About Chapters 2 & 3 of the Yellow Book [1]

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This three chapters present the essential stepping stones of the analytic approach. They require in particular some basic notions concerning Riemann surfaces, algebraic curves and Galois theory.

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Groupe de travail Marches aléatoires, 20 avril 2018.

Covering Manifolds

X and Y are two Riemann surfaces with respective *atlases* $\mathcal{A}_X$ and $\mathcal{A}_Y$ ([2, 5]).

Let $h : Y \to X$ a holomorphic mapping of Y *onto* X.

Note that whenever Y is compact, any holomorphic mapping h is necessarily *onto* and that X is compact.

The pair (Y, h) is called a *cover* and Y is a *covering manifold* of X.

- When h is not constant, $h^{-1}(x)$ is a discrete subset of Y, for any $x \in X$.
- A point $y \in h^{-1}(x)$ is said to *lie over* x (or to *cover* x) and x is the *projection* of y on X.

Definition 1. If, for any neighborhood V of y, the restricted mapping $h_{|V}$ is not one-to-one, y is called a *branch point*.

More precisely, $y \in V$ is a *branch point of order* $n-1$ if there exist two charts $(U,\varphi) \in \mathcal{A}_X$, $(V,\psi) \in \mathcal{A}_Y$, $y \in V$, with $h(V) \subset U$, and a finite number n, such that

$$\boxed{(\varphi \circ h)(v) \equiv \gamma_n \circ \psi(v), \quad \forall v \in V},$$

where the function $\gamma_n : \mathbb{C} \to \mathbb{C}$ satisfies $\gamma_n(z) = z^n$. This means that each point $x \in h(V) \setminus h(y)$ is covered by exactly n points of Y contained in $V \setminus y$, the deleted neighborhood of y.

Often n will be referred to as the *multiplicity*. If $n = 1$, i.e. h is a one-to-one mapping $V \to U$, for some V, then y is called a *regular* point.

▶ *Among all covering maps without branch points, there exists exactly one, up to an isomorphism, which is simply connected and usually referred to as* **the universal covering**.

Definition 2. [Proper mapping] Let (Y, h) be a cover of X. The mapping h is called *proper* if the preimage of every compact set is compact. In particular, if Y is compact then h is always proper.

Proposition 3. *Let X and Y be Riemann surfaces and $h : Y \to X$ be a proper non-constant holomorphic mapping. Then there exists a natural number n such that h takes every value $c \in X$, counting multiplicities, n times.*

Definition 4. [Lifting] Introduce the following objects.

- X, Y, two Riemann surfaces, and Z a topological space;
- $h : Y \to X$, a continuous mapping (see [2]);
- $f : Z \to X$, a continuous function.

▶ A *lifting* of f with respect to h is a continuous mapping $g : Z \to Y$ such that $f = h \circ g$.

Proposition 5. *Let (Y, h) be a cover of X. Let L be an arc of X having its origin at x (i.e. there exists a continuous mapping $f : [0, 1] \to X$, where $L = f[0, 1]$ and $f(0) = x$). Then, $\forall y \in h^{-1}(x)$, $\exists$ an arc in Y which has its origin at y and lies* over L.

Let B be the set of branch-points of h and $A \stackrel{\text{def}}{=} h(B)$. Let also $Y' \stackrel{\text{def}}{=} Y \setminus h^{-1}(A)$ and $X' \stackrel{\text{def}}{=} X \setminus A$. Then the restriction $h_{|Y'} : Y' \to X'$ is a proper unbranched covering, which has a well defined finite number of sheets n.

Proposition 6. ***[Uniqueness of lifting]***

1. *Let (Y, h) be a cover without branch-point. Let g_1 and g_2 be two liftings with respect to h. Then either $g_1(z) \neq g_2(z)$, $\forall z \in Z$, or $g_1 \equiv g_2$. In addition, if f is holomorphic, then any lifting g is also holomorphic.*

2. *In general, when there are branch points, the preceding property holds provided that X, Y and Z are replaced respectively by X', Y' and $Z' = f^{-1}(X')$.*

Remark 7. Proposition 6 will be used either to lift arcs, in which case $Z = [0, 1]$, or also to compare different mappings, taking then $Z \equiv Y$.

Algebraic functions

Let X be a Riemann surface and $\mathcal{M}(X)$ be the field of meromorphic functions on X, and let

$$P(T) = \sum_{i=0}^{n} f_i T^i, \quad f_i \in \mathcal{M}(X),$$

be an irreducible polynomial of degree n in one indeterminate T, with coefficients in $\mathcal{M}(X)$. When X is the complex plane $\mathbb{C}$, the equation

$$P(Y) = \sum_{i=0}^{n} f_i Y^i = 0, \quad f_i \in \mathcal{M}(X), \tag{1}$$

defines a *multi-valued* function $Y(x)$, $x \in \mathbb{C}$, called an *algebraic function*.

Equation (1) has in the neighborhood of every x, except for a finite number of them called *critical points*, exactly n distinct finite roots $Y_1(x), Y_2(x), \ldots, Y_n(x)$, which are the *branches* of the algebraic function $Y(x)$.

At any regular point x_0, there exist n different convergent power series (germs)

$$Y_{x_0,k}(x) = \sum_{l=0}^{\infty} c_l\,(x_0, k)\,(x - x_0)^l\,, \quad k = 1, 2, \dots n,$$

satisfying (1). It is possible to construct a Riemann surface, referred to as the *Riemann surface of the algebraic function* $Y(x)$, by putting together the above germs. One of the main goals pursued by Riemann, when he introduced these abstract surfaces, was to render algebraic functions single-valued. This possibility is contained in the following theorem.

Theorem 8. *There exist a holomorphic n-sheeted proper cover (Y, π) of X and a function $F \in \mathcal{M}(Y)$ such that*

$$\sum_{i=0}^{n} f_i(\pi(y))(F(y))^i = 0, \quad \forall y \in Y. \tag{2}$$

The triple (Y, π, F) is unique up to isomorphism and is sometimes called (see e.g., [2]) the algebraic function defined by the polynomial P. ■

Elements of Galois theory

- Let K and L denote two arbitrary fields such that $K \subset L$. We say that L is a *finite extension* of K if L is a vector space over K of finite dimension n, which is called the *degree of* L *over* K and is denoted by $[L:K]$.

