23
      - 22: reducible to Kreweras/Gessel
        - **D-finite**
      - CA: 
        - **algebraic**
  - $|\mathcal{G}_{\mathfrak{S}}| = \infty$?: 409
    - **non-D-finite?**

[B., Bousquet-Mélou, Kauers, Melczer, 2015]

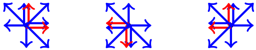
Extensions: Walks in $\mathbb{N}^2$ with small repeated steps



[B., Bousquet-Mélou, Kauers, Melczer, 2015]

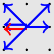
▷ [Du, Hou, Wang, 2015]: proofs that groups are infinite in the 409 cases, and GF are non-D-finite in 366 cases.

▷ [Kauers, Yatchak, 2015]: extension to $4^8 = 65536$ models with mult. ≤ 3. 1457 **D-finite**, 79 **algebraic**, 3 pearls:



A pearl among models in $\mathbb{N}^2$ with small but repeated steps

Theorem [B., Bousquet-Mélou, Kauers, Melczer, 2015]

Let $e_n = \# \Big\{$  – walks of length n in $\mathbb{N}^2$ from $(0,0)$ to $(0,0)\Big\}$

$$(e_n)_{n\geq 0} = (1,\ 0,\ 3,\ 0,\ 26,\ 0,\ 323,\ 0,\ 4830,\ 0,\ 80910, \dots)$$

Then

$$e_{2n} = \frac{6(6n+1)!(2n+1)!}{(3n)!(4n+3)!(n+1)!}.$$

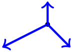
- ▷ Current proof is computer-driven.
- ▷ Open problem: find a *human proof*.

Extensions: Walks in $\mathbb{N}^2$ with large steps

quadrant models with steps in $\{-2,-1,0,1\}^2$: 13 110

- $|\text{orbit}| < \infty$: 240
  - $\text{OS} \neq 0$: 431 — **D-finite**
  - $\text{OS} = 0$: 9 — **D-finite?**
- $|\text{orbit}| = \infty$: 12 870
  - α rational: 16 — **non-D-finite?**
  - α irrational: 12 854 — **non-D-finite**

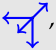
[B., Bousquet-Mélou, Melczer, 2017]

- Example: For the model 

$$xyF(t;x,y) = [x^{>0}y^{>0}]\,\frac{(x-2x^{-2})(y-(x-x^{-2})y^{-1})}{1-t(xy^{-1}+y+x^{-2}y^{-1})}$$

Two pearls among the 9 difficult models with large steps

Conjecture 1 [B., Bousquet-Mélou, Melczer, 2017]

For the model , writing $\phi(t) = \frac{108\,t(1+4t)^2}{(12t-1)^3}$, then $F(t^{1/2};0,0)$ is equal to

$$\frac{1}{3t} - \frac{\sqrt{1-12t}}{6t}\left({}_2F_1\left(\left.\begin{matrix}\frac{1}{6} & \frac{1}{3}\\ & 1\end{matrix}\,\right|\,\phi(t)\right) + {}_2F_1\left(\left.\begin{matrix}-\frac{1}{6} & \frac{2}{3}\\ & 1\end{matrix}\,\right|\,\phi(t)\right)\right).$$

Conjecture 2 [B., Bousquet-Mélou, Melczer, 2017]

For the model , $F(t;0,0)$ is equal to

$$\frac{(1-24\,U+120\,U^2-144\,U^3)\,(1-4\,U)}{(1-3\,U)\,(1-2\,U)^{3/2}\,(1-6\,U)^{9/2}},$$

where $U = t^4 + 53\,t^8 + 4363\,t^{12} + \cdots$ is the unique series in $\mathbb{Q}[[t]]$ satisfying

$$U\,(1-2\,U)^3\,(1-3\,U)^3\,(1-6\,U)^9 = t^4\,(1-4\,U)^4.$$

Extensions: Walks with small steps in $\mathbb{N}^3$

$2^{3^3-1} \approx 67$ million models, of which ≈ 11 million inherently 3D

3D octant models $\mathfrak{S}$ with ≤ 6 steps: 20804

$|\mathcal{G}_{\mathfrak{S}}| < \infty$: 170

$|\mathcal{G}_{\mathfrak{S}}| = \infty$?: 20634

orbit sum $\neq 0$: 108

orbit sum $= 0$: 62

non-D-finite?

