

Diagonals and Creative Telescoping for Lattice Path Combinatorics

Alin Bostan



IHP, May 11, 2018

DIAGONALS

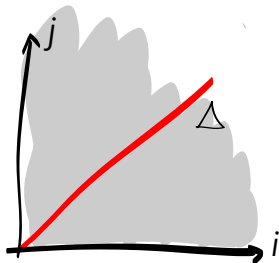
Definition

If F is a formal power series

$$F = \sum_{i_1, \dots, i_n \geq 0} a_{i_1, \dots, i_n} x_1^{i_1} \cdots x_n^{i_n},$$

its *diagonal* is

$$\text{Diag}(F) \stackrel{\text{def}}{=} \sum_i a_{i, \dots, i} t^i.$$



Theorem (Pólya, 1922)

Diagonals of *bivariate* rational functions are *algebraic*.

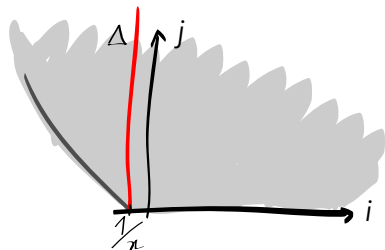
Definition

If F is a formal power series

$$F = \sum_{i_1, \dots, i_n \geq 0} a_{i_1, \dots, i_n} x_1^{i_1} \cdots x_n^{i_n},$$

its *diagonal* is

$$\text{Diag}(F) \stackrel{\text{def}}{=} \sum_i a_{i, \dots, i} t^i.$$



Theorem (Pólya, 1922)

Diagonals of *bivariate* rational functions are *algebraic*.

Proof:

$$\text{Diag}(F) = [x^{-1}] \frac{1}{x} F\left(x, \frac{t}{x}\right) = \frac{1}{2\pi i} \oint_{|x|=\epsilon} F\left(x, \frac{t}{x}\right) \frac{dx}{x}$$

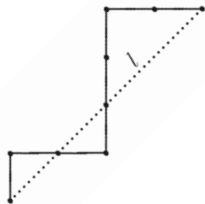
and evaluating the integral by residues concludes (residues are algebraic)

Example: Dyck walks

$$\mathfrak{S} = \{(1, 1), (1, -1)\}$$

Let B_n be the number of **Dyck bridges** (i.e. \mathfrak{S} -walks in \mathbb{Z}^2 starting at $(0, 0)$ and ending on the horizontal axis), of length n

Rotating a Dyck bridge



counterclockwise by $\pi/4$

$B_n =$ number of $\{(1, 0), (0, 1)\}$ -walks in \mathbb{Z}^2 from $(0, 0)$ to (n, n)

$$\implies B(t) = \sum_{n \geq 0} B_n t^n = \text{Diag} \left(\frac{1}{1 - x - y} \right)$$

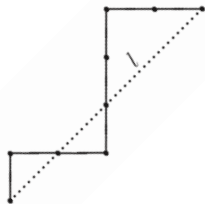
$$B(t) = \frac{1}{2\pi i} \oint_{|x|=\epsilon} \frac{dx}{x - x^2 - t} = \frac{1}{1 - 2x} \Big|_{x=\frac{1-\sqrt{1-4t}}{2}} = \frac{1}{\sqrt{1-4t}}$$

Example: Dyck walks

$$\mathfrak{S} = \{(1,1), (1,-1)\}$$

Let B_n be the number of **Dyck bridges** (i.e. \mathfrak{S} -walks in \mathbb{Z}^2 starting at $(0,0)$ and ending on the horizontal axis), of length n

Rotating a Dyck bridge



counterclockwise by $\pi/4$

$B_n =$ number of $\{(1,0), (0,1)\}$ -walks in \mathbb{Z}^2 from $(0,0)$ to $(n,n) = \binom{2n}{n}$

$$\implies B(t) = \sum_{n \geq 0} B_n t^n = \text{Diag} \left(\frac{1}{1-x-y} \right)$$

$$B(t) = \frac{1}{2\pi i} \oint_{|x|=\epsilon} \frac{dx}{x-x^2-t} = \frac{1}{\sqrt{1-4t}} = \sum_{n \geq 0} \binom{2n}{n} t^n$$

Let $A, B \in \mathbb{K}[x]$ be such that $\deg(A) < \deg(B)$, with B squarefree.
The rational function $F = A/B$ has simple poles only.

If $F = \sum_i \frac{\gamma_i}{x - \beta_i}$, then the **residue γ_i of F at the pole β_i** equals $\gamma_i = \frac{A(\beta_i)}{B'(\beta_i)}$.

Theorem. The residues γ_i of F are roots of the resultant

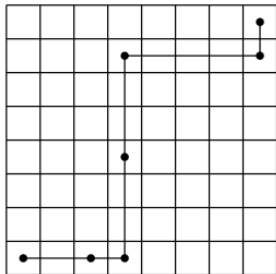
$$R(t) = \text{Res}_x(B(x), A(x) - t \cdot B'(x)).$$

Proof. **Poisson's formula:** $R(t) = \prod_i \left(A(\beta_i) - t \cdot B'(\beta_i) \right)$.

- ▶ Introduced in computer algebra by [[Rothstein-Trager 1976](#)] for the symbolic (indefinite) integration of rational functions.
- ▶ Generalized by [[Bronstein 1992](#)] to multiple poles.

Example: diagonal Rook paths

Question: A chess Rook can move any number of squares horizontally or vertically in one step. How many paths can a Rook take from the lower-left corner square to the upper-right corner square of an $N \times N$ chessboard? Assume that the Rook moves right or up at each step.



1, 2, 14, 106, 838, 6802, 56190, 470010, ...

Example: diagonal Rook paths

Generating function of the sequence

1, 2, 14, 106, 838, 6802, 56190, 470010, ...

is

$$\text{Diag}(F) = [x^0] F(x, t/x) = \frac{1}{2\pi i} \oint F(x, t/x) \frac{dx}{x}, \quad \text{where } F = \frac{1}{1 - \frac{x}{1-x} - \frac{y}{1-y}}.$$

By the [residue theorem](#), $\text{Diag}(F)$ is a sum of roots $y(t)$ of the Rothstein-Trager resultant

- > F:=1/(1-x/(1-x)-y/(1-y)):
- > G:=normal(1/x*subs(y=t/x,F)):
- > factor(resultant(denom(G),numer(G)-y*diff(denom(G),x),x));

$$t^2(1-t)(2y-1)(36ty^2-4y^2+1-t)$$

Answer: Generating series of diagonal Rook paths is $\frac{1}{2} \left(1 + \sqrt{\frac{1-t}{1-9t}} \right)$.

