Functional equations: an important analytic method for random walks in the quarter plane

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The greek word ἀναλυτικός *refers essentially to* "the ability to analyze" *which is indeed the very nature of science...*

Functional equations (FE) arise quite naturally in the analysis of stochastic systems of different kinds : queueing and telecommunication networks, random walks, enumeration of planar lattice walks, etc. The content of this talk mainly refers to Chapter 5 of [1]

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Functional equations of two complex variables

▶ Piecewise homogeneous random walk with sample paths in $\mathbb{Z}_+^2$, the lattice in the positive quarter plane. In the strict interior of $\mathbb{Z}_+^2$, the size of the jumps is 1, and $\{p_{ij}, |i|, |j| \leqslant 1\}$ will denote the generator of the process for this region. Thus a transition $(m,n) \to (m+i, n+j), m, n > 0$, can take place with probability p_{ij}, and

$$\sum_{|i|,|j| \leqslant 1} p_{ij} = 1.$$

▶ No strong assumption about the boundedness of the upward jumps on the axes, neither at $(0,0)$. In addition, the downward jumps on the x [resp. y] axis are bounded by L [resp. M], where L and M are arbitrary finite integers.

▶ Original question : Find an explicit form for the invariant measure π of such process.

The basic functional equation

The invariant measure $\{\pi_{i,j}, i,j \geq 0\}$ satisfies the fundamental bivariate functional equation

$$\boxed{Q(x,y)\pi(x,y) = q(x,y)\pi(x) + \widetilde{q}(x,y)\widetilde{\pi}(y) + \pi_0(x,y)}, \qquad (1)$$

Problem: *Find functions* $\pi(x,y)$, $\pi(x)$, $\widetilde{\pi}(y)$, *satisfying* (1), *holomorphic in* $\mathcal{D} \times \mathcal{D}$ *and continuous in* $\overline{\mathcal{D}} \times \overline{\mathcal{D}}$, *where*

$$\mathcal{D} \stackrel{\text{def}}{=} \{z \in \mathbb{C} : |z| < 1\} \quad \text{and} \quad \overline{\mathcal{D}} \stackrel{\text{def}}{=} \{z \in \mathbb{C} : |z| \leqslant 1\}.$$

- The functions $Q(x,y), q(x,y), \widetilde{q}(x,y)$ are known.
- The function $\pi_0(x,y)$ depends linearly on $L + M - 1$ unknown constants.
- $Q(x,y)$ is often referred to as the kernel of (1).

$$\begin{cases} \pi(x,y) = \sum_{i,j\geqslant 1} \pi_{ij} x^{i-1} y^{j-1}, \\ \pi(x) = \sum_{i\geqslant L} \pi_{i0} x^{i-L}, \quad \widetilde{\pi}(y) = \sum_{j\geqslant M} \pi_{0j} y^{j-M}, \\ Q(x,y) = xy\left[1 - \sum_{i,j\in\mathcal{S}} p_{ij} x^i y^j\right], \quad \sum_{i,j\in\mathcal{S}} p_{ij} = 1, \\ q(x,y) = x^L\left[\sum_{i\geqslant -L, j\geqslant 0} p'_{ij} x^i y^j - 1\right] \equiv x^L(P_{L0}(x,y) - 1), \\ \widetilde{q}(x,y) = y^M\left[\sum_{i\geqslant 0, j\geqslant -M} p''_{ij} x^i y^j - 1\right] \equiv y^M(P_{0M}(x,y) - 1), \\ \pi_0(x,y) = \sum_{i=1}^{L-1} \pi_{i0} x^i \left[P_{i0}(x,y) - 1\right] + \sum_{j=1}^{M-1} \pi_{0j} y^j \left[P_{0j}(x,y) - 1\right] + \pi_{00}(P_{00}(xy) - 1). \end{cases}$$

$\mathcal{S}$ is the set of allowed jumps, and $q, \widetilde{q}, q_0, P_{i0}, P_{0j}$, are given probability generating functions supposed to have suitable analytic continuations (as a rule, they are polynomials when the jumps are bounded).

New approaches toward the solution were discovered by the authors of the book [1], the goal going far beyond the mere obtention of an index theory for the quarter plane...

1. The first step, quite similar to a Wiener–Hopf factorization, consists in considering the above equation on the algebraic curve $\mathcal{A} = \{(x, y) \in \mathbb{C}^2 : Q(x, y) = 0\}$. (which is elliptic in the generic situation), so that we are then left with an equation for two unknown functions of one variable on this curve.

2. Next a crucial idea is to use Galois automorphisms on this algebraic curve. More information is obtained by using the fact that the unknown functions π and $\widetilde{\pi}$ depend solely on x and y respectively.

   It is possible to prove that π and $\widetilde{\pi}$ can be lifted as meromorphic functions onto the universal covering of some Riemann surface $\mathbf{S}$. Here $\mathbf{S}$ corresponds to the algebraic curve $\mathcal{A}$. When $g \stackrel{\text{def}}{=}$ the genus of $\mathbf{S}$ is 1 [Torus] (resp. 0), the universal covering is the complex plane $\mathbb{C}$ (resp. the Riemann sphere).

3. Lifted onto the universal covering, π (and also $\widetilde{\pi}$) satisfies a system of non-local equations having the simple form

$$\begin{cases} \pi(t+\omega_1) = \pi(t), \quad \forall t \in \mathbb{C}\,, \\ \pi(t+\omega_3) = a(t)\pi(t) + b(t), \quad \forall t \in \mathbb{C}\,, \end{cases}$$

where ω_1 [resp. ω_3] is a complex [resp. real] constant. The solution can be presented in terms of infinite series equivalent to Abelian integrals. The backward transformation (projection) from the universal covering onto the initial coordinates can be given in terms of uniformization functions, which for $g = 1$ are Weierstrass $\wp$ elliptic functions.

4. Another direct approach consists in working solely in the complex plane $\mathbb{C}$. After analytic continuation, it appears that the determination of π reduces to a Boundary Value Problem (BVP), belonging to the Riemann–Hilbert–Carleman class, the basic form of which can be formulated as follows.

- $\mathscr{G}(\mathcal{L}) \stackrel{\text{def}}{=}$ interior of the domain bounded by a simple smooth closed contour $\mathcal{L}$.

Find a function Φ^+ holomorphic in $\mathscr{G}(\mathcal{L})$, the limiting values of which are continuous on the contour and satisfy the relation

$$\boxed{\Phi^+(\alpha(t)) = G(t)\Phi^+(t) + g(t), \quad t \in \mathcal{L}.} \tag{2}$$

- g, $G \in \mathbb{H}_\mu(\mathcal{L})$ (*Hölder condition* with parameter μ on $\mathcal{L}$).
- α, the shift, is a function establishing a one-to-one mapping of the contour $\mathcal{L}$ onto itself, such that the direction of traversing $\mathcal{L}$ is changed and

$$\alpha'(t) = \frac{d\alpha(t)}{dt} \in \mathbb{H}_\mu(\mathcal{L}), \quad \alpha'(t) \neq 0, \quad \forall t \in \mathcal{L}.$$

- α is most frequently subject to the so-called Carleman condition

$$\alpha(\alpha(t)) = t, \quad \forall t \in \mathcal{L}, \quad \text{where typically} \quad \alpha(t) = \bar{t}.$$

The point of this method resides in the fact that solutions are given in terms of explicit integral-forms. See e.g. [14, 18].

Brief historical account

- *V. Malyshev*, 1969-1972, at Moscow University. He was the first to be interested in these problems. In particular, in the case of bounded jumps, he answered some questions [asked by A.N. Kolmogorov] about the classification of the walks. He also proposed a solution of (1) via uniformisation.

- *G. Fayolle and R. Iasnogorodski*, 1975-1977, in Paris. Independently, they showed the equivalence to a BVP of Riemann-Hilbert type, yielding solutions in integral form.

These three authors joined their efforts in the so-called **Small Yellow Book** (Маленькая Жёлтая Книга). . .

References

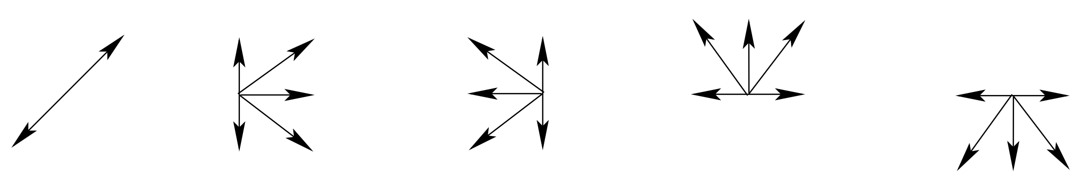
[1] G. Fayolle, R. Iasnogorodski, and V. Malyshev. *Random Walks in the Quarter Plane: Algebraic Methods, Boundary Value Problems, Applications to Queueing Systems and Analytic Combinatorics*. Springer Publishing Company, Incorporated. 1st ed. 1999, 2nd ed., 2017.