kernel method

2D-reducible: 43

not 2D-reducible: 19

D-finite

D-finite

non-D-finite?

[B., Bousquet-Mélou, Kauers, Melczer, 2015]

- Open question: are there non-D-finite models with a finite group?

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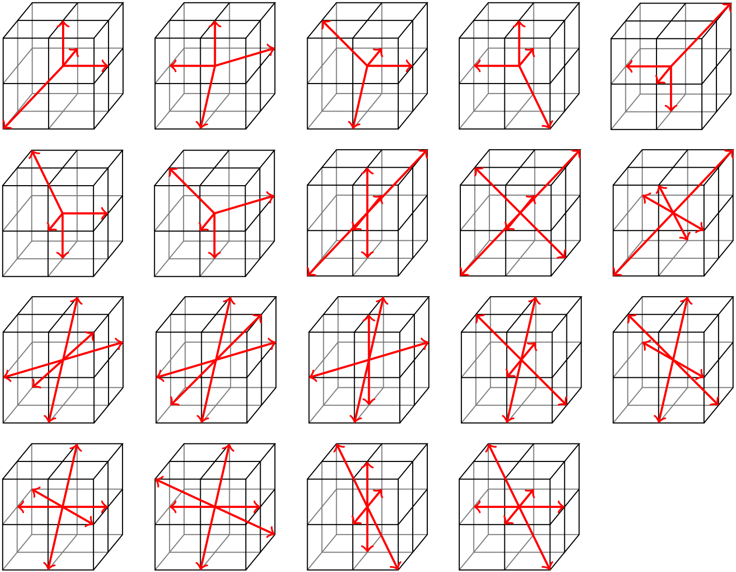
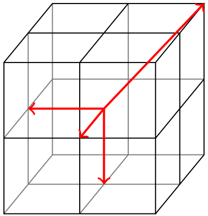
▷ Open question: are there non-D-finite models with a finite group?

▷ [Du, Hou, Wang, 2015]: proofs that groups are infinite in the 20634 cases

▷ [Bacher, Kauers, Yatchak, 2016]: extension to all 3D models; 170 models found with $|\mathcal{G}_{\mathfrak{S}}| < \infty$ and orbit sum 0 (instead of 19)

19 mysterious 3D-models





Two different computations suggest:

$$k_{4n} \approx C \cdot 256^n / n^{3.3257570041744\ldots},$$

so excursions are very probably transcendental (and even non-D-finite)

Conclusion

Computer algebra may solve difficult combinatorial problems

Classification of $F(t;x,y)$ fully completed for 2D small step walks

Robust algorithmic methods, based on efficient algorithms:

- Guess'n'Prove
- Creative Telescoping

Brute-force and/or use of naive algorithms = hopeless.
E.g. size of algebraic equations for $G(t;x,y) \approx$ 30Gb.

Conclusion

Computer algebra may solve difficult combinatorial problems

Classification of $F(t;x,y)$ fully completed for 2D small step walks

Robust algorithmic methods, based on efficient algorithms:

- Guess'n'Prove
- Creative Telescoping

Brute-force and/or use of naive algorithms = hopeless.
E.g. size of algebraic equations for $G(t;x,y) \approx 30$Gb.

Lack of "purely human" proofs for some results.

Open: is $F(t;1,1)$ non-D-finite for all 56 models with infinite group?

Many beautiful open questions for 2D models with repeated or large steps, and in dimension > 2.

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