- Let $K[T]$ be the polynomial ring in one indeterminate T over the field K. The extension L of K is said to be *normal* or a *Galois extension* of K if any polynomial $P \in K[T]$, irreducible and having a root in L, has all its roots in L. The necessary and sufficient condition for L to be normal is that there exists an irreducible $P \in K[T]$ such that the roots of P form a basis of L, as a vector space on K.

- The *Galois group* $G(L/K)$ of the extension L of K is the group of automorphisms of L leaving all elements of K invariant. The order of this group is denoted by $[L:K]$ and any of its elements realizes a permutation of the roots of any polynomial $P \in K[T]$, provided that P is irreducible and has all its roots in L. Any extension of degree 2 is obviously normal.

▶ Let (Y, π) be a cover of X. An automorphism σ of Y is called a *covering automorphism* if $\pi = \pi \circ \sigma$, which means that $\forall y \in Y$, the points y and $\sigma(y)$ have the same projection on X. The set of all covering automorphisms form a multiplicative group denoted by $\mathrm{Aut}(Y/X)$, which is a subgroup of the group of automorphisms of Y.

Definition 9. Let (Y, π) be a cover of X without branch point. One says that π is *normal or Galois* if, for any $y_0, y_1 \in Y$ such that $\pi(y_0) = \pi(y_1)$, there exists a $\sigma \in \mathrm{Aut}(Y/X)$ such that $\sigma(y_0) = y_1$. From Proposition 6, it follows that σ is unique, setting $Z = Y$, $f = \pi$ and $\sigma = g$.

Proposition 10. *Let (Y, π) be a universal cover of X (i.e. simply connected and without branch-point). Then π is* Galois *and Aut (Y/X) is isomorphic to the* fundamental group *of X (i.e. the set of homotopy classes of closed curves going through some arbitrary but fixed point).* (See e.g., [2].) ■

Proposition 11. *Let (Y, π) a cover of X, the mapping π being proper. Then every covering transformation $\sigma' \in \mathrm{Aut}(Y'/X')$ can be extended to a covering transformation $\sigma \in \mathrm{Aut}(Y/X)$ and π will be called* Galois *if $(Y', \pi_{|Y'})$ is* Galois, *in the sense of definition 9, X' and Y' being as in definition 2.* ■

Theorem 12. *Let (Y, π, F) be the algebraic function corresponding to the equation $P(T) = 0$, where P is an irreducible polynomial of degree n, having all its coefficients in $\mathcal{M}(X)$.*

Define the mapping $\pi^* : \mathcal{M}(X) \to \mathcal{M}(Y)$ by the relation

$$[\pi^*(f)]\,(y) = f(\pi(y)), \quad \forall y \in Y \text{ and } f \in \mathcal{M}(X).$$

(In other words, the range of π^* is a subfield of $\mathcal{M}(Y)$ which in this way can be identified with $\mathcal{M}(X)$.) Then the field $\mathcal{M}(Y)$ is an extension of degree n of $\mathcal{M}(X)$ and is isomorphic to the quotient field $\mathcal{M}(X)[T]/P(T)$. Moreover, the groups $\mathrm{Aut}(Y/X)$ and $G(\mathcal{M}(Y)/\mathcal{M}(X))$ are isomorphic via the isomorphism ϕ defined by

$$\phi : \sigma \to \hat{\sigma}, \quad \text{with } \hat{\sigma} \circ f = f \circ \sigma^{-1}, \quad \forall f \in \mathcal{M}(Y).$$

The covering π is Galois *if, and only if, $\mathcal{M}(Y)$ is a* Galois *extension of $\mathcal{M}(X)$.*

■

Universal Covering and Uniformization

Every Riemann surface of a polynomial with coefficients in $\mathcal{M}(\mathbb{P}^1)$ is topologically isomorphic to a sphere with g *handles* attached to it. The number g is called the *genus* of the Riemann surface.

Let S be a compact Riemann surface with its associated polynomial Q and (Ω, λ) its universal covering. λ is unique up to an automorphism of Ω. It is known [3] that only three possible situations can take place.

1. $\Omega = \mathbb{P}^1$, the Riemann sphere, then $g = 0$.
2. $\Omega = \mathbb{C}$, the (finite) complex plane, then $g = 1$.
3. $\Omega = \mathcal{D}$, the open unit disc, then $g > 1$.

The next result can be found, for example, in [5].

Proposition 13. *There exist two functions $f, g \in \mathcal{M}(S)$ such that*

1. $Q(f, g) = 0$;

2. *any function belonging to $\mathcal{M}(S)$ is a rational function of f and g.*

■

We shall also use the following, see e.g., [2].

Proposition 14. *Every meromorphic function on $\mathbb{P}^1$ is rational.* ■

Remembering that $\lambda : \Omega \to S$, it follows that the functions $\widetilde{f}$ and $\widetilde{g}$, respectively given by

$$\widetilde{f} = f \circ \lambda, \quad \widetilde{g} = g \circ \lambda,$$

with f, g given in Proposition 13, are meromorphic on Ω and invariant with respect to the group of covering automorphisms. It is customary to state that the pair of functions $\widetilde{f}, \widetilde{g}$ gives a uniformization of $\mathcal{M}(S)$, or, equivalently, uniformizes any $h \in \mathcal{M}(S)$.

Restriction of the functional equation to the Riemann surface

Assuming that the polynomial Q is irreducible. Then the equations

$$Q(x, Y) = 0 \quad \text{and} \quad Q(X, y) = 0$$

define a priori two Riemann surfaces (with the two corresponding algebraic functions $Y(x)$ and $X(y)$). In fact (see e.g., [5]), these two surfaces have the same genus, and hence are conformally equivalent.

- Hence, we shall consider a single Riemann surface, denoted by $\mathbf{S}$, which is the Riemann surface for both $Y(x)$ and $X(y)$, but has two different coverings

$$h_x : \mathbf{S} \to \mathbb{C}_x, \quad h_y : \mathbf{S} \to \mathbb{C}_y, \tag{3}$$

where $\mathbb{C}_z$ denotes the complex plane with respect to the z-coordinate.

Any function f on a domain $V \subset \mathbb{C}_x$ can be *lifted* onto $h_x^{-1}(V) \subset \mathbf{S}$, thus yielding a new function $\hat{f} \stackrel{\text{def}}{=} f \circ h_x$. Correspondingly, one has $\hat{g} \stackrel{\text{def}}{=} g \circ h_y$.