Étude de la courbe algébrique $Q(x,y)=0$

$Q(x,y)$: polynôme de degré 2 par rapport à chaque variable et de degré 4 en (x,y). Par exemple

$$Q(x,y)=\begin{cases} a(x)y^2+b(x)y+c(x) \\ \widetilde{a}(y)x^2+\widetilde{b}(y)x+\widetilde{c}(y). \end{cases}$$

1. Marche dite singulière si Q est soit réductible, soit de degre 1 par rapport à au moins une variable.



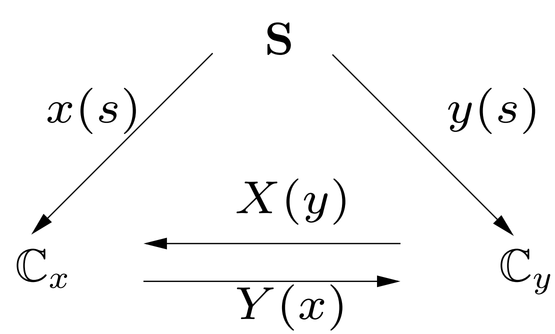
a $\quad$ b $\quad$ c $\quad$ d $\quad$ e

2. **Cas général : Q irreductible**. Alors $\{Q(x,y)=0\}$ est associé à une surface de Riemann compacte $\mathbf{S}$ de genre g, avec $g=0$ ou $g=1$.

- Si $g=0$, alors $\mathbf{S}$ est la sphère de Riemann $\mathbb{P}^1$ et la *couverture universelle* Ω est le plan complexe $\mathbb{C}$. Uniformisation par des fonctions rationnelles.

- Si $g=1$, alors $\mathbf{S}$ est un tore et la *couverture universelle* Ω est le plan complexe (fini) noté $\mathbb{C}_\omega$. Uniformisation à l'aide de fonctions elliptiques.

$$h_x : \mathbf{S} \to \mathbb{C}_x, \quad h_y : \mathbf{S} \to \mathbb{C}_y, \quad x(s) \stackrel{\text{def}}{=} h_x(s), \quad y(s) \stackrel{\text{def}}{=} h_y(s), \ s \in \mathbf{S}.$$



- On a ainsi trois niveaux de recouvrement (par ordre croissant d'altitude !) :

$$\{\mathbb{C}_x, \mathbb{C}_y\} \Longrightarrow \mathbf{S} \Longrightarrow \mathbb{C}_\omega.$$

Solving (1) by reduction to a BVP

To find functions $\pi(x,y)$, $\pi(x)$, $\widetilde{\pi}(y)$ satisfying (1), holomorphic in $\mathcal{D}\times\mathcal{D}$ and continuous in $\overline{\mathcal{D}}\times\overline{\mathcal{D}}$, 3 steps are proposed.

Sketch of the 3 main steps.

- Restrict (1) to the algebraic curve $\mathcal{A}=\{(x,y)\in\mathbb{C}^2 : Q(x,y)=0\}$.

  When Q is irreducible, $\mathcal{A}$ is associated with a compact Riemann surface of genus g, where $g=1$ [Torus] or $g=0$ [Sphere].

- Eliminate one function, say $\widetilde{\pi}(y)$.

- Formulate a Boundary Value Problem for $\pi(x)$.

Step 1 [Restriction to $\mathcal{A} = \{(x, y) \in \mathbb{C}^2 : Q(x, y) = 0\}$, assuming $g = 1$]

Q(x,y) is a polynomial of degree 2 w.r.t. each variable, and of degree 4 in (x, y).

$$\boxed{Q(x, y) = a(x)y^2 + b(x)y + c(x) = \widetilde{a}(y)x^2 + \widetilde{b}(y)x + \widetilde{c}(y)}, \tag{3}$$

Let the discriminants of the kernel, in the respective $\mathbb{C}_x$ and $\mathbb{C}_y$ complex planes,

$$D(x) = b(x)^2 - 4a(x)c(x), \qquad \widetilde{D}(y) = \widetilde{b}(y)^2 - 4\widetilde{a}(y)\widetilde{c}(y). \tag{4}$$

Then $D(x)$ [resp. $\widetilde{D}(y)$] has 4 real roots in $\mathbb{C}_x$ (resp. $\mathbb{C}_y$) satisfying

$$|x_1| < x_2 < 1 < x_3 < |x_4| \leqslant \infty,$$
$$|y_1| < y_2 < 1 < y_3 < |y_4| \leqslant \infty,$$

Moreover, the slits $[x_1, x_2]$ and $[y_1, y_2]$ are included in the segment $[-1, +1]$.

Let the algebraic functions $X(y)$ and $Y(x)$ be defined by

$$\boxed{Q(X(y), y) = Q(x, Y(x)) = 0}.$$

- $X(\cdot)$ has two branches $X_0(y), X_1(y)$, which are meromorphic in $\mathbb{C}_y$ cut along $[y_1, y_2] \cup [y_3, y_4]$.
  - These two branches can be separated to ensure $|X_0(y)| \leqslant |X_1(y)|$, $\forall y \in \mathbb{C}_y$.
  - On the cut $[y_1, y_2] \cup [y_3, y_4]$, we have $X_0 = \overline{X}_1$ (complex conjugate).
- On the circle $|y| = 1$, we have $|X_0(y)| \leqslant 1$.

Similarly, $Y_0(x), Y_1(x)$ are the two branches of $Y(\cdot)$ defined in $\mathbb{C}_x$ with the respective properties. . .

Step 2 [Elimination of $\widetilde{\pi}(y)$]

- Starting from the region $\mathcal{A} \cap (\overline{\mathcal{D}} \times \overline{\mathcal{D}})$, the study of the branches allows to continue $\pi(x)$ and $\widetilde{\pi}(y)$ as meromorphic functions to the whole complex plane, cut respectively along $[x_3, x_4]$ and $[y_3, y_4]$.

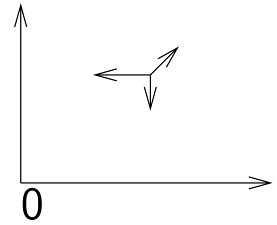
- When y tends successively from above (y^+) and from below (y^-) to an arbitrary point y of the cut $[y_1, y_2]$, $\widetilde{\pi}(y)$ remains continuous, and thus can be eliminated by computing the difference $\widetilde{\pi}(y^+) - \widetilde{\pi}(y^-) = 0$.

- Let $[\overrightarrow{\underleftarrow{y_1, y_2}}]$ stand for a contour representing the slit $[y_1, y_2)]$ traversed from y_1 to y_2 along the upper edge, and then back to y_1 along the lower edge. The object

$$\mathcal{M} = X_0([\overrightarrow{\underleftarrow{y_1, y_2}}]) = \overline{X}_1([\overleftarrow{\underrightarrow{y_1, y_2}}])$$

is a smooth closed contour in $\mathbb{C}_x$, which is part of a quartic curve when $g = 1$.

The curves $\mathcal{L}$, $\mathcal{L}_{ext}$, $\overrightarrow{\underleftarrow{y_1 y_2}}$ and $\overrightarrow{\underleftarrow{y_3 y_4}}$ belong to the same homotopy class on the Riemann surface $\mathbf{S}$. This indicates *a priori* that the curves $\mathcal{L}$ or $\mathcal{L}_{ext}$ contain in their interior at most one slit.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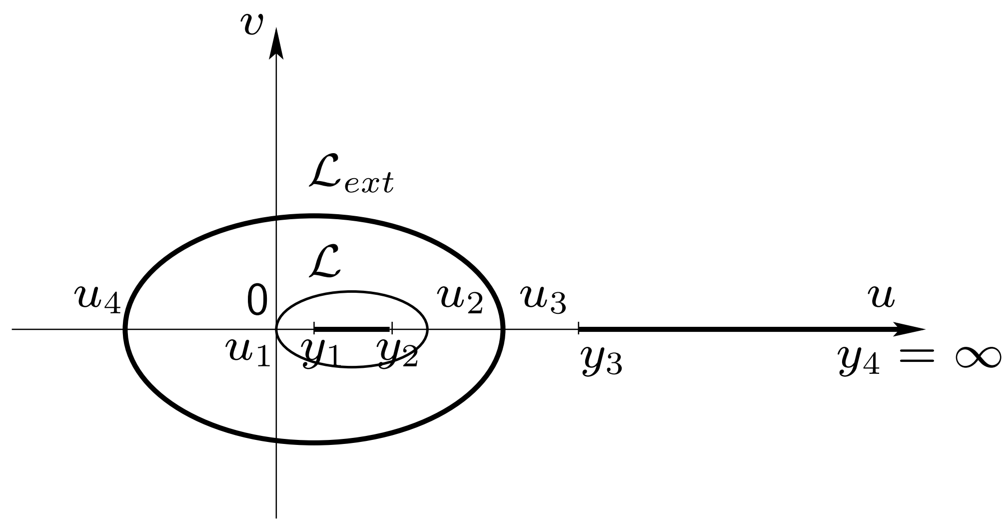
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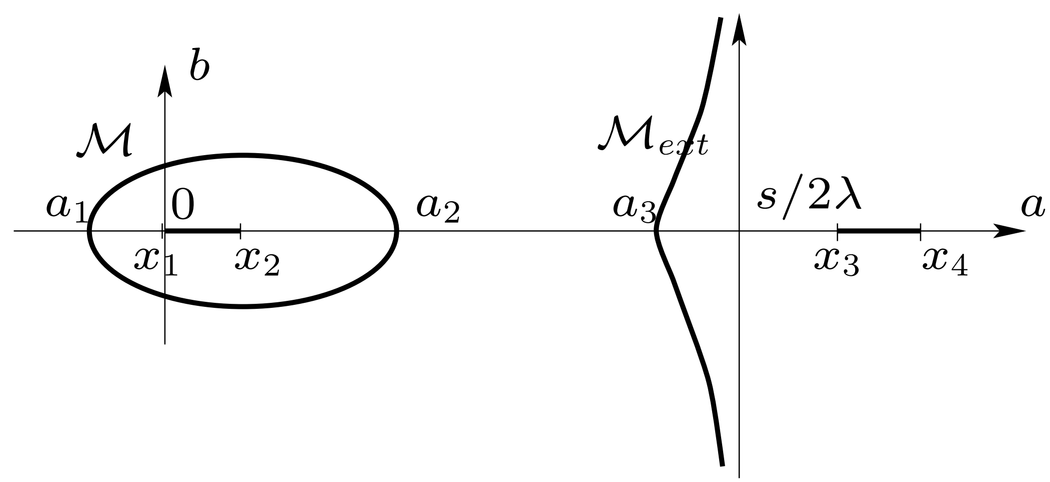
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Step 3 [Formulation of the Riemann-Hilbert-Carleman BVP with a shift]

From the argument used in **Step 2**, we can write

$$\boxed{\pi(t)A(t) - \pi(\bar{t})A(\bar{t}) = g(t), \quad t \in \mathcal{M}},$$

which is a problem of type (2) for the domain $\mathscr{G}(\mathcal{M})$ bounded by the closed contour $\mathcal{M}$, the function $\pi(x)$ being meromorphic (at most one pole) in $\mathscr{G}(\mathcal{M})$.