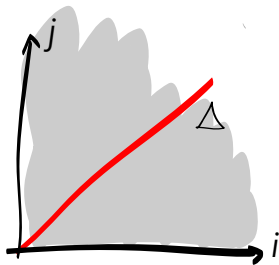
Definition

If F is a formal power series

$$F = \sum_{i_1, \dots, i_n \geq 0} a_{i_1, \dots, i_n} x_1^{i_1} \cdots x_n^{i_n},$$

its *diagonal* is

$$\text{Diag}(F) \stackrel{\text{def}}{=} \sum_i a_{i, \dots, i} t^i.$$



Theorem (Pólya, 1922)

Diagonals of *bivariate* rational functions are *algebraic*.^a

^aThe converse is also true [Furstenberg, 1967]

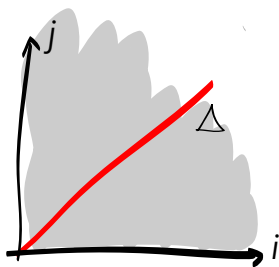
Definition

If F is a formal power series

$$F = \sum_{i_1, \dots, i_n \geq 0} a_{i_1, \dots, i_n} x_1^{i_1} \cdots x_n^{i_n},$$

its *diagonal* is

$$\text{Diag}(F) \stackrel{\text{def}}{=} \sum_i a_{i, \dots, i} t^i.$$



Theorem (Pólya, 1922)

Diagonals of *bivariate* rational functions are *algebraic*.

▷ This is false for more than 2 variables. E.g.

$$\text{Diag} \left(\frac{1}{1-x-y-z} \right) = \sum_{n \geq 0} \frac{(3n)!}{n!^3} t^n = {}_2F_1 \left(\frac{1}{3}, \frac{2}{3} \middle| 27t \right) \text{ is transcendental}$$

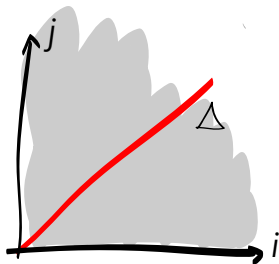
Definition

If F is a formal power series

$$F = \sum_{i_1, \dots, i_n \geq 0} a_{i_1, \dots, i_n} x_1^{i_1} \cdots x_n^{i_n},$$

its *diagonal* is

$$\text{Diag}(F) \stackrel{\text{def}}{=} \sum_i a_{i, \dots, i} t^i.$$



Theorem (Pólya, 1922)

Diagonals of *bivariate* rational functions are *algebraic* and thus *D-finite*.

- ▷ Algebraic equation has **exponential size** [B., Dumont, Salvy, 2015]
- ▷ Differential equation has **polynomial size** [B., Chen, Chyzak, Li, 2010]

Lipshitz's theorem

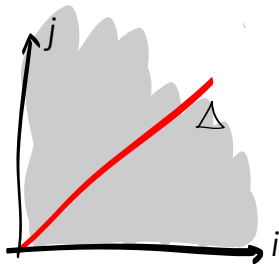
Definition

If F is a formal power series

$$F = \sum_{i_1, \dots, i_n \geq 0} a_{i_1, \dots, i_n} x_1^{i_1} \cdots x_n^{i_n},$$

its *diagonal* is

$$\text{Diag}(F) \stackrel{\text{def}}{=} \sum_i a_{i, \dots, i} t^i.$$



Theorem (Lipshitz, 1988)

Diagonals of rational functions are D-finite.

Example: Diagonal Rook paths on a 3D chessboard

Question [Erickson 2010]

How many ways can a Rook move from $(0,0,0)$ to (N,N,N) , where each step is a positive integer multiple of $(1,0,0)$, $(0,1,0)$, or $(0,0,1)$?

1, 6, 222, 9918, 486924, 25267236, 1359631776, 75059524392, ...

Answer [B.-Chyzak-Hoeij-Pech 2011]: GF of 3D diagonal Rook paths is

$$G(t) = 1 + 6 \cdot \int_0^t \frac{{}_2F_1\left(\begin{matrix} 1/3 & 2/3 \\ 2 \end{matrix} \middle| \frac{27x(2-3x)}{(1-4x)^3}\right)}{(1-4x)(1-64x)} dx$$

Proof of Lipshitz's theorem on the 3D Rooks example

Problem: Show that $\text{Diag}(F)$ is D-finite, where $F(x, y, z)$ is

$$\left(1 - \sum_{n \geq 1} x^n - \sum_{n \geq 1} y^n - \sum_{n \geq 1} z^n\right)^{-1} = \frac{(1-x)(1-y)(1-z)}{1-2(x+y+z)+3(xy+yz+zx)-4xyz}$$

Proof of Lipshitz's theorem on the 3D Rooks example

Problem: Show that $\text{Diag}(F)$ is D-finite, where $F(x, y, z)$ is

$$\left(1 - \sum_{n \geq 1} x^n - \sum_{n \geq 1} y^n - \sum_{n \geq 1} z^n\right)^{-1} = \frac{(1-x)(1-y)(1-z)}{1-2(x+y+z)+3(xy+yz+zx)-4xyz}$$

Idea: If one is able to find a nonzero differential operator of the form

$$L(t, \partial_t, \partial_x, \partial_y) = P(t, \partial_t) + (\text{higher-order terms in } \partial_x \text{ and } \partial_y)$$

that annihilates $G = \frac{1}{xy} \cdot F\left(x, \frac{y}{x}, \frac{t}{y}\right)$, then $P(t, \partial_t)$ annihilates $\text{Diag}(F)$.

Proof of Lipshitz's theorem on the 3D Rooks example

Problem: Show that $\text{Diag}(F)$ is D-finite, where $F(x, y, z)$ is

$$\left(1 - \sum_{n \geq 1} x^n - \sum_{n \geq 1} y^n - \sum_{n \geq 1} z^n\right)^{-1} = \frac{(1-x)(1-y)(1-z)}{1-2(x+y+z)+3(xy+yz+zx)-4xyz}$$

Idea: If one is able to find a nonzero differential operator of the form

$$L(t, \partial_t, \partial_x, \partial_y) = P(t, \partial_t) + (\text{higher-order terms in } \partial_x \text{ and } \partial_y)$$

that annihilates $G = \frac{1}{xy} \cdot F\left(x, \frac{y}{x}, \frac{t}{y}\right)$, then $P(t, \partial_t)$ annihilates $\text{Diag}(F)$.

Proof:

- ① $\text{Diag}(F) = [x^0 y^0] F\left(x, \frac{y}{x}, \frac{t}{y}\right)$
- ② $0 = L(G) = P(G) + \partial_x(\cdot) + \partial_y(\cdot)$
- ③ $0 = [x^{-1} y^{-1}] L(G) = [x^{-1} y^{-1}] P(G) = P([x^{-1} y^{-1}] G) = P(\text{Diag}(F))$

Proof of Lipshitz's theorem on the 3D Rooks example

Problem: Show that $\text{Diag}(F)$ is D-finite, where $F(x, y, z)$ is

$$\left(1 - \sum_{n \geq 1} x^n - \sum_{n \geq 1} y^n - \sum_{n \geq 1} z^n\right)^{-1} = \frac{(1-x)(1-y)(1-z)}{1-2(x+y+z)+3(xy+yz+zx)-4xyz}$$

Idea: If one is able to find a nonzero differential operator of the form

$$L(t, \partial_t, \partial_x, \partial_y) = P(t, \partial_t) + (\text{higher-order terms in } \partial_x \text{ and } \partial_y)$$

that annihilates $G = \frac{1}{xy} \cdot F\left(x, \frac{y}{x}, \frac{t}{y}\right)$, then $P(t, \partial_t)$ annihilates $\text{Diag}(F)$.