Hence, taking for f and g the identity functions, we can put

$$x(s) \stackrel{\text{def}}{=} h_x(s), \quad y(s) \stackrel{\text{def}}{=} h_y(s), \quad s \in \mathbf{S}. \tag{4}$$

The general flowchart is given in Fig. 1.

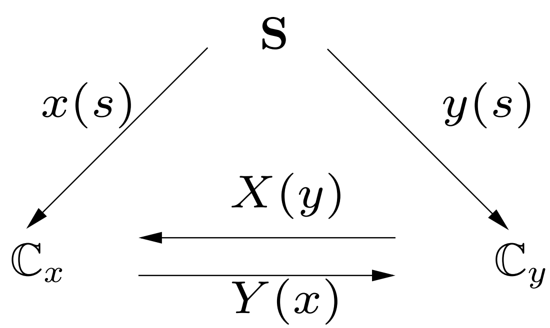

Figure 1: lifting the complex planes onto $\mathbf{S}$.

On **S**, we have $Q(x(s), y(s)) = 0$ and it will be convenient to write

$$\begin{aligned} \hat{\pi}(s) &\stackrel{\text{def}}{=} \pi(x(s)), \quad \hat{\widetilde{\pi}}(s) \stackrel{\text{def}}{=} \widetilde{\pi}(y(s)), \\ \hat{q}(s) &\stackrel{\text{def}}{=} q(x(s), y(s)), \quad \text{etc.} \end{aligned}$$

Then, for all $s \in h_x^{-1}(\mathcal{D}) \cap h_y^{-1}(\mathcal{D})$, we have the equation

$$\boxed{\hat{\pi}(s)\hat{q}(s) + \hat{\widetilde{\pi}}(s)\hat{\widetilde{q}}(s) + \hat{\pi}_0(s) = 0}, \tag{5}$$

where $\hat{q}$, $\hat{\widetilde{q}}$, $\hat{q}_0$ are meromorphic functions on $\mathbf{S}$, as resulting from the composition of meromorphic functions.

▶ Two constraints are of primary importance: the solution $\hat{\pi}$ [resp. $\hat{\widetilde{\pi}}$] of (5) has to be a function of the sole quantity $x(s)$ [resp. $y(s)$]. These properties of $\hat{\pi}$ and $\hat{\widetilde{\pi}}$ will lead to the central idea, based on the use of Galois automorphisms of $\mathbf{S}$.

Galois Automorphisms and the Group of the Random Walk

▶ Choose an arbitrary non singular-random walk. Hence, $Q(x,y)$ considered as a polynomial in y is irreducible over the field $\mathbb{C}(x)$ of rational functions of x.

▶ Let the vector space over $\mathbb{C}(x)$ generated by the constant 1 and one zero $y(x)$ of Q. This vector space is a field, which is the extension of order 2 of $\mathbb{C}(x)$ and will be denoted by $\mathbb{C}(x)[y(x)]$.
Each element of $\mathbb{C}(x)[y(x)]$ can be written in a unique way as $u(x)+v(x)y(x)$, where u and v are elements of $\mathbb{C}(x)$. It is quite natural to identify $\mathbb{C}(x)[y(x)]$ with the quotient field $\mathbb{C}(x)[T]/Q(x,T)$. Similarly, exchanging x and y, one can define $\mathbb{C}(y)[x(y)]$ and $\mathbb{C}(y)[T]/Q(T,y)$.

▶ Let $\mathbb{C}(x,y)$ be the field of rational functions in (x,y) over $\mathbb{C}$. Since Q is irreducible, the quotient ring of $\mathbb{C}(x,y)$ is a field denoted by $\mathbb{C}_Q(x,y)$.

Proposition 15. *The fields $\mathbb{C}(x)[T]/Q(x,T)$ and $\mathbb{C}(y)[T]/Q(T,y)$ are isomorphic to $\mathbb{C}_Q(x,y)$.*

▶ The Galois group of $\mathbb{C}(x)[y(x)]$ (resp. $\mathbb{C}(y)[x(y)]$) is cyclic of order 2 and its generic element will be denoted by ξ (resp. η). Here Q(x,y) is a polynomial of degree 2 w.r.t. each variable

$$\boxed{Q(x,y) = a(x)y^2 + b(x)y + c(x) = \widetilde{a}(y)x^2 + \widetilde{b}(y)x + \widetilde{c}(y)}. \qquad (6)$$

We have

$$\xi(u(x)) = u(x), \qquad \forall u \in \mathbb{C}(x), \qquad (7)$$

$$\xi(y(x)) = \frac{c(x)}{y(x)a(x)} = -\frac{b(x)}{a(x)} - y(x), \qquad (7')$$

$$\eta(w(y)) = w(y), \qquad \forall w \in \mathbb{C}(y), \qquad (8)$$

$$\eta(x(y)) = \frac{\widetilde{c}(y)}{x(y)\widetilde{a}(y)} = -\frac{\widetilde{b}(y)}{\widetilde{a}(y)} - x(y). \qquad (8')$$

▶ Note that ξ [resp. η] permutes the two roots in y [resp. x] of $Q(x,y)=0$.

▶ By Proposition 15, one can define two automorphisms $\widetilde{\xi}$ and $\widetilde{\eta}$ of $\mathbb{C}_Q(x,y)$ such that

$$\begin{cases} \widetilde{\xi}(f(x,y)) & = f(x,\widetilde{\xi}(y)) \mod Q, \quad \forall f \in \mathbb{C}(x,y), \\ \widetilde{\eta}(g(x,y)) & = g(\widetilde{\eta}(x),y) \mod Q, \quad \forall g \in \mathbb{C}(x,y), \end{cases} \tag{9}$$

$$\widetilde{\xi}(y) = \frac{c(x)}{ya(x)}, \qquad \widetilde{\eta}(x) = \frac{\widetilde{c}(y)}{x\widetilde{a}(y)}.$$

Definition 16. The group of the random walk $\mathcal{H}$ is the group $\mathcal{H}$ of automorphisms of $\mathbb{C}_Q(x,y)$ generated by $\widetilde{\xi}$ and $\widetilde{\eta}$.

The next lemma gives the structure of all groups with generators ξ and η, such that $\xi^2 = \eta^2 = I_d$.