Theorem 1. *Under ergodicity condition (6), the function π is given by*

$$\pi(x) = \frac{U(x)H(x)}{2i\pi} \int_{\mathcal{M}_d} \frac{K(t)w'(t)dt}{H^+(t)(w(t)-w(x))} + V(x), \ \forall x \in \mathscr{G}(\mathcal{M}). \tag{5}$$

1. *$\mathcal{M}_d$ is the portion of the curve $\mathcal{M}$ located in the lower half-plane $\Im z \leqslant 0$.*
2. *U, V, K are known functions, all involving some specific zeros of $\widetilde{q}(X_0(y), y)$ and $q(x, Y_0(x))$ inside $\mathscr{G}(\mathcal{M})$; moreover U, V are rational fractions.*
3. *w is a* gluing function, *which realizes the conformal mapping of $\mathscr{G}(\mathcal{M})$ onto the complex plane cut along a segment, and has an explicit form via the Weierstrass $\wp$-function.*

Theorem 2. *Introduce the following two quantities:*

$$\delta \stackrel{def}{=} \begin{cases} 0, \text{ if } Y_0(1) < 1 \text{ or } \left\{ Y_0(1) = 1 \text{ and } \dfrac{dq(x, Y_0(x))}{dx}_{|x=1} > 0 \right\}, \\ 1, \text{ if } Y_0(1) = 1 \text{ and } \dfrac{dq(x, Y_0(x))}{dx}_{|x=1} < 0. \end{cases}$$

$$\widetilde{\delta} \stackrel{def}{=} \begin{cases} 0, \text{ if } X_0(1) < 1 \text{ or } \left\{ X_0(1) = 1 \text{ and } \dfrac{d\widetilde{q}(X_0(y), y)}{dy}_{|y=1} > 0 \right\}, \\ 1, \text{ if } X_0(1) = 1 \text{ and } \dfrac{d\widetilde{q}(X_0(y), y)}{dy}_{|y=1} < 0. \end{cases}$$

Then (1) *admits a probabilistic solution if, and only if,*

$$\delta + \widetilde{\delta} = \mathbb{1}_{\{X_0(1)=1, Y_0(1)=1\}} + 1, \qquad (6)$$

which are the necessary and sufficient conditions for the random walk to be ergodic. ∎

Introduction to BVPs in the Complex Plane

B. Riemann was probably the first in his dissertation to have mentioned the following general problem:

Find a function which is holomorphic in some domain $\mathcal{D}$ and continuous on the boundary of $\mathcal{D}$, for a given relation between the limiting values of its real and imaginary parts.

This problem was studied by Hilbert in 1905, who gave a partial answer, by reduction to integral equations. Then H. Poincaré in his *Leçons de Mécanique Céleste* came up with a boundary value problem for harmonic functions, which led him to study singular integral equations in more detail. . .

In this overall presentation, we essentially follow Muskhelischvili [23], Gakhov [14] and Litvintchuk [17, 18].

Let $\mathcal{L}$ be a simple smooth line or curve in the complex plane, i.e. an arc (open or closed) with a continuously varying tangent (see [23]). A function f will be said to satisfy the *Hölder condition* on the curve $\mathcal{L}$ if, for any two points t_1, t_2 of $\mathcal{L}$,

$$|f(t_2) - f(t_1)| \leqslant A|t_2 - t_1|^\mu, \tag{7}$$

where A and μ are positive constants. Clearly for $\mu > 1$ the above condition (7) implies $f =$ constant. Hence we only consider the interesting situation $0 < \mu \leqslant 1$.

When (7) holds, we shall write, $\varphi \in \mathbb{H}_\mu(\mathcal{L})$, for all φ bounded on $\mathcal{L}$. It is also important to note that $\mathbb{H}_\mu(\mathcal{L})$ can be endowed with the norm

$$\|\varphi\| = \sup_{t\in\mathcal{L}} |\varphi(t)| + \sup_{s,t\in\mathcal{L}} \frac{|\varphi(s) - \varphi(t)|}{|s-t|^\mu}.$$

▶ The Sokhotski (1873)–Plemelj (1908) formulae

Theorem 3. *The function*

$$\Phi(z) = \frac{1}{2i\pi}\int_{\mathcal{L}} \frac{\varphi(t)dt}{t-z}\,, z \notin \mathcal{L}, \tag{8}$$

is continuous on $\mathcal{L}$ *from the left and from the right, with the exception of the ends. Moreover, the corresponding limiting values, denoted respectively by* Φ^+ *and* Φ^-*, are in the class* $\mathbb{H}_\mu(\mathcal{L})$*, and they satisfy the so-called Sokhotski–Plemelj formulae, for* $t \in \mathcal{L}$*,*

$$\begin{cases} \Phi^+(t) &= \dfrac{1}{2}\varphi(t) + \dfrac{1}{2i\pi}\displaystyle\int_{\mathcal{L}} \frac{\varphi(s)ds}{s-t}, \\ \Phi^-(t) &= -\dfrac{1}{2}\varphi(t) + \dfrac{1}{2i\pi}\displaystyle\int_{\mathcal{L}} \frac{\varphi(s)ds}{s-t}, \end{cases} \tag{9}$$

where the integrals are understood in the sense of Cauchy-principal value. ■

▶ Subtracting and adding formulae (9), one obtains two other equivalent and fundamental formulae, for $t \in \mathcal{L}$, namely

$$\begin{cases} \Phi^+(t) - \Phi^-(t) & = & \varphi(t), \\ \Phi^+(t) + \Phi^-(t) & = & \dfrac{1}{i\pi}\displaystyle\int_L \frac{\varphi(s)ds}{s-t}. \end{cases} \tag{10}$$

In fact, (10) can be shown to be valid for $\varphi(t)$ simply continuous on $\mathcal{L}$, provided some restrictions are put on the way of approaching $\mathcal{L}$ (which should not be, roughly said, too *tangential*); on the other hand, cusp points and corner points, when they exist, compel to modify (9) (see e.g., [23]).

▶ The Riemann Boundary Value Problem for a Closed Contour

Find a sectionally holomorphic function Φ *of finite degree at infinity,under the boundary condition on* $\mathcal{L}$

$$\boxed{\Phi^+(t) = G(t)\Phi^-(t) + g(t), \quad t \in \mathcal{L},} \qquad (11)$$

where $G \in \mathbb{C}(\mathcal{L})$ denotes the space of continuous functions on $\mathcal{L}$, and $g \in L_p(\mathcal{L})$. Note that a priori g and G are only defined on $\mathcal{L}$.

Notation: We Introduce the following important quantity

$$\boxed{\chi = \mathcal{I}nd[G]_{\mathcal{L}} \stackrel{\text{def}}{=} \frac{1}{2\pi}[\arg G]_{\mathcal{L}} = \frac{1}{2i\pi}[\log G]_{\mathcal{L}}}, \qquad (12)$$

which is called the *index* and represents the variation of the argument of $G(t)$, as t moves along the contour $\mathcal{L}$ in the positive direction.

Then the function

$$\log[t^{-\chi}G(t)], \quad t \in \mathcal{L},$$

has zero index, is single-valued, and satisfies a Hölder condition.

Let

$$\begin{cases} \Gamma(z) & = & \dfrac{1}{2i\pi}\displaystyle\int_{\mathcal{L}} \dfrac{\log(t^{-\chi}G(t))dt}{t-z}, \quad z \notin \mathcal{L}, \\ X^+(z) & = & e^{\Gamma(z)}, \quad z \in \mathcal{L}^+, \\ X^-(z) & = & z^{-\chi}e^{\Gamma(z)}, \quad z \in \mathcal{L}^-. \end{cases} \tag{13}$$

The functions X^+ and X^- are usually referred to as the *canonical functions* of the BVP. Thus $G(t)$ admits the following factorization by (10)

$$X^+(t) = G(t)X^-(t), \quad t \in \mathcal{L}, \tag{14}$$

which gives, *en passant*, a solution of the *homogeneous* BVP (i.e. when $g(t) \equiv 0$) having order $-\chi$ at infinity (i.e. behaving as $z^{-\chi}$). Putting (14) into (11), we get

$$\frac{\Phi^+(t)}{X^+(t)} = \frac{\Phi^-(t)}{X^-(t)} + \frac{g(t)}{X^+(t)}, \quad t \in \mathcal{L}. \tag{15}$$

Since $\frac{g}{X^+} \in \mathbb{H}_\mu(\mathcal{L})$, see [23], we can write, using the Sokhotski–Plemelj formula (10),

$$\psi^+(t) - \psi^-(t) = \frac{g(t)}{X^+(t)}, \quad t \in \mathcal{L},$$

where

$$\psi(z) = \frac{1}{2i\pi} \int_{\mathcal{L}} \frac{g(s)ds}{X^+(s)(s-z)}, \quad z \notin \mathcal{L}. \tag{16}$$

Thus (15) can be rewritten as

$$\frac{\Phi^+(t)}{X^+(t)} - \psi^+(t) = \frac{\Phi^-(t)}{X^-(t)} - \psi^-(t), \quad t \in \mathcal{L}.$$

Two cases must now be distinguished.