▷ **Remaining task:** Show that such an L does exist.

Counting argument: By Leibniz's rule, the $\binom{N+4}{4}$ rational functions

$$t^i \partial_t^j \partial_x^k \partial_y^\ell (G), \quad 0 \leq i + j + k + \ell \leq N$$

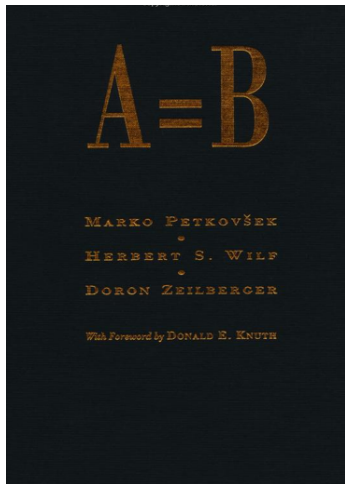
are contained in the \mathbb{Q} -vector space of dimension $\leq 18(N+1)^3$ spanned by

$$\frac{t^i x^j y^k}{\text{denom}(G)^{N+1}}, \quad 0 \leq i \leq 2N+1, \quad 0 \leq j \leq 3N+2, \quad 0 \leq k \leq 3N+2.$$

- ▷ If $\binom{N+4}{4} > 18(N+1)^3$, then there exists $L(t, \partial_t, \partial_x, \partial_y)$ (resp. $P(t, \partial_t)$) of total degree at most N , such that $LG = 0$ (resp. $P(\text{Diag}(F)) = 0$).
- ▷ $N = 425$ is the smallest integer satisfying $\binom{N+4}{4} > 18(N+1)^3$
- ▷ Finding the operator P by Lipshitz' argument would require solving a linear system with 1,391,641,251 unknowns and 1,391,557,968 equations!
- ▷ A better solution is provided by **creative telescoping**.

CREATIVE TELESCOPING

General framework in computer algebra –initiated by Zeilberger in the '90s–
for proving identities on multiple integrals and sums with parameters.



Examples I: hypergeometric summation

- $\sum_{k \in \mathbb{Z}} (-1)^k \binom{a+b}{a+k} \binom{a+c}{c+k} \binom{b+c}{b+k} = \frac{(a+b+c)!}{a!b!c!}$ [Dixon 1891]

- $A_n = \sum_{k=0}^n \binom{n}{k}^2 \binom{n+k}{k}^2$ satisfies the recurrence [Apéry 1978]:

$$(n+1)^3 A_{n+1} = (34n^3 + 51n^2 + 27n + 5)A_n - n^3 A_{n-1}.$$

(Neither Cohen nor I had been able to prove this in the intervening two months [Van der Poorten 1979])

- $\sum_{k=0}^n \binom{n}{k}^2 \binom{n+k}{k}^2 = \sum_{k=0}^n \binom{n}{k} \binom{n+k}{k} \sum_{j=0}^k \binom{k}{j}^3$ [Strehl 1992]

Examples I: hypergeometric summation

- $\sum_{k \in \mathbb{Z}} (-1)^k \binom{a+b}{a+k} \binom{a+c}{c+k} \binom{b+c}{b+k} = \frac{(a+b+c)!}{a!b!c!}$ [Dixon 1891]

- $A_n = \sum_{k=0}^n \binom{n}{k}^2 \binom{n+k}{k}^2$ satisfies the recurrence [Apéry 1978]:

$$(n+1)^3 A_{n+1} = (34n^3 + 51n^2 + 27n + 5)A_n - n^3 A_{n-1}.$$

(The specific problem was mentioned to Don Zagier, who solved it with irritating speed [Van der Poorten 1979])

- $\sum_{k=0}^n \binom{n}{k}^2 \binom{n+k}{k}^2 = \sum_{k=0}^n \binom{n}{k} \binom{n+k}{k} \sum_{j=0}^k \binom{k}{j}^3$ [Strehl 1992]

Examples II: Integrals and Diagonals

$$\bullet \int_0^1 \frac{\cos(zu)}{\sqrt{1-u^2}} du = \int_1^{+\infty} \frac{\sin(zu)}{\sqrt{u^2-1}} du = \frac{\pi}{2} J_0(z);$$

$$\bullet \int_0^{+\infty} x J_1(ax) I_1(ax) Y_0(x) K_0(x) dx = -\frac{\ln(1-a^4)}{2\pi a^2}$$

[Glasser-Montaldi 1994];

$$\bullet \frac{1}{2\pi i} \oint \frac{(1+2xy+4y^2) \exp\left(\frac{4x^2y^2}{1+4y^2}\right)}{y^{n+1}(1+4y^2)^{\frac{3}{2}}} dy = \frac{H_n(x)}{[n/2]!}$$

[Doetsch 1930];

$$\bullet \text{Diag} \frac{1}{(1-x-y)(1-z-u)-xyzu} = \sum_{n \geq 0} A_n t^n$$

[Straub 2014].

$$I_n := \sum_{k=0}^n \binom{n}{k} = 2^n.$$

IF one knows Pascal's triangle:

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1} = 2\binom{n}{k} + \binom{n}{k-1} - \binom{n}{k},$$

then summing over k gives

$$I_{n+1} = 2I_n.$$

The initial condition $I_0 = 1$ concludes the proof.

$$F_n = \sum_k u_{n,k} = ?$$

IF one knows $P(n, S_n)$ and $R(n, k, S_n, S_k)$ such that

$$(P(n, S_n) + \Delta_k R(n, k, S_n, S_k)) \cdot u_{n,k} = 0$$

(where Δ_k is the difference operator, $\Delta_k \cdot v_{n,k} = v_{n,k+1} - v_{n,k}$),
then the sum “telescopes”, leading to

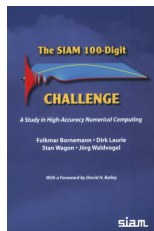
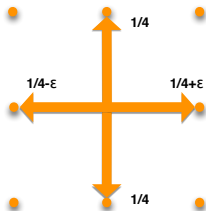
$$P(n, S_n) \cdot F_n = 0.$$

Input: a **hypergeometric** term $u_{n,k}$, i.e., $u_{n+1,k}/u_{n,k}$ and $u_{n,k+1}/u_{n,k}$ rational functions in n and k ;

Output:

- a linear recurrence (P) satisfied by $F_n = \sum_k u_{n,k}$
- a **certificate** (Q), such that checking the result is easy from $P(n, S_n) \cdot u_{n,k} = \Delta_k Q \cdot u_{n,k}$.