Lemma 17. *Let*

$$\delta \stackrel{def}{=} \eta\xi. \tag{10}$$

Then $\mathcal{H}$ has a normal cyclic subgroup $\mathcal{H}_0 = \{\delta^n, n \in \mathbb{Z}\}$, which is finite or infinite, and $\mathcal{H}/\mathcal{H}_0$ is a cyclic group of order 2. ■

▶ Construction of the Automorphisms $\hat{\xi}$ and $\hat{\eta}$ on $\mathbf{S}$

We use Theorem 12, taking $X = \mathbb{P}_x$, $Y = \mathbf{S}$, $n = 2$. Then, by Definition 9 and Propositions 11, 14, $\mathcal{M}(\mathbb{P}_x) = \mathbb{C}(x)$ and $\mathcal{M}(\mathbf{S})$ is an extension of degree 2 of $\mathbb{C}(x)$, so that it is *Galois*. Moreover, Theorem 12 and Proposition 15 imply that the three objects

$$\mathcal{M}(\mathbf{S}),\ \mathbb{C}_Q(x,y),\ \mathbb{C}(x)[y(x)]$$

are isomorphic, and can consequently be identified. According to statements (3) and (4), $\mathbb{C}(x)$ can be embedded into $\mathcal{M}(\mathbf{S})$ by the correspondence

$$\hat{f} : \mathbf{S} \to \mathcal{M}(\mathbf{S}), \quad \text{with } \hat{f}(s) \stackrel{\text{def}}{=} f(x(s)), \quad \forall f \in \mathbb{C}(x).$$

Thus ξ, which is a generator of $G\big(\mathbb{C}(x)[y(x)]/\mathbb{C}(x)\big)$, can be viewed as acting on $\mathcal{M}(\mathbf{S})$, and the automorphism $\hat{\xi} \in$Aut$(\mathbf{S}/\mathbb{P}_x)$ will be defined by the relation

$$f \circ \hat{\xi} = \xi^{-1} \circ f, \quad \forall f \in \mathcal{M}(\mathbf{S}).$$

Similarly,

$$f \circ \hat{\eta} = \eta^{-1} \circ f, \quad \forall f \in \mathcal{M}(\mathbf{S}).$$

▶ Letting I_d be the identity operator, the following set of relations hold:

$$h_x \circ \hat{\xi} = h_x \qquad \hat{\xi}^2 = I_d, \tag{11}$$

$$h_y \circ \hat{\eta} = h_y \qquad \hat{\eta}^2 = I_d. \tag{12}$$

Indeed, the points s and $\hat{\xi}(s)$ [resp. $\hat{\eta}(s)$] have the same projections onto $\mathbb{P}_x$ [resp. $\mathbb{P}_y$]. In particular, the branch points of h_x [resp. h_y] are the fixed points of $\hat{\xi}$ [resp. $\hat{\eta}$], which read $h^{-1}(x_i)$ [resp. $h^{-1}(y_i)$], $i = 1, \ldots, 4$.

Definition 18. The *group* of the random walk is the group $\mathcal{H}$ of automorphisms of $\mathbb{C}_Q(x, y)$ generated by $\widetilde{\xi}$ and $\widetilde{\eta}$ (given in equation (9). This group is isomorphic to a subgroup of automorphisms of $\mathbf{S}$ generated by $\hat{\xi}$ and $\hat{\eta}$.

▶ **We shall identify the automorphisms ξ, η with their respective counterparts $\widetilde{\xi}, \widetilde{\eta}, \hat{\xi}, \hat{\eta}$.**

Reduction of the Main Equation to the Riemann Torus

- $\Gamma \stackrel{\text{def}}{=} \{z \in \mathbb{C} : |z| = 1\}, \quad \mathcal{D} \stackrel{\text{def}}{=} \{z \in \mathbb{C} : |z| < 1\}, \quad \overline{\mathcal{D}} \stackrel{\text{def}}{=} \{z \in \mathbb{C} : |z| \leqslant 1\}.$

- Mean jump vector (drift) $\overrightarrow{\mathbf{M}} = (M_x, M_y) = \left(\sum ip_{ij}, \sum jp_{ij}\right)$.

- From now on, the Riemann surface is supposed to have genus 1 (which excludes non-singular random walks). Starting from equations (4) and (5), we study the geometric properties of the domains $h_x^{-1}(\mathcal{D})$, $h_y^{-1}(\mathcal{D})$ and of their boundaries $h_x^{-1}(\Gamma)$, $h_y^{-1}(\Gamma)$. There are essentially three main situations.

1. $M_y < 0, \quad M_x < 0,$
2. $M_y < 0, \quad M_x > 0,$
3. $M_y < 0, \quad M_x = 0.$

▶ Since $\mathbf{S}$ has genus 1, $h_x^{-1}(\mathcal{D})$ is connected. To see this, it suffices to check that $h_x^{-1}(\mathcal{D})$ is arcwise connected.

Proof. There two branch points x_1 and x_2 inside $\mathcal{D}$. Choose two arbitrary points u, v in $h_x^{-1}(\mathcal{D})$ on $\mathbf{S}$ and draw two arcs $[h_x(u), x_1]$ and $[x_1, h_x(v)]$ in $\mathcal{D}$. Arcwise connectivity now follows from Proposition 5 and the fact that x_1 corresponds to the unique point s_1 on $\mathbf{S}$. ■

▶ Analogous properties hold for $h_y^{-1}(\mathcal{D})$ and $h_x^{-1}(\mathcal{D}) \cap h_y^{-1}(\mathcal{D}) \neq \emptyset$, since $Y(x)$ and $X(y)$ can simultaneously take on values inside $\mathcal{D}$.

▶ Let Γ_0 and Γ_1, denote the respective connected components of the boundary $h_x^{-1}(\Gamma)$ of $h_x^{-1}(\mathcal{D})$, such that

$$\begin{cases} \Gamma_0 \subset h_x^{-1}(\Gamma) \cap \{s : |y(s)| \leqslant 1\} = \{|x(s)| = 1\} \cap \{|y(s)| \leqslant 1\}, \\ \Gamma_1 \subset h_x^{-1}(\Gamma) \cap \{s : |y(s)| \geqslant 1\} = \{|x(s)| = 1\} \cap \{|y(s)| \geqslant 1\}. \end{cases}$$

Since by hypothesis $M_y \neq 0$, Γ_0 and Γ_1 are closed analytic curves without self-intersections. The *homology class* of Γ_0 and Γ_1 is one of the normal homology bases on the torus (see [5]), the same being true for $\widetilde{\Gamma_0}$ and $\widetilde{\Gamma_1}$. The curves $\widetilde{\Gamma_0}$ and $\widetilde{\Gamma_1}$ are defined in an analogous way. For example, $\widetilde{\Gamma_0} \subset h_y^{-1}(\Gamma)$ and, on it, $|y(s)| = 1$, $|x(s)| \leqslant 1$.