(1) $\chi \geq 0$. From the principle of analytic continuation and Liouville's theorem, we obtain

$$\Phi(z) = X(z)\psi(z) + X(z)P_\chi(z), \quad \forall z \notin \mathcal{L}, \tag{17}$$

where $X(z)$ and $\psi(z)$ are given by (13) and (16), and P_χ is an arbitrary polynomial of degree χ. Thus one sees that, in the case of a closed contour $\mathcal{L}$, the solutions belonging to the class of functions bounded at infinity depend on $\chi + 1$ arbitrary complex constants. The second term on the right of (17) represents the general solution of the homogeneous Riemann BVP.

(2) $\chi < 0$. In this case, the ratio $\dfrac{\Phi^-}{X^-}$ vanishes at infinity, so that $P_\chi \equiv 0$ and by Liouville's theorem

$$\Phi(z) = X(z)\psi(z).$$

But in view of (13), Φ^- has a pole of order $-\chi - 1$ at infinity. In order for Φ to be holomorphic at infinity (in particular bounded), it is necessary and sufficient that ψ has a zero of order not smaller than $-\chi - 1$. By expanding the Cauchy-type integral representing ψ (see (16)) in power series of z around the point at infinity, we get the following conditions of solubility

$$\int_{\mathcal{L}} \frac{g(t)t^{k-1}}{X^+(t)} dt = 0, \quad k = 1, 2, \ldots, -\chi - 1. \tag{18}$$

Theorem 4. *The number of linearly independent (non-trivial) solutions of the homogeneous (i.e. $g = 0$) Riemann BVP (11) is given by*

$$\ell = \begin{cases} \max(0, \chi + 1), & \chi \neq -1, \\ 1, & \chi = -1, \end{cases} \qquad (19)$$

and the number of conditions of solubility is

$$p = \max(0, -\chi - 1). \qquad (20)$$

■

- For $\chi = -1$, the non-homogeneous problem is always soluble and has a unique solution.
- For $\chi < -1$, the non-homogeneous problem has in general no solution. It has exactly one if, and only if, the free term g satisfies the $p = -\chi - 1$ conditions given in (18).

▶ The Riemann-Carleman Boundary Value Problem

We shall need the solution of the generalized BVP (2) sometimes referred to as the Carleman problem, see [14] [17], hereafter presented under the name *Riemann–Carleman problem*.

For the sake of completeness, we recall its formulation.

Let $\mathcal{L}$ *be a simple smooth closed contour,* $\mathscr{G}(\mathcal{L})$ *denoting its interior. Find a function* Φ^+ *holomorphic in* $\mathscr{G}(\mathcal{L})$ *the limiting values of which on the contour are continuous and satisfy the relation*

$$\boxed{\Phi^+(\alpha(t)) = G(t)\Phi^+(t) + g(t), \quad t \in \mathcal{L},} \tag{21}$$

where $\alpha(t)$ is a Carleman automorphism defined on $\mathcal{L}$.

Theorem 5. *The Carleman–Dirichlet problem*

$$\Phi^+(\alpha(t)) - \Phi^+(t) = g(t), \quad t \in \mathcal{L}, \tag{22}$$

where $g \in \mathbb{H}_\mu(\mathcal{L})$ *satisfies the relation*

$$g(t) + g(\alpha(t)) = 0,$$

has a unique solution given up to an arbitrary additive constant by

$$\boxed{\Phi^+(z) = \frac{1}{2i\pi}\int_{\mathcal{L}} \frac{\varphi(\alpha(s))ds}{s-z} + C}, \tag{23}$$

where C *is an arbitrary constant and* $\varphi(t)$ *is the unique solution of the integral equation*

$$(\mathcal{B}\varphi)(t) = g(t),$$

$\mathcal{B}$ *being the operator*

$$(\mathcal{B}\varphi)(t) \equiv \varphi(t) + \frac{1}{2i\pi}\int_{\mathcal{L}} \left[\frac{1}{s-t} - \frac{\alpha'(s)}{\alpha(s)-\alpha(t)}\right]\varphi(s)ds. \tag{24}$$

Theorem 6. *Let $\alpha(t)$ be a Carleman automorphism of the curve $\mathcal{L}$. Then there exists a function w, holomorphic in $\mathcal{L}^+$, except at one point $z = z_0 \in \mathcal{L}^+$, where w has a simple pole, such that*

$$w(\alpha(t)) - w(t) = 0, \quad t \in \mathcal{L}. \tag{25}$$

Moreover, w establishes a conformal mapping of the domain $\mathcal{L}^+$ onto the domain Δ, which consists of the plane cut along an open smooth arc $\mathcal{U}$.

Proof. The proof of this theorem can be found in [17]. ■

From Theorem 6, one can write

$$w(z) = \frac{1}{z - z_0} + \psi^+(z),$$

where $\psi^+(z)$ is holomorphic in $\mathcal{L}^+$ and satisfies the condition

$$\psi^+(\alpha(t)) - \psi^+(t) = \frac{1}{t - z_0} - \frac{1}{\alpha(t) - z_0} \stackrel{def}{\equiv} g(t), \quad t \in \mathcal{L}.$$

Hence ψ^+ is given by the formula (23).

▶ Some remarks about the conformal gluing function (CGF) $w(\cdot)$

From Theorem 6, it follows that the function $w(\cdot)$ [see (25] has an inverse denoted by $z(\cdot)$, satisfying

$$\begin{cases} z^+(w) &= \alpha(t) \\ z^-(w) &= t \end{cases}, \quad \text{for } \omega \in \mathcal{U} \text{ and } t \in \mathcal{L}_d.$$

On $\mathcal{L}_u$, one must exchange z^+ and z^- and $\mathcal{L} \underset{z(w)}{\overset{w(z)}{\rightleftarrows}} \mathcal{U}$.

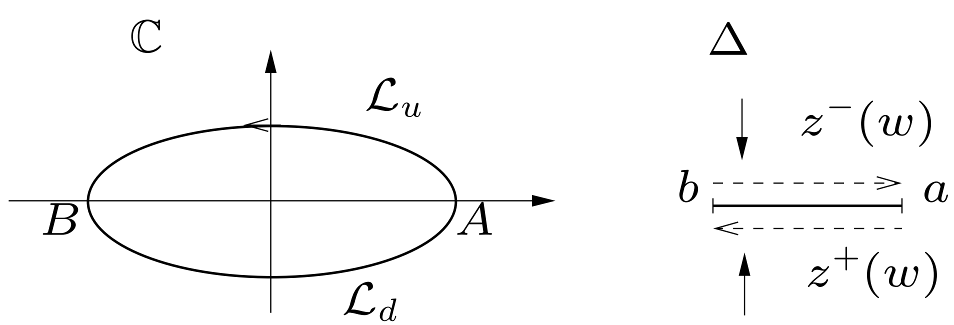


Figure 1: general symmetric form of the curve $\mathcal{L}$.

▶ An explicit form for the(CGF) $w(\cdot)$

Setting $z \stackrel{\text{def}}{=} 2a(x)y + b(x)$, the algebraic curve $\mathcal{A}$ can be uniformized as follows.

- If $d_4 \neq 0$ (4 finite branch points $x_1, \ldots, x_4$) then $D'(x_4) > 0$ and

$$\begin{cases} x(\omega) = x_4 + \dfrac{D'(x_4)}{\wp(\omega) - \frac{1}{6}D''(x_4)}, \\ z(\omega) = \dfrac{D'(x_4)\wp'(\omega)}{2\left(\wp(\omega) - \frac{1}{6}D''(x_4)\right)^2}. \end{cases} \tag{26}$$

- If $d_4 = 0$ (3 finite branch points x_1, x_2, x_3, and $x_4 = \infty$) then

$$\begin{cases} x(\omega) = \dfrac{\wp(\omega) - \dfrac{d_2}{3}}{d_3}, \\ z(\omega) = -\dfrac{\wp'(\omega)}{2d_3}. \end{cases}$$

Let

$$f(t) \stackrel{\text{def}}{=} \begin{cases} \dfrac{D''(x_4)}{6} + \dfrac{D'(x_4)}{t - x_4} & \text{if} \quad x_4 \neq \infty, \\ \dfrac{1}{6}(D''(0) + D'''(0)t) & \text{if} \quad x_4 = \infty. \end{cases}$$

Then a CGF for the domain $\mathscr{G}(\mathcal{M})$ bounded by $\mathcal{M}$ is given by the function

$$\boxed{w(t) = \wp_{1,3}(\wp_{1,2}^{-1}(f(t)) - [\omega_1 + \omega_2]/2)},$$

where $\wp_{i,j}$ denotes the Weierstrass elliptic function with periods ω_i, ω_j which are known.