Example: SIAM flea



$$U_{n,k} := \binom{2n}{2k} \binom{2k}{k} \binom{2n-2k}{n-k} \left(\frac{1}{4} + c\right)^k \left(\frac{1}{4} - c\right)^k \frac{1}{4^{2n-2k}}$$

$$p_n = \sum_{k=0}^n U_{n,k} = \text{probability of return to } (0,0) \text{ at step } 2n.$$

> p:=SumTools[Hypergeometric][Zeilberger](U,n,k,Sn);

$$\begin{aligned} & [(4n^2 + 16n + 16)Sn^2 + (-4n^2 + 32c^2n^2 + 96c^2n - 12n + 72c^2 - 9)Sn \\ & \quad + 128c^4n + 64c^4n^2 + 48c^4, \dots(\text{BIG certificate})\dots] \end{aligned}$$

$$I(t) = \oint_{\gamma} H(t, x) dx = ?$$

IF one knows $P(t, \partial_t)$ and $R(t, x, \partial_t, \partial_x)$ such that

$$(P(t, \partial_t) + \partial_x R(t, x, \partial_t, \partial_x)) \cdot H(t, x) = 0,$$

then the integral “telescopes”, leading to

$$P(t, \partial_t) \cdot I(t) = 0.$$

Example: diagonal Rook paths again

Generating function of the sequence

1, 2, 14, 106, 838, 6802, 56190, 470010, ...

is

$$\text{Diag}(F) = [x^0] F(x, t/x) = \frac{1}{2\pi i} \oint F(x, t/x) \frac{dx}{x}, \quad \text{where } F = \frac{1}{1 - \frac{x}{1-x} - \frac{y}{1-y}}.$$

By **creative telescoping**, $\text{Diag}(F)$ satisfies the differential equation

- > F:=1/(1-x/(1-x)-y/(1-y)):
- > G:=normal(1/x*subs(y=t/x,F)):
- > Zeilberger(G, t, x, Dt)[1];

$$(9t^2 - 10t + 1)\partial_t^2 + (18t - 14)\partial_t$$

Answer: Generating series of diagonal Rook paths is $\frac{1}{2} \left(1 + \sqrt{\frac{1-t}{1-9t}} \right)$.

CT for Multiple rational integrals

Problem:

$\mathbf{x} = x_1, \dots, x_n$ — integration variables

t — parameter

$H(t, \mathbf{x})$ — rational function

γ — n -cycle in \mathbb{C}^n

$$\left. \begin{array}{l} \mathbf{x} = x_1, \dots, x_n \\ t \\ H(t, \mathbf{x}) \\ \gamma \end{array} \right\} \oint_{\gamma} H(t, \mathbf{x}) d\mathbf{x}$$

Principle of creative telescoping

$$\underbrace{\sum_{k=0}^r c_k(t) \frac{\partial^k H}{\partial t^k}}_{\text{telescopic relation}} = \underbrace{\sum_{i=1}^n \frac{\partial A_i}{\partial x_i}}_{\text{certificate}} \implies \underbrace{\left(\sum_{k=0}^r c_k(t) \frac{\partial^k}{\partial t^k} \right)}_{\text{telescoper}} \cdot \oint_{\gamma} H d\mathbf{x} = 0$$

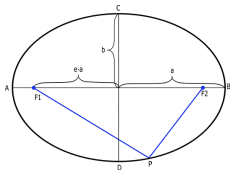
Task:

- ① find the $c_k(t)$ which satisfy a telescopic relation,
- ② ideally, without computing the certificate (A_i).

Example: Perimeter of an ellipse

Perimeter of an ellipse of eccentricity e , semi-major axis 1 [Euler, 1733]

$$p(e) = 4 \int_0^1 \sqrt{\frac{1 - e^2 x^2}{1 - x^2}} dx = 2\pi - \frac{\pi}{2} e^2 - \frac{3\pi}{32} e^4 - \dots$$



Creative telescoping finds the telescopic relation

$$\begin{aligned} & \left((e - e^3) \partial_e^2 + (1 - e^2) \partial_e + e \right) \cdot \left(\frac{1}{1 - \frac{1 - e^2 x^2}{(1 - x^2) y^2}} \right) = \\ & \partial_x \left(- \frac{e(-1 - x + x^2 + x^3) y^2 (-3 + 2x + y^2 + x^2 (-2 + 3e^2 - y^2))}{(-1 + y^2 + x^2 (e^2 - y^2))^2} \right) \\ & \quad + \partial_y \left(\frac{2e(-1 + e^2) x (1 + x^3) y^3}{(-1 + y^2 + x^2 (e^2 - y^2))^2} \right) \end{aligned}$$

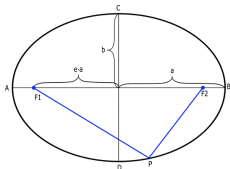
Thus $(e - e^3)p'' + (1 - e^2)p' + ep = 0$.

▷ Note the size of the certificate (compared to that of the telescoper).

Example: Perimeter of an ellipse

Perimeter of an ellipse of eccentricity e , semi-major axis 1 [Euler, 1733]

$$p(e) = 4 \int_0^1 \sqrt{\frac{1 - e^2 x^2}{1 - x^2}} dx = 4 \oint \frac{dx dy}{1 - \frac{1 - e^2 x^2}{(1 - x^2) y^2}},$$



Creative telescoping finds the telescopic relation

$$\begin{aligned} & \left((e - e^3) \partial_e^2 + (1 - e^2) \partial_e + e \right) \cdot \left(\frac{1}{1 - \frac{1 - e^2 x^2}{(1 - x^2) y^2}} \right) = \\ & \partial_x \left(- \frac{e(-1 - x + x^2 + x^3) y^2 (-3 + 2x + y^2 + x^2 (-2 + 3e^2 - y^2))}{(-1 + y^2 + x^2 (e^2 - y^2))^2} \right) \\ & \quad + \partial_y \left(\frac{2e(-1 + e^2) x (1 + x^3) y^3}{(-1 + y^2 + x^2 (e^2 - y^2))^2} \right) \end{aligned}$$

Thus $(e - e^3)p'' + (1 - e^2)p' + ep = 0$.

▷ Note the size of the certificate (compared to that of the telescoper).