Define s_0 such that $x(s_0) = y(s_0) = 1$. Three following possibilities exist according to the orientation of $\overrightarrow{\mathbf{M}}$.

- $\boxed{M_y < 0, \quad M_x < 0.}$

Then $\Gamma_1 \cap \widetilde{\Gamma_1} = \emptyset$, and $\Gamma_0 \cap \widetilde{\Gamma_0} = h_x^{-1}(\Gamma) \cap h_y^{-1}(\Gamma)$ consists of the single point s_0 [See Fig. 2].

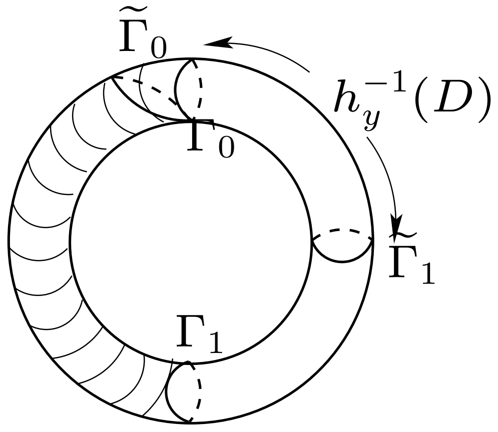

Figure 2: $M_y < 0, \quad M_x < 0.$

- $\boxed{M_y < 0, \quad M_x > 0.}$

Then $\Gamma_0 \cap \widetilde{\Gamma_1} = \{s_0\}$ [See Fig. 3].

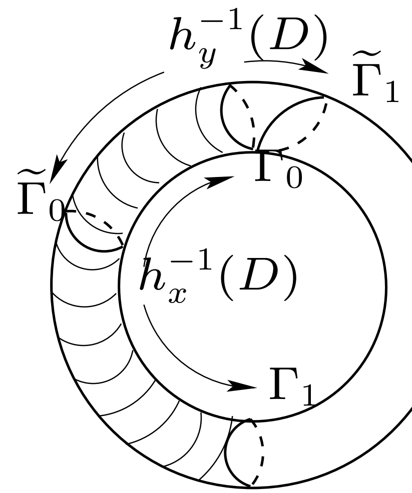

Figure 3: $M_y < 0, \quad M_x > 0.$

- $\boxed{M_y < 0, \quad M_x = 0.}$

Then Γ_0 and Γ_1 do not intersect and one can define $\widetilde{\Gamma_0} = h_x^{-1}(\overline{\mathcal{D}}) \cap h_y^{-1}(\Gamma)$ and $\widetilde{\Gamma_1} = h_y^{-1}(\Gamma) \cap (\mathbf{S} \backslash h_x^{-1}(\mathcal{D}))$, so that any pair of curves belonging to the triple $(\Gamma_0, \widetilde{\Gamma_0}, \widetilde{\Gamma_1})$ intersect at the point s_0 only [See Fig. 4]

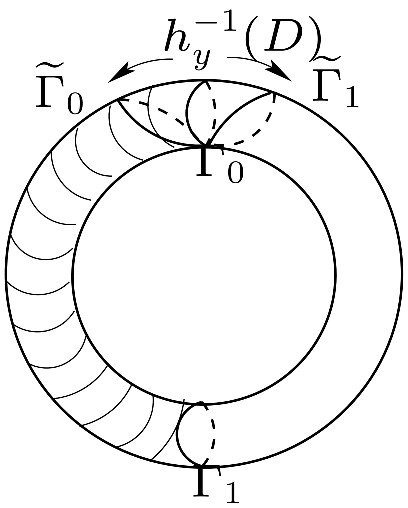

Figure 4: $M_y < 0, \quad M_x = 0.$

The proof in Case 3 is just the same, by letting in Case 1 $\varepsilon \to 0$ in $h_y^{-1}(\Gamma_\varepsilon)$, where $\Gamma_\varepsilon = \{x : |x| = 1 + \varepsilon\}$.

▶ $\hat{\pi}$ and $\hat{\tilde{\pi}}$ are analytic in $h_x^{-1}(\mathcal{D}) \cap h_y^{-1}(\mathcal{D})$. They admit a meromorphic continuation to the domains $h_y^{-1}(\mathcal{D})$ and $h_x^{-1}(\mathcal{D})$ respectively, provided that in these domains they are defined by

$$\hat{\pi} = -\frac{\hat{\tilde{q}}\,\hat{\tilde{\pi}} + \hat{q}_0\pi_{00}}{\hat{q}}, \quad \hat{\tilde{\pi}} = -\frac{\hat{q}\,\hat{\pi} + \hat{q}_0\pi_{00}}{\hat{\tilde{q}}}.$$

In $h_y^{-1}(\mathcal{D})$ [resp. $h_x^{-1}(\mathcal{D})$], the only possible poles of $\hat{\pi}$ [resp. $\hat{\tilde{\pi}}$] are the zeros of $\hat{q}$ [resp. $\hat{\tilde{q}}$], since $\hat{q}$ [resp. $\hat{\tilde{q}}$] is a meromorphic function on $\mathbf{S}$.

In other words, $\hat{\pi}$ and $\hat{\tilde{\pi}}$ are meromorphic in $h_x^{-1}(\mathcal{D}) \cup h_y^{-1}(\mathcal{D})$ and in this region equation (5) still holds. One can reformulate the problem set on the Riemann surface $\mathbf{S}$.

▶ **Problem** Find two meromorphic functions $\hat{\pi}$ and $\hat{\tilde{\pi}}$ in $h_x^{-1}(\mathcal{D})$ and $h_y^{-1}(\mathcal{D})$ respectively, such that

$$\hat{\pi}(s) = \hat{\pi}(\hat{\xi}(s)), \qquad \hat{\tilde{\pi}}(s) = \hat{\tilde{\pi}}(\hat{\eta}(s)),$$

where $(s, \hat{\xi}(s))$ and $(s, \hat{\eta}(s))$ belong to $h_x^{-1}(\mathcal{D}) \cup h_y^{-1}(\mathcal{D})$.

Lifting the Fundamental Equation onto the Universal Covering

Since $\mathbf{S}$ is of genus 1, its covering manifold is the complex plane $\mathbb{C}$ and its universal covering has the form $(\mathbb{C}\,,\lambda)$, where, by Proposition 10, λ is *Galois* and Aut$(\mathbb{C}\,/\mathbf{S})$ is isomorphic to the fundamental group of $\mathbf{S}$, and denoted by $\mathbf{T}$.