Example 1: Two parallel coupled M/M/1 queues, with infinite capacities.

- Arrivals form two independent Poisson processes with parameters λ_1, λ_2.
- Service discipline FIFO (first-in-first-out) in each queue.
- Service times are distributed exponentially with instantaneous service rates S_1 and S_2 depending on the state of the system.

1. If both queues are busy, then $S_1 = \mu_1$ and $S_2 = \mu_2$.
2. If queue 2 is empty, then $S_1 = \mu_1^*$.
3. If queue 1 is empty, then $S_2 = \mu_2^*$.

$$\boxed{T(x,y)F(x,y) = a(x,y)F(0,y) + b(x,y)F(x,0) + c(x,y)F(0,0)}. \qquad (27)$$

$$\begin{cases} T(x,y) = \lambda_1(1-x) + \lambda_2(1-y) + \mu_1\left(1-\frac{1}{x}\right) + \mu_2\left(1-\frac{1}{y}\right), \\ a(x,y) = \mu_1\left(1-\frac{1}{x}\right) + q\left(1-\frac{1}{y}\right), \quad b(x,y) = \mu_2\left(1-\frac{1}{y}\right) + p\left(1-\frac{1}{x}\right), \\ c(x,y) = p\left(\frac{1}{x}-1\right) + q\left(\frac{1}{y}-1\right), \quad p = \mu_1 - \mu_1^*, \quad q = \mu_2 - \mu_2^*. \end{cases}$$

- Pleasant fact here : BVP set on the circle of radius $\mathcal{C}(0, \sqrt{\frac{\mu_2}{\lambda_2}})$.
- **Particular case 1:** $pq = \mu_1\mu_2$, that is, for $0 \le \xi \le 1$,

$$\begin{cases} \mu_1 = \xi\mu_1^*, \\ \mu_2 = (1-\xi)\mu_2^*, \end{cases} \qquad (28)$$

which corresponds to the head of line processor sharing discipline. Under the ergodicity condition $1 - \lambda_1/\mu_1^* - \lambda_2/\mu_2^* > 0$, we get the following integral.

$$F\left(0,\sqrt{\frac{\mu_2}{\lambda_2}}z\right)=\frac{1}{\pi}\int_0^{\pi}\frac{z\sin\theta\, v(\theta)d\theta}{z^2-2z\cos\theta+1}+F(0,0),\quad |z|<1,$$

where

$$v(\theta)=\frac{-\lambda_2\sin\theta\, K(\theta)}{\xi[\rho_1^*(\mu_2^*-\mu_1^*)K^2(\theta)+(\mu_1^*-\mu_2^*+\lambda_1+\lambda_2)K(\theta)-\mu_1^*]},$$

$$K(\theta)=\frac{\lambda_1+\mu_1+\beta-\sqrt{[(\sqrt{(\lambda_2}+\sqrt{\mu_2})^2+\beta][(\sqrt{\lambda_2}-\sqrt{\mu_2})^2+\beta]}}{2\lambda_1},$$

$$\beta=\lambda_2+\mu_2-2\sqrt{\lambda_2\mu_2}\cos\theta.$$

In [8], the functions $F(0,y)$ and $F(x,0)$ have been completely expressed in terms of elliptic integrals of the third kind

$$\int\frac{dw}{(1-nw^2)\sqrt{(1-w^2)(1-kw^2)}}.$$

▶ **Particular case 2:** $p + q = 0$, that is $\mu_1 + \mu_2 = \mu_1^* + \mu_2^*$.

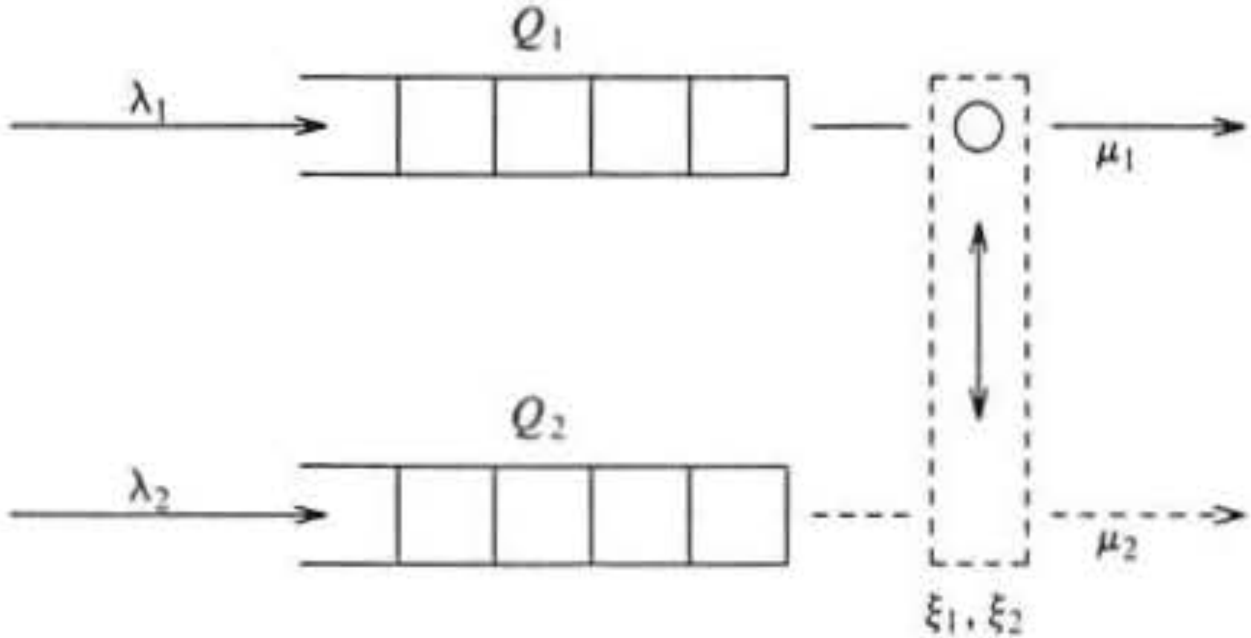
Here the solution is particularly simple ! Indeed,

$$F(x,y) = \frac{F(0,0)}{(1-\alpha x)(1-\beta y)},$$

where α and β satisfy respectively the following second degree equations

$$\begin{cases} p\mu_1^*\alpha^2 - [p(\lambda_1+\lambda_2) + \mu_1^*\mu_2^*]\alpha + \lambda_1\mu_2^* = 0, \\ q\mu_2^*\beta^2 - [q(\lambda_1+\lambda_2) + \mu_1^*\mu_2^*]\beta + \lambda_2\mu_1^* = 0. \end{cases} \tag{29}$$

Example 2: Two M/M/1 queues with alternative service periods (see [4])



- A single server alternates service between Q_1 and Q_2, for respective periods of time T_1, T_2, which are independent exponential r.v. with parameters ξ_1, ξ_2. For $l = 1, 2$, the state probabilities are defined by

$$p_l(i,j) = \mathbb{P}(\text{server at } Q_l,\ i\,\text{customers in } Q_1,\ j\,\text{customers in } \mathrm{Q}_2),$$

with $p(0,0) = p_1(0,0) = p_2(0,0)$.

- The ergodicity condition $\rho_1 + \rho_2 < 1$ is assumed, where $\rho_i = \lambda_i/\mu_i$.

$$G_1(x,y) \stackrel{\text{def}}{=} \sum_{i\geq 1, j\geq 0} p_1(i,j)x^{i-1}y^j, \quad G_2(x,y) \stackrel{\text{def}}{=} \sum_{i\geq 0, j\geq 1} p_2(i,j)x^i y^{j-1}.$$

▶ Simple algebraic manipulations allows to switch from a system of 2 FEs to a single FE of type (1), involving the two unknown functions $H(x)$ and $K(y)$.

$$\begin{cases} R_1(x,y) = \lambda_1(1-x) + \lambda_2(1-y) + \mu_1\big(1-\frac{1}{x}\big), \\ R_2(x,y) = \lambda_1(1-x) + \lambda_2(1-y) + \mu_2\big(1-\frac{1}{y}\big), \\ \Delta(x,y) = (R_1(x,y)+\xi_1)(R_2(x,y)+\xi_2) - \xi_1\xi_2, \\ H(x) = x(R_1(x,0)+\xi_1)G_1(x,0), \\ K(y) = y(R_2(0,y)+\xi_2)G_2(0,y) + \mu_2 G_2(0,0) - p(0,0)\lambda_2 y, \\ g(x,y) = \dfrac{\lambda_1(1-x)+\lambda_2(1-y)}{R_2(x,y)}\xi_2 p(0,0). \end{cases}$$

Lemma 7. *For $\Delta(x,y)=0$, with $|x|\leq 1$ and $|y|\leq 1$, we have*

$$H(x) - K(y) = g(x,y). \tag{30}$$

The polynomial $xy\Delta(x,y)$ is assumed to be irreducible. The Riemann surface $\mathbf{S}$ corresponding to $\Delta(x,y)=0$ has in general a genus greater than 1, as it reduces to a polynomial equation of degree 3 in x and in y (3-sheeted covering).