Task: Given G in $\mathbb{Q}(t, x, y)$, construct a linear differential operator $P(t, \partial_t)$, and two rational functions R and S in $\mathbb{Q}(t, x, y)$ such that

$$P(G) = \frac{\partial R}{\partial x} + \frac{\partial S}{\partial y}.$$

Solution: Creative telescoping!

```
> G:=subs(y=y/x,z=t/y,1/(1-x/(1-x)-y/(1-y)-z/(1-z)))/y/x:
> P,R,S:=op(op(Mgfun:-creative_telescoping(G,t::diff,[x::diff,y::diff]))):
> P;
```

$$P = t(t-1)(64t-1)(3t-2)(6t+1)\partial_t^3 + (4608t^4 - 6372t^3 + 813t^2 + 514t - 4)\partial_t^2 + 4(576t^3 - 801t^2 - 108t + 74)\partial_t$$

- ▷ The whole computation takes < 10 seconds on a personal laptop.
- ▷ Proves a recurrence conjectured by [Erickson 2010]

General-purpose creative telescoping algorithms:

- using linear algebra [Lipshitz, 1988];
 - using non-commutative Gröbner bases:
 - and elimination [Takayama, 1990; Zeilberger, 1990];
 - and rational resolution of differential equations [Chyzak, 2000];
 - and very efficient heuristics [Koutschan, 2009, 2010].
- ▷ Drawbacks: Bad or unknown complexity; all compute certificates.

Rational case:

- univariate integrals [B., Chen, Chyzak, Li, 2010];
- double integrals [Chen, Kauers, Singer, 2012];
- multiple integrals [B., Lairez, Salvy, 2013], [Lairez 2015].

Problem: Given $H = P/Q \in \mathbb{K}(t, x)$ compute $\oint_{\gamma} H(t, x) dx$

Hermite reduction: H can be written in **reduced form**

$$H = \partial_x(g) + \frac{a}{Q^*},$$

where Q^* is the squarefree part of Q and $\deg_x(a) < d^* := \deg_x(Q^*)$.

Algorithm [B., Chen, Chyzak, Li, 2010]

(1) For $i = 0, 1, \dots, d^*$ compute Hermite reduction of $\partial_t^i(H)$:

$$\partial_t^i(H) = \partial_x(g_i) + \frac{a_i}{Q^*}, \quad \deg_x(a_i) < d^*$$

(2) Find the first linear relation over $\mathbb{K}(t)$ of the form $\sum_{k=0}^r c_k a_k = 0$.

▷ $L = \sum_{k=0}^r c_k \partial_t^k$ is a **telescoper** (and $\sum_{k=0}^r c_k g_k$ the corresponding certificate).

$H = \frac{P}{Q}$ — a rational function in t and $\mathbf{x} = x_1, \dots, x_n$

$d_{\mathbf{x}}$ — the degree of Q w.r.t. \mathbf{x}

d_t — $\max(\deg_t P, \deg_t Q)$

Theorem ([B., Lairez, Salvy, 2013])

A telescoper for H can be computed using $\tilde{O}(e^{3n} d_{\mathbf{x}}^{8n} d_t)$ operations.

The minimal telescoper has order $\leq d_{\mathbf{x}}^n$ and degree $O(e^n d_{\mathbf{x}}^{3n} d_t)$.

These size bounds are *generically reached*.

- ▷ First polynomial time algorithm for rational creative telescoping.
- ▷ It avoids the costly computation of certificates.
- ▷ Generically, certificates have size $\Omega(d_{\mathbf{x}}^{n^2/2})$.
- ▷ General-purpose algorithms have doubly-exponential complexity.
- ▷ Applies to diagonals: $\text{Diag}(F)(t) = \frac{1}{(2\pi i)^n} \oint F\left(\frac{t}{x_1 \dots x_n}, x_1, \dots, x_n\right)$.

Griffiths–Dwork method for the generic case

Linear reduction classical in algebraic geometry;
Generalization of Hermite's reduction.

Fast linear algebra on polynomial matrices

Macaulay matrices encoding Gröbner bases computations;
Sophisticated algorithms due to Villard, Storjohann, Zhou, etc.

Deformation technique for the general case
























Input perturbation using a new free variable.

▷ Recent, highly non-trivial, extension by [\[Lairez, 2015\]](#) tremendously improves the efficiency of the algorithm in [\[B., Lairez, Salvy, 2013\]](#)

WALKS IN THE QUARTER PLANE


























The 19 D-finite cases with nonzero orbit sum

Task: Prove Cases 1–19 in the tables [B. & Kauers 2009] for $F(t; 1, 1)$

	OEIS	\mathfrak{S}	Pol size	ODE size		OEIS	\mathfrak{S}	Pol size	ODE size
1	A005566		—	3, 4	13	A151275		—	5, 24
2	A018224		—	3, 5	14	A151314		—	5, 24
3	A151312		—	3, 8	15	A151255		—	4, 16
4	A151331		—	3, 6	16	A151287		—	5, 19
5	A151266		—	5, 16	17	A001006		2, 2	2, 3
6	A151307		—	5, 20	18	A129400		2, 2	2, 3
7	A151291		—	5, 15	19	A005558		—	3, 5
8	A151326		—	5, 18					
9	A151302		—	5, 24	20	A151265		6, 8	4, 9
10	A151329		—	5, 24	21	A151278		6, 8	4, 12
11	A151261		—	4, 15	22	A151323		4, 4	2, 3
12	A151297		—	5, 18	23	A060900		8, 9	3, 5

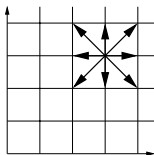
Equation sizes = {order, degree}@{algeq, diffeq}

Task: Prove Cases 1–19 in the tables [B. & Kauers 2009] for $F(t; 1, 1)$

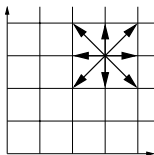
	OEIS		\mathfrak{S} alg?	asympt		OEIS		\mathfrak{S} alg?	asympt
1	A005566		N	$\frac{4}{\pi} \frac{4^n}{n}$	13	A151275		N	$\frac{12\sqrt{30}}{\pi} \frac{(2\sqrt{6})^n}{n^2}$
2	A018224		N	$\frac{2}{\pi} \frac{4^n}{n}$	14	A151314		N	$\frac{\sqrt{6}\lambda\mu C^{5/2}}{5\pi} \frac{(2C)^n}{n^2}$
3	A151312		N	$\frac{\sqrt{6}}{\pi} \frac{6^n}{n}$	15	A151255		N	$\frac{24\sqrt{2}}{\pi} \frac{(2\sqrt{2})^n}{n^2}$
4	A151331		N	$\frac{8}{3\pi} \frac{8^n}{n}$	16	A151287		N	$\frac{2\sqrt{2}A^{7/2}}{\pi} \frac{(2A)^n}{n^2}$
5	A151266		N	$\frac{1}{2} \sqrt{\frac{3}{\pi}} \frac{3^n}{n^{1/2}}$	17	A001006		Y	$\frac{3}{2} \sqrt{\frac{3}{\pi}} \frac{3^n}{n^{3/2}}$
6	A151307		N	$\frac{1}{2} \sqrt{\frac{5}{2\pi}} \frac{5^n}{n^{1/2}}$	18	A129400		Y	$\frac{3}{2} \sqrt{\frac{3}{\pi}} \frac{6^n}{n^{3/2}}$
7	A151291		N	$\frac{4}{3\sqrt{\pi}} \frac{4^n}{n^{1/2}}$	19	A005558		N	$\frac{8}{\pi} \frac{4^n}{n^2}$
8	A151326		N	$\frac{2}{\sqrt{3\pi}} \frac{6^n}{n^{1/2}}$					
9	A151302		N	$\frac{1}{3} \sqrt{\frac{5}{2\pi}} \frac{5^n}{n^{1/2}}$	20	A151265		Y	$\frac{2\sqrt{2}}{\Gamma(1/4)} \frac{3^n}{n^{3/4}}$
10	A151329		N	$\frac{1}{3} \sqrt{\frac{7}{3\pi}} \frac{7^n}{n^{1/2}}$	21	A151278		Y	$\frac{3\sqrt{3}}{\sqrt{2}\Gamma(1/4)} \frac{3^n}{n^{3/4}}$
11	A151261		N	$\frac{12\sqrt{3}}{\pi} \frac{(2\sqrt{3})^n}{n^2}$	22	A151323		Y	$\frac{\sqrt{23}^{3/4}}{\Gamma(1/4)} \frac{6^n}{n^{3/4}}$
12	A151297		N	$\frac{\sqrt{3}B^{7/2}}{2\pi} \frac{(2B)^n}{n^2}$	23	A060900		Y	$\frac{4\sqrt{3}}{3\Gamma(1/3)} \frac{4^n}{n^{2/3}}$