In addition, it is known that

$$\mathbf{T} = \mathbb{Z}\,\omega_{\mathbf{1}} \oplus \mathbb{Z}\,\omega_{\mathbf{2}}\,,$$

where ω_1 and ω_2 are two complex numbers, linearly independent on $\mathcal{R}$, acting as generators of $\mathbf{T}$. Thus, any $t \in \mathbf{T}$ can be written in the form

$$t = m_1\omega_1 + m_2\omega_2, \quad m_1, m_2 \in \mathbb{Z}\,,$$

and, $\forall \tau \in$ Aut$(\mathbb{C}/\mathbf{S})$, there exists a unique $t \in \mathbf{T}$, with

$$\tau(\omega) = \omega + t, \quad \omega \in \mathbb{C}.$$

Along lines quite similar to those leading to equation (4), any meromorphic function f defined on $\mathbf{S}$ can be lifted onto the universal covering $\mathbb{C}$ by the mapping

$$f^* = f \circ \lambda. \tag{13}$$

Clearly, any such f^* is meromorphic and satisfies the functional equation

$$f^*(\omega) = f^*(\omega + m_1\omega_1 + m_2\omega_2), \quad \forall m_1, m_2 \in \mathbb{Z}, \tag{14}$$

so that it is elliptic with periods ω_1, ω_2.

Thus the field $\mathbb{C}(\mathbf{S})$ of meromorphic functions on $\mathbf{S}$ is isomorphic, by (13), to the field of elliptic functions with the corresponding periods.

In addition, the period parallelogram

$$\Pi \stackrel{\text{def}}{=} \{\alpha_1\omega_1 + \alpha_2\omega_2 : 0 \leqslant \alpha_i < 1\},$$

provided that its opposite sides are identified, is isomorphic to $\mathbf{S}$. Then $\lambda[0, \omega_1]$ and $\lambda[0, \omega_2]$ are homologous to the homotopy classes on the torus.

Fix the notation so that $\lambda\{[0,\omega_1]\}$ is homologous to Γ_0 and, therefore, also to the curves $\Gamma_1, \widetilde{\Gamma}_0, \widetilde{\Gamma}_1$ and consider the domain

$$\Delta \stackrel{\text{def}}{=} h_x^{-1}(\mathcal{D}) \cup h_y^{-1}(\mathcal{D}) \subset \mathbf{S}.$$

- The inverse image $\lambda^{-1}(\Delta)$ belongs to $\mathbb{C}$ and consists of a denumerable number of curvilinear strips, which differ from each other by a translation of vector ω_2.
- Each strip is simply connected and invariant by translation of amplitude ω_1.

Let Δ^* the connected component of $\lambda^{-1}(\Delta)$ having a non empty intersection with Π. Then one can use the mapping λ to lift π and $\widetilde{\pi}$ (meromorphic in Δ) onto Δ^*, just setting

$$\pi^* = \hat{\pi} \circ \lambda, \quad \widetilde{\pi}^* = \hat{\widetilde{\pi}} \circ \lambda.$$

Hence $\pi(\omega)$ and $\widetilde{\pi}(\omega)$ are well defined and meromorphic in Δ^*, where, from their single valuedness along Γ_0 and $\widetilde{\Gamma}_0$ respectively, they satisfy the equations

$$\begin{cases} \pi(\omega+\omega_1) = \pi(\omega), \quad \widetilde{\pi}(\omega) = \widetilde{\pi}(\omega+\omega_1), \\ q(\omega)\pi(\omega) + \widetilde{q}(\omega)\widetilde{\pi}(\omega) + q_0(\omega)\pi_{00} = 0. \end{cases} \tag{15}$$

It suffices to analyze equation (15) in the domain $\Delta^* \cap \Pi$.

Lifting of the Branch Points

Starting from the branch points $x_i, y_i \in \mathbb{C}$ previously introduced, let $s_i, \widetilde{s}_i \in \mathbf{S}$ be such that

$$h_x(s_i) = x_i, \quad h_y(\widetilde{s}_i) = y_i, \quad \forall i = 1, \dots, 4.$$

Clearly, s_i and $\widetilde{s}_i$ are uniquely defined, so that there exist unique points

$$a_i \stackrel{\text{def}}{=} \lambda^{-1}(s_i) \cap \Pi, \quad b_i \stackrel{\text{def}}{=} \lambda^{-1}(\widetilde{s}_i) \cap \Pi, \quad \forall i = 1, \dots, 4.$$

- For $i = 1, 2$, the points a_i and b_i belong to $\Delta^* \cap \Pi$, since $s_i, \widetilde{s}_i \in \Delta$.
- For $i = 3, 4$, the points a_i and b_i belong to $\Pi \setminus \Delta^*$.

Lifting of the Automorphisms on the Universal Covering

Let $\hat{\alpha}$ be an arbitrary automorphism of $\mathbf{S}$. The mapping

$$f \stackrel{\text{def}}{=} \hat{\alpha}^{-1} \circ \lambda : \mathbb{C} \to \mathbf{S}$$

is holomorphic, so that, choosing in Definition 4

$$X = \mathbf{S}, \quad Y = Z = \mathbb{C}, \quad h = \lambda,$$

we get a lifting of f with respect to λ, denoted by α^*, such that $f = \lambda \circ \alpha^*$.

Since $(\mathbb{C}, \lambda)$ is a cover of $\mathbf{S}$ without branch point, it follows from Proposition 6 that α^* is holomorphic. Taking now $\hat{\alpha} \equiv \hat{\xi}$, where $\hat{\xi}$ was constructed above, we get the corresponding automorphism ξ^* on $\mathbb{C}$, which satisfies the relation

$$\lambda \circ \xi^* = \hat{\xi}^{-1} \circ \lambda. \tag{16}$$

Since

$$\lambda(a_i) = s_i, \quad \hat{\xi}(s_i) = s_i, \quad \forall i = 1, \ldots, 4,$$

it follows that

$$\lambda(a_i) = \lambda(\xi^*(a_i)).$$

Hence, there exist integers $k_{1,i}, k_{2,i} \in \mathbb{Z}$ with

$$\xi^*(a_i) = a_i + k_{1,i}\omega_1 + k_{2,i}\omega_2, \quad \forall i = 1, \ldots, 4.$$

Still using Proposition 6, we know that ξ^* is unique, provided that one of its values is fixed. In particular, it is always possible to use a translation to make a_1 a fixed point of ξ^*, so that

$$\xi^*(a_1) = a_1.$$

Equation (16) yields immediately $\hat{\xi}^{-2} \circ \lambda = \lambda \circ \xi^{*2}$, whence, since the range of $\xi^{*2} - I_d$ belongs to $\mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2$ and $\hat{\xi}^2 = I_d$,

$$\xi^{*2} - I_d = K,$$

where K is a constant. But, $K = 0$ since $\xi^*(a_1) = a_1$.