Mixing uniformization and BVP. Let

$$\boxed{R_1(x,y)+\xi_1=\xi_1 z, \qquad R_2(x,y)+\xi_2=\frac{\xi_2}{z}}. \tag{31}$$

$$\Delta(x,y)=0 \iff \text{(31) holds.}$$

▶ Setting $\lambda=\lambda_1+\lambda_2$ and $\nu=r_1x+r_2y$, where $r_i=\lambda_i/\lambda$, we relate (x,y) to (ν,z) by

$$x(\nu,z)=\frac{\mu_1}{\mu_1+\lambda(1-\nu)+\xi_1(1-z)}, \quad y(\nu,z)=\frac{\mu_2}{\mu_2+\lambda(1-\nu)+\xi_2(1-\frac{1}{z})}, \tag{32}$$

with

$$\boxed{\nu=\frac{r_1\mu_1}{\mu_1+\lambda(1-\nu)+\xi_1(1-z)}+\frac{r_2\mu_2}{\mu_2+\lambda(1-\nu)+\xi_2(1-\frac{1}{z})}}. \tag{33}$$

Lemma 8.

1. *For any z with $|z| = 1$, there exists exactly one $\nu = \nu(z)$ such that $|v| \leq 1$ and $\Delta(x, y) = 0$, with $|x|, |y| \leq 1$, $x = x(v, z)$, $y = y(v, z)$. For $z = 1$, this value is $v = 1$.*

2. *Let $|\nu| = 1$ and $\xi_1\rho_1 \geq \xi_2\rho_2$. Then (33) has exactly one root $z(\nu)$ satisfying $|z| \geq 1$; the equality $|z| = 1$ holds only for $\nu = z(\nu) = 1$.* ■

▶ Setting $X(\nu) \stackrel{\text{def}}{=} X(\nu, z(\nu))$ and $Y(\nu) \stackrel{\text{def}}{=} Y(\nu, z(\nu))$, the desired simplification has been realized, from a cubic in y (or x) to a quadratic in z in (33).

▶ The function $z(\nu)$ has two (real) branch points $\nu' < \nu''$ in the unit circle $|\nu| = 1$. As ν moves around the cut $[\nu' < \nu'']$, $X(\nu)$ and $Y(\nu)$ traverse simple closed contours Γ_x and Γ_y in their respective planes $\mathbb{C}_x$ and $\mathbb{C}_y$.

▶ Rewriting now (30) as

$$\boxed{H(X(\nu)) - K(Y(\nu)) = f(\nu)}, \tag{34}$$

$H(\cdot)$ and $K(\cdot)$ can be analytically continued.

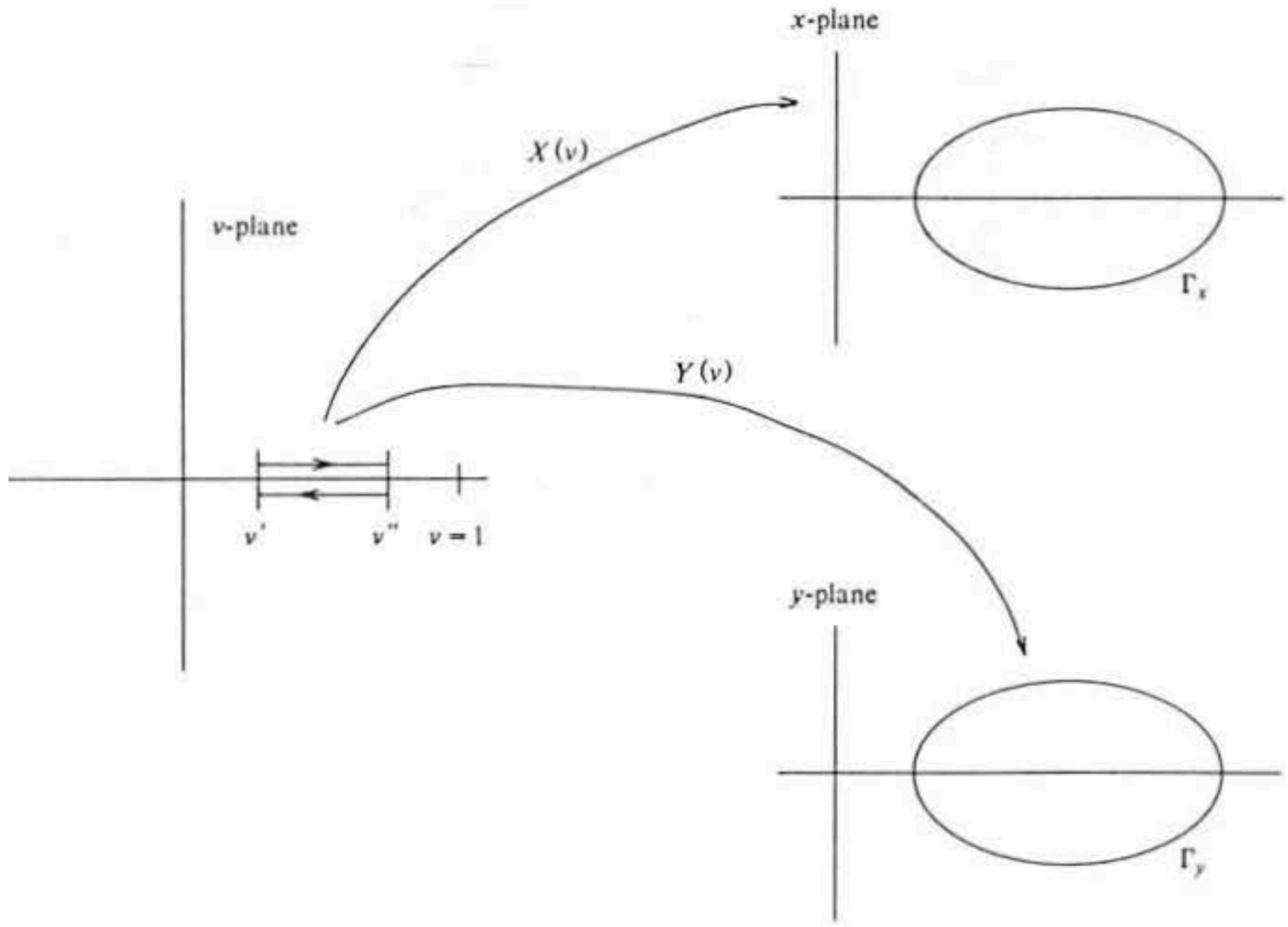


Figure 2: The mappings $X(\nu)$ and $Y(\nu)$

The BVP (34) leads to the integral

$$\boxed{K(y) = \frac{1}{2i\pi}\int_L \frac{\psi(s)Y'(s)ds}{Y(s)-y}},$$

where $\psi(s)$ satisfies the non singular Fredholm integral equation,

$$\boxed{\psi(s) - \frac{1}{2i\pi}\int_L \psi(s)\frac{\partial}{\partial s}\log\frac{X(s)-X(\nu)}{Y(s)-Y(\nu)}ds = \frac{1}{2i\pi}\int_L f(s)\frac{\partial}{\partial s}\log\frac{X(s)-X(\nu)}{s-\nu}ds}.$$

which admits a unique solution by the Fredholm alternative.

Example 3: Sojourn time in a 3-node Jackson network with overtaking (see[9, 1])

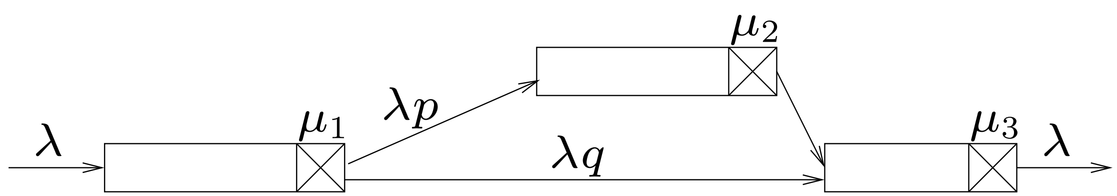


Figure 3: Network with overtaking

- An inherent overtaking phenomenon renders things slightly complicated...
- Cutting the Gordian Knot amounts to finding the function

$G(x, y, z, s) \stackrel{\text{def}}{=}$ Laplace transform of the conditional waiting time distribution of a tagged customer at a departure instant of the first queue.

The following non-homogeneous FE holds, for $|x|, |y|, |z| \leq 1, \Re(s) \geq 0$,

$$K(x,y,z,s)G(x,y,z,s) = \left(\mu_1 - \frac{\lambda}{x}\right) G(0,y,z,s) + \left(\mu_3 - q\mu_1\frac{x}{z} - \mu_2\frac{y}{z}\right) G(x,y,0,s) + \frac{\mu_2\mu_3}{(1-x)[s+\mu_3(1-z)]},$$

where p, q are routing probabilities with $p+q=1$, λ is the external arrival rate, μ_i is the service rate at queue i, and

$$K(x,y,z,s) = s + \lambda\left(1 - \frac{1}{x}\right) + \mu_1\left(1 - px - q\frac{x}{z}\right) + \mu_2\left(1 - \frac{y}{z}\right) + \mu_3(1-z).$$

Taking y and s as parameters, the above FE becomes

$$\boxed{K(x,y,z,s)\widetilde{G}(x,z) = A(x)\widetilde{G}(0,z) + B(x,y,z,s)\widetilde{G}(x,0) + C(x,y,z,s)}.$$

Finally, the LST of the total sojourn time of an arbitrary customer is given by

$$(1-\rho_1)(1-\rho_2)(1-\rho_3)\frac{\mu_1}{\mu_1+s}G\left[\frac{\mu_1}{\mu_1+s}, \rho_2, \rho_3, s\right].$$

Example 4: Joining the shorter queue.