$$A = 1 + \sqrt{2}, \quad B = 1 + \sqrt{3}, \quad C = 1 + \sqrt{6}, \quad \lambda = 7 + 3\sqrt{6}, \quad \mu = \sqrt{\frac{4\sqrt{6}-1}{19}}$$

The group of a model

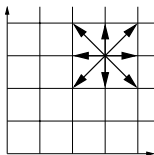


The polynomial $\chi_{\mathfrak{S}} := \sum_{(i,j) \in \mathfrak{S}} x^i y^j = \sum_{i=-1}^1 B_i(y) x^i = \sum_{j=-1}^1 A_j(x) y^j$



The polynomial $\chi_{\mathfrak{S}} := \sum_{(i,j) \in \mathfrak{S}} x^i y^j = \sum_{i=-1}^1 B_i(y) x^i = \sum_{j=-1}^1 A_j(x) y^j$ is left invariant under

$$\psi(x, y) = \left(x, \frac{A_{-1}(x)}{A_{+1}(x)} \frac{1}{y} \right), \quad \phi(x, y) = \left(\frac{B_{-1}(y)}{B_{+1}(y)} \frac{1}{x}, y \right),$$

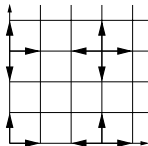


The polynomial $\chi_{\mathfrak{G}} := \sum_{(i,j) \in \mathfrak{G}} x^i y^j = \sum_{i=-1}^1 B_i(y) x^i = \sum_{j=-1}^1 A_j(x) y^j$ is left invariant under

$$\psi(x, y) = \left(x, \frac{A_{-1}(x)}{A_{+1}(x)} \frac{1}{y} \right), \quad \phi(x, y) = \left(\frac{B_{-1}(y)}{B_{+1}(y)} \frac{1}{x}, y \right),$$

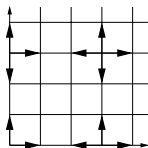
and thus under any element of the group

$$\mathcal{G}_{\mathfrak{G}} := \langle \psi, \phi \rangle.$$



$J = 1 - t \cdot \sum_{(i,j) \in \mathfrak{S}} x^i y^j = 1 - t \left(x + \frac{1}{x} + y + \frac{1}{y} \right)$ is **invariant** under the change of (x, y) into, respectively:

$$\left(\frac{1}{x}, y \right), \left(\frac{1}{x}, \frac{1}{y} \right), \left(x, \frac{1}{y} \right).$$

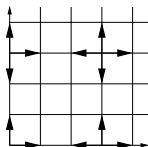


$J = 1 - t \cdot \sum_{(i,j) \in \mathfrak{S}} x^i y^j = 1 - t \left(x + \frac{1}{x} + y + \frac{1}{y} \right)$ is invariant under the change of (x, y) into, respectively:

$$\left(\frac{1}{x}, y \right), \left(\frac{1}{x}, \frac{1}{y} \right), \left(x, \frac{1}{y} \right).$$

Kernel equation:

$$J(t; x, y)xyF(t; x, y) = xy - txF(t; x, 0) - tyF(t; 0, y)$$

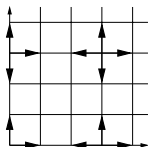


$J = 1 - t \cdot \sum_{(i,j) \in \mathfrak{S}} x^i y^j = 1 - t \left(x + \frac{1}{x} + y + \frac{1}{y} \right)$ is invariant under the change of (x, y) into, respectively:

$$\left(\frac{1}{x}, y \right), \left(\frac{1}{x}, \frac{1}{y} \right), \left(x, \frac{1}{y} \right).$$

Kernel equation:

$$\begin{aligned} J(t; x, y)xyF(t; x, y) &= xy - txF(t; x, 0) - tyF(t; 0, y) \\ - J(t; x, y)\frac{1}{x}yF(t; \frac{1}{x}, y) &= -\frac{1}{x}y + t\frac{1}{x}F(t; \frac{1}{x}, 0) + tyF(t; 0, y) \end{aligned}$$

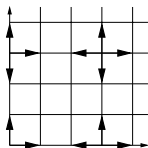


$J = 1 - t \cdot \sum_{(i,j) \in \mathfrak{S}} x^i y^j = 1 - t \left(x + \frac{1}{x} + y + \frac{1}{y} \right)$ is invariant under the change of (x, y) into, respectively:

$$\left(\frac{1}{x}, y \right), \left(\frac{1}{x}, \frac{1}{y} \right), \left(x, \frac{1}{y} \right).$$

Kernel equation:

$$\begin{aligned} J(t; x, y)xyF(t; x, y) &= xy - txF(t; x, 0) - tyF(t; 0, y) \\ - J(t; x, y)\frac{1}{x}yF(t; \frac{1}{x}, y) &= -\frac{1}{x}y + t\frac{1}{x}F(t; \frac{1}{x}, 0) + tyF(t; 0, y) \\ J(t; x, y)\frac{1}{x}\frac{1}{y}F(t; \frac{1}{x}, \frac{1}{y}) &= \frac{1}{x}\frac{1}{y} - t\frac{1}{x}F(t; \frac{1}{x}, 0) - t\frac{1}{y}F(t; 0, \frac{1}{y}) \end{aligned}$$

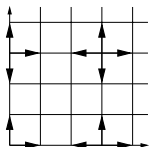


$J = 1 - t \cdot \sum_{(i,j) \in \mathfrak{S}} x^i y^j = 1 - t \left(x + \frac{1}{x} + y + \frac{1}{y} \right)$ is invariant under the change of (x, y) into, respectively:

$$\left(\frac{1}{x}, y \right), \left(\frac{1}{x}, \frac{1}{y} \right), \left(x, \frac{1}{y} \right).$$