- Thus we have just proved that ξ^* is an automorphism satisfying $\xi^{*2} = I_d$.

Since any automorphism ζ^* of the complex plane has the form

$$\zeta^*(\omega) = \alpha\omega + \beta, \quad \forall \omega \in \mathbb{C},$$

one easily obtains

$$\xi^*(\omega) = -\omega + 2a_1. \tag{17}$$

Hence, $\forall i = 1, \dots, 4$,

$$\xi^*(a_i) = -a_i + 2a_1 = a_i + k_{1,i}\omega_1 + k_{2,i}\omega_2,$$

and, using (17),

$$a_1 - a_i = \frac{k_{1,i}\omega_1 + k_{2,i}\omega_2}{2},$$

where $k_{1,i}, k_{2,i} \in \{-1, 0, 1\}$, since all a_i's are located in Π. Remembering that $\lambda([0,\omega_1])$ was chosen to be homologous to Γ_0 on $\mathbf{S}$, one concludes that the open curve $\lambda([a_1, a_2])$ is homologous to *one half* of Γ_0. Thus $|a_1 - a_2| = \dfrac{\omega_1}{2}$ and we will take

$$a_2 - a_1 = \frac{\omega_1}{2}. \tag{18}$$

Mutatis mutandis, one can prove along the same lines that it is possible to lift η, to obtain an automorphism η^* of $\mathbb{C}$, such that

$$\eta^*(\omega) = -\omega + 2b_1, \quad b_2 - b_1 = \frac{\omega_1}{2}. \tag{19}$$

Setting

$$\omega_3 = 2(b_1 - a_1), \tag{20}$$

we can write, in accordance with equation (10),

$$\delta^* = (\eta\xi)^* = \eta^*\xi^*,$$

so that

$$\delta^*(\omega) = \omega + \omega_3. \tag{21}$$

Analytic continuation

Theorem 19. *π and $\widetilde{\pi}$ can be continued as meromorphic functions to the whole universal covering, where they satisfy system (15) together with the relations*

$$\pi(\omega) = \pi(\xi^*(\omega)), \quad \widetilde{\pi}(\omega) = \widetilde{\pi}(\eta^*(\omega)), \quad \forall \omega \in \mathbb{C}. \tag{22}$$

■

Proof. Δ^* is simply connected and $\Delta^* \cap \Pi$ is bounded by the curves $\lambda^{-1}(\Gamma_1) \cap \Pi$ and $\lambda^{-1}(\widetilde{\Gamma}_1) \cap \Pi$. The following automorphy relationships on $\mathbf{S}$ hold:

$$\hat{\xi}(\Gamma_1) = \Gamma_0, \quad \hat{\delta}(\Gamma_1) = \hat{\eta}(\Gamma_0) \subset \overline{\Delta}, \quad \hat{\xi}(\widetilde{\Gamma}_0) \subset \overline{\Delta}, \tag{23}$$

Lifting now the relations in (23) onto the universal covering, we have

$$\xi^*(\lambda^{-1}(\Gamma_1) \cap \Pi) = \lambda^{-1}(\Gamma_0) \cap \Pi, \quad \eta^*(\lambda^{-1}(\widetilde{\Gamma}_1) \cap \Pi) = \lambda^{-1}(\widetilde{\Gamma}_0) \cap \Pi,$$

$$\delta^*(\lambda^{-1}(\Gamma_1) \cap \Pi) \subset \overline{\Delta^*} \cap \Pi, \quad \delta^{*-1}(\lambda^{-1}(\widetilde{\Gamma}_1) \cap \Pi) \subset \overline{\Delta^*} \cap \Pi. \tag{24}$$

- Thus the domain

$$\Delta^* \bigcup \delta^*(\Delta^*)$$

is simply connected and it follows by induction that

$$\bigcup_{n=-\infty}^{\infty} \delta^{*n}(\Delta^*) \tag{25}$$

covers the whole complex plane $\mathbb{C}$. Moreover,

$$\begin{cases} \pi(\omega) = \pi(\xi^*(\omega)), & \forall (\omega, \xi^*(\omega)) \in \Delta^*, \\ \widetilde{\pi}(\omega) = \widetilde{\pi}(\eta^*(\omega)), & \forall (\omega, \eta^*(\omega)) \in \Delta^*, \end{cases}$$

and these equalities, by the principle of analytic continuation, indeed holds for all $\omega \in \mathbb{C}$, so that (22) is proved.

- Finally, we have the systerm

$$\begin{cases} q(\omega)\pi(\omega) + \widetilde{q}(\omega)\widetilde{\pi}(\omega) + q_0(\omega)\pi_{00} = 0, & \forall\omega, \in \Delta^*, \\ q(\xi^*(\omega))\pi(\omega) + \widetilde{q}(\xi^*(\omega))\widetilde{\pi}(\delta^*(\omega)) + q_0(\xi^*(\omega))\pi_{00} = 0, & \forall(\omega, \delta^*(\omega)) \in \Delta^*, \end{cases}$$

which yields

$$\begin{aligned} \widetilde{\pi}(\delta^*(\omega)) = \widetilde{\pi}(\omega + \omega_3) &= \frac{q(\xi^*(\omega))\widetilde{q}(\omega)}{\widetilde{q}(\xi^*(\omega))q(\omega)}\widetilde{\pi}(\omega) \\ &+ \frac{q(\xi^*(\omega))}{\widetilde{q}(\xi^*(\omega))}\left[-\frac{q_0(\xi^*(\omega))}{q(\xi(\omega))} + \frac{q_0(\omega)}{q(\omega)}\right]\pi_{00}. \end{aligned} \tag{26}$$

Equation (26) allows us to continue $\widetilde{\pi}(\omega)$, first to the domain $\delta^*(\Delta^*)$, and then, by induction, to the region introduced in (25). The proof of Theorem 19 is complete. ■

Theorem 20. *The functions* π *and* $\widetilde{\pi}$ *can be continued to the whole Riemann surface* $\mathbf{S}$.