- The problem is far more difficult, due to the non space-homogeneity.

Let the probability

$$p_t(m,n) \stackrel{\text{def}}{=} \mathbb{P}(M_t = m, N_t = n)$$

that, at time t, there are m customers in queue 1 and n customers in queue 2.

- Kolmogorov's equations for the stationary probabilities

$$p(m,n) \stackrel{\text{def}}{=} \lim_{t\to\infty} p_t(m,n)$$

are written separately in the two distinct regions

$$\mathcal{U}_1 \stackrel{\text{def}}{=} \{(m,n), m \le n\} \quad \text{and} \quad \mathcal{U}_2 \stackrel{\text{def}}{=} \{(m,n), n \le m\}.$$

$$
\begin{cases}
F_1(x,y) = \sum_{i,j\geq 0} p(i,i+j)x^i y^j, \quad P_1(x) = \sum_{i,\geq 0} p(i,i+1)x^i, \\
F_2(x,y) = \sum_{i,j\geq 0} p(i+j,i)x^i y^j, \quad P_2(x) = \sum_{i,\geq 0} p(i+1,i)x^i, \\
Q(x) = F_1(x,0) = F_2(x,0) = \sum_{i,\geq 0} p(i,i)x^i, \\
A_1(x) = (\alpha + \lambda x)P_2(x), \quad A_2(x) = (\beta + \lambda x)P_1(x), \\
G_i(y) = F_i(0,y), \ i = 1,2, \\
T_1(x,y) = \lambda\left(1 - \frac{x}{y}\right) + \alpha\left(1 - \frac{y}{x}\right) + \beta\left(1 - \frac{1}{y}\right), \quad R_1(x,y) = xy\,T_1(x,y), \\
T_2(x,y) = \lambda\left(1 - \frac{x}{y}\right) + \beta\left(1 - \frac{y}{x}\right) + \alpha\left(1 - \frac{1}{y}\right), \quad R_2(x,y) = xy\,T_2(x,y), \\
s = \lambda + \alpha + \beta.
\end{cases}
$$

The following system holds.

$$\begin{cases} T_1(x,y)F_1(x,y) & = \alpha\left(1-\frac{y}{x}\right)G_1(y) + \left(\frac{\lambda\pi_2 y^2 - \lambda x - \beta}{y}\right)Q(x) + A_1(x), \\ T_2(x,y)F_2(x,y) & = \beta\left(1-\frac{y}{x}\right)G_2(y) + \left(\frac{\lambda\pi_1 y^2 - \lambda x - \alpha}{y}\right)Q(x) + A_2(x), \\ sQ(x) & = A_1(x) + A_2(x). \end{cases} \tag{35}$$

- Assuming $\alpha \neq \beta$, one has to deal with two kernels $T_1(x,y)$ and $T_2(x,y)$ corresponding to algebraic curves of genus 0.

- By using the zeros of these kernels in $\overline{\mathcal{D}} \times \overline{\mathcal{D}}$, we are ultimately left with two unknown functions of one complex variable, say for instance $G_i(y)$, or $A_i(x), i = 1,2$.

Notation and assumption: From now on, $\beta > \alpha$ (the case $\beta = \alpha$ will be considered separately). In a superscript or subscript position, the pair $\alpha\beta$ (resp. $\beta\alpha$) refers to any quantity related to the kernel $R_1(x,y)$ (resp. $R_2(x,y)$).
For instance, branches $Y_0^{\alpha\beta}, X_1^{\beta\alpha}$, etc.

- The functions $Y_i^{\alpha\beta}(x), i = 0, 1$, have exactly two branch points (genus is 0), which are always located inside the unit disc $\mathcal{D}$,

$$x = 0, \qquad x^*_{\alpha\beta} = \frac{4\alpha\beta}{s^2 - 4\alpha\lambda} < 1. \tag{36}$$

- $\Psi^{\alpha\beta}$ is the ellipse obtained by the mapping $[\overleftarrow{\underrightarrow{0, x^*_{\alpha\beta}}}] \xrightarrow{Y^{\alpha\beta}} \Psi^{\alpha\beta}$.

- Similarly, $X_i^{\alpha\beta}(x), i = 0, 1$, admits two branch-points

$$y_1^{\alpha\beta} = \frac{\beta}{s + 2\sqrt{\alpha\lambda}} < y_2^{\alpha\beta} = \frac{\beta}{s - 2\sqrt{\alpha\lambda}}, \quad 0 < y_1^{\alpha\beta} < y_2^{\alpha\beta} < 1.$$

- $\Phi^{\alpha\beta}$ is the ellipse obtained by the mapping

$$[\overleftarrow{\underrightarrow{y_1^{\alpha\beta}, y_2^{\alpha\beta}}}] \xrightarrow{X^{\alpha\beta}} \Phi^{\alpha\beta}.$$

- $\beta > \alpha \iff x^*_{\alpha\beta} < x^*_{\beta\alpha}$.

Theorem 9. *The functions $Q(\cdot), G_i(\cdot), A_i(\cdot), i = 1, 2,$ can be continued as meromorphic functions to the whole complex plane. In particular, $\forall y \in \mathbb{C}_y$, the function $G_1(y)$ satisfies the FE*

$$\boxed{\begin{array}{l} L_1(X_0^{\beta\alpha}(y))G_1(Y_1^{\alpha\beta} \circ X_0^{\beta\alpha}(y)) - L_1(X_1^{\beta\alpha}(y))G_1(Y_1^{\alpha\beta} \circ X_1^{\beta\alpha}(y)) = \\ K_1(X_0^{\beta\alpha}(y))G_1(Y_0^{\alpha\beta} \circ X_0^{\beta\alpha}(y)) - K_1(X_1^{\beta\alpha}(y))G_1(Y_0^{\alpha\beta} \circ X_1^{\beta\alpha}(y)) \end{array}}, \quad (37)$$

which involves an algebraic curve with 4 branches. ■

The proof (first established in 1979) can be found in [1] and relies on the following

Lemma 10. *Let $\mathcal{D}_n$ be the domain recursively defined by*

$$\begin{cases} \mathcal{D}_0 & = \mathcal{D}, \\ \mathcal{D}_{n+1} & = \inf \left\{ (X_1 \circ Y_1)^{\alpha\beta}(\mathcal{D}_n), (X_1 \circ Y_1)^{\beta\alpha}(\mathcal{D}_n) \right\}. \end{cases}$$

Then $\mathcal{D}_n \subset \mathcal{D}_{n+1}$ and $\lim_{n\to\infty} \mathcal{D}_n = \mathbb{C}$, where $\mathbb{C}$ denotes the complex plane.

From (37), explicit intricate recursive computations of the poles and residues of $G_1(\cdot)$ can be achieved (see [5])...

Proposition 11.

1. *Equation* (37) *for* $G_1(\cdot)$ *is equivalent to a BVP of the form*

$$\boxed{U^+(t) - U^-(t) = H(t)\overline{U^+(t)} + C, \quad t \in w(\Phi^{\alpha\beta})}, \tag{38}$$

where $H(t)$ *is known, and* $w(z)$ *is a conformal gluing of the domain inside the ellipse* $\Phi^{\alpha\beta}$ *onto the complex plane cut along an open smooth arc. The BVP* (38) *defines a Noetherian operator with index* 0 *(Fredholm operator), having a unique solution analytic in* $\mathcal{D}$ *if and only if*

$$\boxed{\Delta'(1) > 0 \quad \Longleftrightarrow \quad \lambda < \alpha + \beta}.$$

2. *The function* $Q(x)$ *satisfies the real integral equation*

$$\boxed{Z(x)Q(x) = \int_0^{x^*_{\beta\alpha}} Q(u)S(x,u)du + W(x), \quad \forall x \in [0, x^*_{\beta\alpha}]}, \tag{39}$$

containing both a regular part and a singular part, where $Z(x), S(x,u), W(x)$ *are known quantities.* ■

Explicit integral forms for equal service rates ($\alpha = \beta$)

The problem simplifies in a breathtaking way, since $T_1(x,y) = T_2(x,y)$.

$F \stackrel{\text{def}}{=} F_1 + F_2, \quad G \stackrel{\text{def}}{=} G_1 + G_2, \quad \Delta(x,y) \stackrel{\text{def}}{=} s - \left(\frac{2\alpha}{x} + \lambda\right) y$. Then, from (35),

$$\boxed{F(x,y)T(x,y) = \alpha\left(1 - \frac{y}{x}\right) G(y) - \Delta(x,y)Q(x)}.$$

Proposition 12. *For $\alpha = \beta$, the system is ergodic if and only if $\lambda < 2\alpha$. Then*

$$Q(x) = K\, e^{\Gamma(x)}, \tag{40}$$

$$\begin{cases} \Gamma(x) & = \dfrac{1}{2i\pi} \displaystyle\int_{\Phi} \dfrac{\log w(t)\theta'(t)dt}{\theta(t) - \theta(x)}, \quad x \in \Phi_{int}, \\ w(t) & = \dfrac{\overline{i\xi(t)}}{i\xi(t)}, \quad \xi(t) = \dfrac{\Delta(t)}{1 - \frac{Y_0(t)}{t}}, \quad \Delta(t) = \Delta(t, Y_0(t)), \end{cases}$$

where K is a positive constant, Φ_{int} is the interior domain bounded by the ellipse Φ, and $\theta(.)$ denotes the conformal mapping of Φ_{int} onto the unit disc.