Kernel equation:

$$\begin{aligned} J(t; x, y)xyF(t; x, y) &= xy - txF(t; x, 0) - tyF(t; 0, y) \\ - J(t; x, y)\frac{1}{x}yF(t; \frac{1}{x}, y) &= -\frac{1}{x}y + t\frac{1}{x}F(t; \frac{1}{x}, 0) + tyF(t; 0, y) \\ J(t; x, y)\frac{1}{x}\frac{1}{y}F(t; \frac{1}{x}, \frac{1}{y}) &= \frac{1}{x}\frac{1}{y} - t\frac{1}{x}F(t; \frac{1}{x}, 0) - t\frac{1}{y}F(t; 0, \frac{1}{y}) \\ - J(t; x, y)x\frac{1}{y}F(t; x, \frac{1}{y}) &= -x\frac{1}{y} + txF(t; x, 0) + t\frac{1}{y}F(t; 0, \frac{1}{y}) \end{aligned}$$



$J = 1 - t \cdot \sum_{(i,j) \in \mathcal{G}} x^i y^j = 1 - t \left(x + \frac{1}{x} + y + \frac{1}{y} \right)$ is invariant under the change of (x, y) into, respectively:

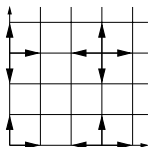
$$\left(\frac{1}{x}, y \right), \left(\frac{1}{x}, \frac{1}{y} \right), \left(x, \frac{1}{y} \right).$$

Kernel equation:

$$\begin{aligned} J(t; x, y)xyF(t; x, y) &= xy - txF(t; x, 0) - tyF(t; 0, y) \\ - J(t; x, y)\frac{1}{x}yF(t; \frac{1}{x}, y) &= -\frac{1}{x}y + t\frac{1}{x}F(t; \frac{1}{x}, 0) + tyF(t; 0, y) \\ J(t; x, y)\frac{1}{x}\frac{1}{y}F(t; \frac{1}{x}, \frac{1}{y}) &= \frac{1}{x}\frac{1}{y} - t\frac{1}{x}F(t; \frac{1}{x}, 0) - t\frac{1}{y}F(t; 0, \frac{1}{y}) \\ - J(t; x, y)x\frac{1}{y}F(t; x, \frac{1}{y}) &= -x\frac{1}{y} + txF(t; x, 0) + t\frac{1}{y}F(t; 0, \frac{1}{y}) \end{aligned}$$

Summing up yields the orbit equation:

$$\sum_{\theta \in \mathcal{G}} (-1)^\theta \theta(xy F(t; x, y)) = \frac{xy - \frac{1}{x}y + \frac{1}{x}\frac{1}{y} - x\frac{1}{y}}{J(t; x, y)}$$



$J = 1 - t \cdot \sum_{(i,j) \in \mathcal{G}} x^i y^j = 1 - t \left(x + \frac{1}{x} + y + \frac{1}{y} \right)$ is invariant under the change of (x, y) into, respectively:

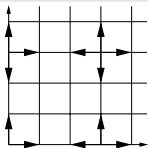
$$\left(\frac{1}{x}, y \right), \left(\frac{1}{x}, \frac{1}{y} \right), \left(x, \frac{1}{y} \right).$$

Kernel equation:

$$\begin{aligned} J(t; x, y)xyF(t; x, y) &= xy - txF(t; x, 0) - tyF(t; 0, y) \\ - J(t; x, y)\frac{1}{x}yF(t; \frac{1}{x}, y) &= -\frac{1}{x}y + t\frac{1}{x}F(t; \frac{1}{x}, 0) + tyF(t; 0, y) \\ J(t; x, y)\frac{1}{x}\frac{1}{y}F(t; \frac{1}{x}, \frac{1}{y}) &= \frac{1}{x}\frac{1}{y} - t\frac{1}{x}F(t; \frac{1}{x}, 0) - t\frac{1}{y}F(t; 0, \frac{1}{y}) \\ - J(t; x, y)x\frac{1}{y}F(t; x, \frac{1}{y}) &= -x\frac{1}{y} + txF(t; x, 0) + t\frac{1}{y}F(t; 0, \frac{1}{y}) \end{aligned}$$

Taking positive parts yields:

$$[x^>y^>] \sum_{\theta \in \mathcal{G}} (-1)^\theta \theta(xy F(t; x, y)) = [x^>y^>] \frac{xy - \frac{1}{x}y + \frac{1}{x}\frac{1}{y} - x\frac{1}{y}}{J(t; x, y)}$$



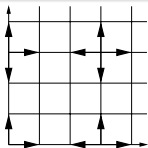
$J = 1 - t \cdot \sum_{(i,j) \in \mathfrak{S}} x^i y^j = 1 - t \left(x + \frac{1}{x} + y + \frac{1}{y} \right)$ is invariant under the change of (x, y) into, respectively:

$$\left(\frac{1}{x}, y \right), \left(\frac{1}{x}, \frac{1}{y} \right), \left(x, \frac{1}{y} \right).$$

Kernel equation:

$$\begin{aligned} J(t; x, y)xyF(t; x, y) &= xy - txF(t; x, 0) - tyF(t; 0, y) \\ - J(t; x, y)\frac{1}{x}yF(t; \frac{1}{x}, y) &= -\frac{1}{x}y + t\frac{1}{x}F(t; \frac{1}{x}, 0) + tyF(t; 0, y) \\ J(t; x, y)\frac{1}{x}\frac{1}{y}F(t; \frac{1}{x}, \frac{1}{y}) &= \frac{1}{x}\frac{1}{y} - t\frac{1}{x}F(t; \frac{1}{x}, 0) - t\frac{1}{y}F(t; 0, \frac{1}{y}) \\ - J(t; x, y)x\frac{1}{y}F(t; x, \frac{1}{y}) &= -x\frac{1}{y} + txF(t; x, 0) + t\frac{1}{y}F(t; 0, \frac{1}{y}) \end{aligned}$$

$$GF = \text{PosPart} \left(\frac{\text{OS}}{\text{kernel}} \right)$$



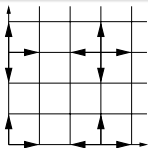
$J = 1 - t \cdot \sum_{(i,j) \in \mathfrak{S}} x^i y^j = 1 - t \left(x + \frac{1}{x} + y + \frac{1}{y} \right)$ is invariant under the change of (x, y) into, respectively:

$$\left(\frac{1}{x}, y \right), \left(\frac{1}{x}, \frac{1}{y} \right), \left(x, \frac{1}{y} \right).$$