Theorem 21. $\pi(x)$ *is a meromorphic function in the complex plane* $\mathbb{C}_x$ *cut along* $[x_3, x_4]$*. A similar statement holds for* $\widetilde{\pi}(y)$*, with the corresponding cut* $[y_3, y_4]$ *in* $\mathbb{C}_y$.

Corollary 22. $\pi(x)$ *and* $\widetilde{\pi}(y)$ *are holomorphic in the neighborhood of the unit circle, in their respective complex planes.*

Corollary 23. *The following conditions are equivalent:*

1. π *is not rational;*

2. $\widetilde{\pi}$ *is not rational;*

3. π *(resp.* $\widetilde{\pi}$*) is not meromorphic in* $\mathbb{C}_x$ *(resp.* $\mathbb{C}_y$*);*

4. x_3 *(resp.* y_3*) is a non-polar singularity of* π *(resp.* $\widetilde{\pi}$*).*

■

Uniformization

The polynomial Q(x,y) has been introduced in (6).

Set for a while

$$\begin{cases} D(x) = b^2(x) - 4a(x)c(x) = d_4x^4 + d_3x^3 + d_2x^2 + d_1x + d_0, \\ z = 2a(x)y + b(x), \end{cases}$$

It was shown in Chapter 2 that $D(x)$ has four real zeros x_i, $i = 1, \ldots, 4$. In addition x_2 and x_3 are always positive and we have

$$\begin{cases} -1 \leqslant x_1 < 1 \leqslant x_2 < x_3 < x_4, & \text{if} \quad d_4 > 0, \\ x_4 < -1 \leqslant x_1 < 1 \leqslant x_2 < x_3, & \text{if} \quad d_4 < 0, \\ x_4 = \infty, & \text{if} \quad d_4 = 0. \end{cases}$$

Lemma 24. *When* **S** *is of genus 1, the algebraic curve* $Q(x,y)=0$ *admits a uniformization given in terms of the Weierstrass function* $\wp(\omega;\omega_1,\omega_2)$, *and its derivative* $\wp'$ *by the following formulas:*

1. ▶ *If* $d_4 \neq 0$, *then* $D'(x_4) > 0$ *and*

$$\begin{cases} x(\omega) = x_4 + \dfrac{D'(x_4)}{\wp(\omega) - \frac{1}{6}D''(x_4)}, \\ z(\omega) = \dfrac{D'(x_4)\wp'(\omega)}{2\left(\wp(\omega) - \frac{1}{6}D''(x_4)\right)^2}; \end{cases} \tag{27}$$

2. ▶ *If* $d_4 = 0$, *then* $x_4 = \infty$ *and*

$$\begin{cases} x(\omega) = \dfrac{\wp(\omega) - \dfrac{d_2}{3}}{d_3}, \\ z(\omega) = -\dfrac{\wp'(\omega)}{2d_3}. \end{cases} \tag{28}$$

In the above formulas any x_i *might have been chosen, but* x_4 *is in some sense the best candidate, since it depends on* d_4 *in a very direct way.* ■

It is well known that $\wp(\omega)$ satisfies the differential equation

$$\wp'^2(\omega) = 4\wp^3(\omega) - g_2\wp(\omega) - g_3,$$

where g_2 and g_3 are the so-called invariants. We have

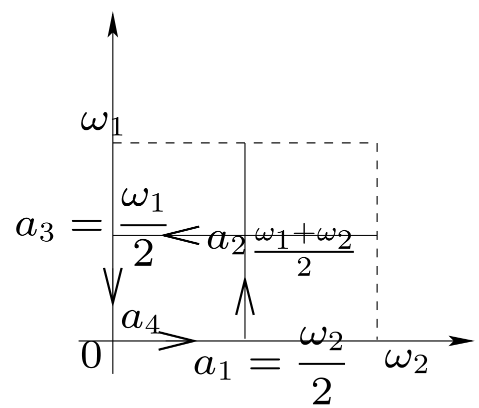

Figure 5: variation of the Weierstrass $\wp$-function.

$$x\left(\frac{\omega_2}{2}\right) = x_1, \quad x\left(\frac{\omega_1+\omega_2}{2}\right) = x_2, \quad x\left(\frac{\omega_1}{2}\right) = x_3, \quad x(0) = x_4. \tag{29}$$

- The following admissible choice of the periods was computed in [1]:

$$\omega_1 = 2i \int_{x_1}^{x_2} \frac{\mathrm{d}x}{\sqrt{-D(x)}}, \qquad \omega_2 = 2 \int_{x_2}^{x_3} \frac{\mathrm{d}x}{\sqrt{D(x)}}, \qquad \omega_3 = 2 \int_{X(y_1)}^{x_1} \frac{\mathrm{d}x}{\sqrt{D(x)}}.$$

- Note that ω_1 is purely imaginary, while $0 < \omega_3 < \omega_2$.

On the universal covering $\mathbb{C}$ (the finite complex plane), the automorphisms ξ, η, δ become

$$\xi^*(\omega) = -\omega + \omega_2, \qquad \eta^*(\omega) = -\omega + \omega_2 + \omega_3, \qquad \delta^*(\omega) = \eta^*\xi^* = \omega + \omega_3.$$

Here $\delta = \eta\xi$ corrresponds to $\delta^* = \eta^*\xi^*$, and, for any $f(x,y) \in C_Q(x,y)$,

$$\delta(f(x,y)) = f(\delta(x), \delta(y)) = f(x(\delta^*(\omega)), y(\delta^*(\omega))), \; \omega \in \mathbb{C}.$$

In particular,

$$\begin{cases} \delta(x) = x(\delta^*(\omega)) = x(\omega + \omega_3), \\ \eta(x) = x(\eta^*(\omega)) = x(-\omega + \omega_2 + \omega_3) = x(\omega - \omega_3). \end{cases} \tag{30}$$

- It follows immediately that the group $\mathcal{H}$ is finite of order $2n$ if and only if

$$n\omega_3 = 0 \quad \mod (\omega_1, \omega_2),$$

or, since ω_3 is real,

$$n\omega_3 = 0 \quad \mod (\omega_2), \tag{31}$$

where n stands for the minimal positive integer with this property.

In a future (!?) presentation, we shall see that there are explicit criterions for the finiteness of $\mathcal{H}$.

Thank you for your patience in listening..!

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