Counting lattice walks in the quarter plane $\mathbb{Z}_+^2$

For a given set $\mathcal{S}$ of admissible steps (or jumps), it is a matter of counting the number of paths of a certain length, which start and end at some arbitrary points, and might even be restricted to some region of the plane.

Three natural questions arise.

- Q1: How many such paths do exist?

- Q2: What is the asymptotic behavior, as their length goes to infinity, of the number of walks ending at some given point or domain (for instance one axis)?

- Q3: What is the nature of the generating function of the numbers of walks (*rational*, *algebraic*, *holonomic*)? If paths remain in a half-plane, then the CGFs have explicit forms and can only be rational or algebraic (see e.g., [3]). The situation happens to be much richer if the walks are confined to $\mathbb{Z}_+^2$. . .

Questions Q1, Q2 for walks in $\mathbb{Z}_+^2$ with small steps.

- Walks start at the origin.

- The set $\mathcal{S}$ of admissible steps is included in the set of the 8 nearest neighbors, i.e., $\mathcal{S} \subset \{-1,0,1\}^2 \setminus \{(0,0)\}$. By using an extended Kronecker's delta, we shall write

$$\delta_{i,j} = \begin{cases} 1 & \text{if} \quad (i,j) \in \mathcal{S}, \\ 0 & \text{if} \quad (i,j) \notin \mathcal{S}. \end{cases}$$

- Walks are supposed to be continuous on the boundary of $\mathbb{Z}_+^2$: allowed jumps are the natural ones and jumps that would take the walk out of $\mathbb{Z}_+^2$ are discarded.

A priori, there are 2^8 such models. But, by using simple geometrical symmetries (see [2]), there are in fact only 79 types of essentially distinct walks.

Let $f(i,j,k) \stackrel{\text{def}}{=}$ the number of paths in $\mathbb{Z}_+^2$ starting from $(0,0)$ and ending at (i,j) after k steps. Then

$$F(x,y,z) \stackrel{\text{def}}{=} \sum_{i,j,k \geqslant 0} f(i,j,k) x^i y^j z^k \tag{41}$$

satisfies the functional equation

$$\boxed{K(x,y,z)F(x,y,z) = c(x)F(x,0,z) + \widetilde{c}(y)F(0,y,z) + c_0(x,y,z)}, \tag{42}$$

a priori valid in the domain $|x| \leqslant 1, |y| \leqslant 1, |z| < 1/|\mathcal{S}|$, where

$$\begin{cases} K(x,y;z) = xy\left[\displaystyle\sum_{(i,j)\in\mathcal{S}} x^i y^j - 1/z\right], \\ c(x) = \displaystyle\sum_{i \leqslant 1} \delta_{i,-1} x^{i+1}, \quad \widetilde{c}(y) = \sum_{j \leqslant 1} \delta_{-1,j} y^{j+1}, \\ c_0(x,y,z) = -\delta_{-1,-1} F(0,0,z) - xy/z, \end{cases}$$

Proposition 13. *For* $x \in \mathscr{G}(\mathcal{M}_z)$,

$$\boxed{c(x)F(x,0,z) - c(0)F(0,0,z) = \frac{1}{2\pi i z}\int_{\mathcal{M}_z} tY_0(t,z)\frac{w'(t,z)}{w(t,z)-w(x,z)}\,\mathrm{d}t}, \tag{43}$$

where $w(x,z)$ *is the gluing function for the domain* $\mathscr{G}(\mathcal{M}_z)$ *in the* $\mathbb{C}_x$*-plane. A similar expression can be written for* $F(0,y,z)$.

By symmetry and classical arguments, only the real singularities of $F(0,0,z)$, $F(1,0,z)$, $F(0,1,z)$ and $F(1,1,z)$ with respect to z will play a role in the asymptotics. From the expression (43), the main source of all possible singularities can be explained. We simply quote the main result (see [12]).

Proposition 14. *The smallest positive singularity of* $F(0,0,z)$ *is*

$$\boxed{z_g = \inf\{z > 0 : y_2(z) = y_3(z)\}}. \tag{44}$$

For the simple walk [i.e. $(i,j) \in \mathcal{S}$ if and only if $ij = 0$] formulas are pleasant, since the curve $\mathcal{M}_z$ is a circle.

Proposition 15. *For the simple walk,*

$$\begin{aligned} F(0,0,z) &= \frac{1}{\pi}\int_{-1}^{1} \frac{1-2uz-\sqrt{(1-2uz)^2-4z^2}}{z^2}\sqrt{1-u^2}\,\mathrm{d}u, \\ F(1,0,z) &= \frac{1}{2\pi}\int_{-1}^{1} \frac{1-2uz-\sqrt{(1-2uz)^2-4z^2}}{z^2}\sqrt{\frac{1+u}{1-u}}\,\mathrm{d}u. \end{aligned}$$

$F(0,0,z)$ counts the number of excursions starting from $(0,0)$ and returning to $(0,0)$, while $F(1,0,z)$ counts the number of walks starting from $(0,0)$ and ending at the horizontal axis. By symmetry, $F(0,1,z) = F(1,0,z)$.

Proposition 16.

$$f(0,0,2n) \sim \frac{16^{2n+1}}{\pi\, n^3}, \quad \sum_{i\geqslant 0} f(i,0,n) \sim \frac{2.4^{n+1}}{\pi\, n^2}, \quad \sum_{i,j\geqslant 0} f(i,j,n) \sim \frac{4^{n+1}}{\pi\, n}.$$

Questions Q3 [nature of the functions]

▶ Let $\mathbb{C}(x), \mathbb{C}(y)$ and $\mathbb{C}(x,y)$ denote the respective fields of rational functions of x, y and (x,y) over $\mathbb{C}$. In general Q is assumed to be irreducible, and the quotient field $\mathbb{C}(x,y)$ with respect to Q will be denoted by $\mathbb{C}_Q(x,y)$.

Definition 17. The group of the random walk is the Galois group $\mathcal{H} = \langle \xi, \eta \rangle$ of automorphisms of $\mathbb{C}_Q(x,y)$ generated by ξ and η given by

$$\xi(x,y) = \left(x, \frac{c(x)}{y\, a(x)}\right), \qquad \eta(x,y) = \left(\frac{\widetilde{c}(y)}{x\, \widetilde{a}(y)}, y\right).$$

ξ and η are involutions satisfying $\xi^2 = \eta^2 = I$.

Lemma 18. *Let*

$$\delta \stackrel{def}{=} \eta\xi. \tag{45}$$

$\mathcal{H}$ has a normal cyclic subgroup $\mathcal{H}_0 = \{\delta^n, n \in \mathbb{Z}\}$, which is finite or infinite, and $\mathcal{H}/\mathcal{H}_0$ is a cyclic group of order 2. The group $\mathcal{H}$ is finite of order $2n$ if and only if $\delta^n = I$.

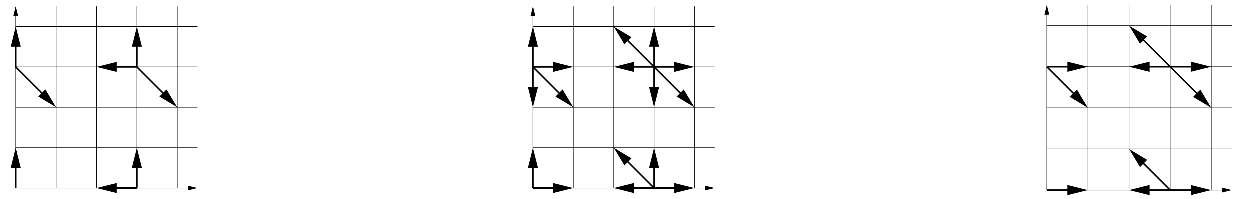


Figure 4: Left, 2 walks with group 6. Right, 1 walk with group 8.

Figure 5: Left, 3 walks with group 6. Right, 1 walk with group 8.

Theorem 19 (see [2]). *For the 16 walks with a group of order 4 and for the 3 walks in figure 4, the formal trivariate series* (41) *is holonomic non-algebraic. For the 3 walks on the left in figure 5, the trivariate series* (41) *is algebraic.*

Theorem 20 (see [1]). *For the so-called* Gessel's walk *on the right in figure 5, the formal trivariate series* (41) *is algebraic.*

They use a less general group acting only on $\mathbb{C}(x, y)$! In [11], a direct proof with $\mathcal{H}$, is provided, together with the fact that $\pi(x)$ in equation (1) is always holonomic when the groupe is finite.

About generalizations

- Arbitrary finite jumps of size n. In this case, the crucial first step is (as before) the analytic continuation process. A priori $2n$ unknown functions of one variable. A functional equation can be obtained on a Riemann surface $\mathbf{S}$ of arbitrary genus, implying the manipulation of hyperelliptic curves.
  The ultimate goal might be to set a generalized BVP on a single curve for a vector of analytic functions: this remains a doable challenge (see [13]).

- Space inhomogeneity. Here, one must frequently deal with systems of functional equations (see e.g., the example JSQ).

- Larger dimensions. In $\mathbb{Z}_+^n, n \geq 3$, most of the questions are largely open.
  For a first step in this direction, see [24], where explicit integral formulas for the resolvent of the discrete Laplace operator in an orthant are given.

  Yet, a global solution to the essential problems seems out of reach, even computationally : analytic continuation, index calculation, BVP for n complex variables...

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