Kernel equation:

$$\begin{aligned} J(t; x, y)xyF(t; x, y) &= xy - txF(t; x, 0) - tyF(t; 0, y) \\ - J(t; x, y)\frac{1}{x}yF(t; \frac{1}{x}, y) &= -\frac{1}{x}y + t\frac{1}{x}F(t; \frac{1}{x}, 0) + tyF(t; 0, y) \\ J(t; x, y)\frac{1}{x}\frac{1}{y}F(t; \frac{1}{x}, \frac{1}{y}) &= \frac{1}{x}\frac{1}{y} - t\frac{1}{x}F(t; \frac{1}{x}, 0) - t\frac{1}{y}F(t; 0, \frac{1}{y}) \\ - J(t; x, y)x\frac{1}{y}F(t; x, \frac{1}{y}) &= -x\frac{1}{y} + txF(t; x, 0) + t\frac{1}{y}F(t; 0, \frac{1}{y}) \end{aligned}$$

$$\text{GF} = \text{PosPart} \left(\frac{\text{OS}}{\text{ker}} \right) = \text{D-finite [Lipshitz'88]}$$



$J = 1 - t \cdot \sum_{(i,j) \in \mathfrak{S}} x^i y^j = 1 - t \left(x + \frac{1}{x} + y + \frac{1}{y} \right)$ is invariant under the change of (x, y) into, respectively:

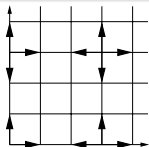
$$\left(\frac{1}{x}, y \right), \left(\frac{1}{x}, \frac{1}{y} \right), \left(x, \frac{1}{y} \right).$$

Kernel equation:

$$\begin{aligned} J(t; x, y)xyF(t; x, y) &= xy - txF(t; x, 0) - tyF(t; 0, y) \\ - J(t; x, y)\frac{1}{x}yF(t; \frac{1}{x}, y) &= -\frac{1}{x}y + t\frac{1}{x}F(t; \frac{1}{x}, 0) + tyF(t; 0, y) \\ J(t; x, y)\frac{1}{x}\frac{1}{y}F(t; \frac{1}{x}, \frac{1}{y}) &= \frac{1}{x}\frac{1}{y} - t\frac{1}{x}F(t; \frac{1}{x}, 0) - t\frac{1}{y}F(t; 0, \frac{1}{y}) \\ - J(t; x, y)x\frac{1}{y}F(t; x, \frac{1}{y}) &= -x\frac{1}{y} + txF(t; x, 0) + t\frac{1}{y}F(t; 0, \frac{1}{y}) \end{aligned}$$

$$\text{GF} = \text{PosPart} \left(\frac{\text{OS}}{\text{ker}} \right) = \text{D-finite [Lipshitz'88]}$$

▷ This is an algebraic version of the reflection principle



$J = 1 - t \cdot \sum_{(i,j) \in \mathfrak{S}} x^i y^j = 1 - t \left(x + \frac{1}{x} + y + \frac{1}{y} \right)$ is invariant under the change of (x, y) into, respectively:

$$\left(\frac{1}{x}, y \right), \left(\frac{1}{x}, \frac{1}{y} \right), \left(x, \frac{1}{y} \right).$$

Kernel equation:

$$\begin{aligned} J(t; x, y)xyF(t; x, y) &= xy - txF(t; x, 0) - tyF(t; 0, y) \\ - J(t; x, y)\frac{1}{x}yF(t; \frac{1}{x}, y) &= -\frac{1}{x}y + t\frac{1}{x}F(t; \frac{1}{x}, 0) + tyF(t; 0, y) \\ J(t; x, y)\frac{1}{x}\frac{1}{y}F(t; \frac{1}{x}, \frac{1}{y}) &= \frac{1}{x}\frac{1}{y} - t\frac{1}{x}F(t; \frac{1}{x}, 0) - t\frac{1}{y}F(t; 0, \frac{1}{y}) \\ - J(t; x, y)x\frac{1}{y}F(t; x, \frac{1}{y}) &= -x\frac{1}{y} + txF(t; x, 0) + t\frac{1}{y}F(t; 0, \frac{1}{y}) \end{aligned}$$

$$\text{GF} = \text{PosPart} \left(\frac{\text{OS}}{\text{ker}} \right) = \text{D-finite [Lipshitz'88]}$$

▷ **Creative telescoping** finds a differential equation for $\text{PosPart}(\text{OS}/\text{ker})$

Theorem [Bousquet-Mélou & Mishna, 2010]

Let \mathfrak{S} be one of the step sets 1–19. Then, the invariant group \mathcal{G} is finite and:

$$xyt F(t; x, y) = [x^> y^>] \frac{\sum_{\theta \in \mathcal{G}} (-1)^{\theta} \theta(xy)}{J(t; x, y)}.$$

In particular, $F(t; x, y)$ is D-finite.

Theorem [Bousquet-Mélou & Mishna, 2010]

Let \mathfrak{S} be one of the step sets 1–19. Then, the invariant group \mathcal{G} is finite and:

$$xyt F(t; x, y) = [x \succ y \succ] \frac{\sum_{\theta \in \mathcal{G}} (-1)^\theta \theta(xy)}{J(t; x, y)}.$$

In particular, $F(t; x, y)$ is D-finite.

Proof: Use [Lipshitz'88] for positive parts of rational functions: “positive parts are Hadamard products, and thus diagonals”.

▷ Constructive proof, but it leads to a **highly inefficient** algorithm to get an ODE for $F(t; x, y)$.

Theorem [Bousquet-Mélou & Mishna, 2010]

Let \mathfrak{S} be one of the step sets 1–19. Then, the invariant group \mathcal{G} is finite and:

$$xyt F(t; x, y) = [x^> y^>] \frac{\sum_{\theta \in \mathcal{G}} (-1)^{\theta} \theta(xy)}{J(t; x, y)}.$$

In particular, $F(t; x, y)$ is D-finite.

Proof: Use [Lipshitz'88] for positive parts of rational functions: “positive parts are Hadamard products, and thus diagonals”.

- ▷ Constructive proof, but it leads to a **highly inefficient** algorithm to get an ODE for $F(t; x, y)$; in fact, any such ODE is probably
TOO LARGE TO BE MERELY WRITTEN!

Theorem [B.-Chyzak-van Hoeij-Kauers-Pech, 2015]

Let \mathfrak{S} be one of the step sets 1–19. Then, the generating series $F(t; x, y)$ is expressible using iterated integrals of ${}_2F_1$ expressions.

Explicit Expressions for the Cases 1–19

Theorem [B.-Chyzak-van Hoeij-Kauers-Pech, 2015]

Let \mathfrak{S} be one of the step sets 1–19. Then, the generating series $F(t; 1, 1)$ is expressible using iterated integrals of ${}_2F_1$ expressions.

Example: King walks in the quarter plane (A025595)

$$\begin{aligned} F(t; 1, 1) &= \frac{1}{t} \int_0^t \frac{1}{(1+4x)^3} \cdot {}_2F_1\left(\frac{3}{2}, \frac{3}{2} \mid \frac{16x(1+x)}{(1+4x)^2}\right) dx \\ &= 1 + 3t + 18t^2 + 105t^3 + 684t^4 + 4550t^5 + 31340t^6 + 219555t^7 + \dots \end{aligned}$$

Theorem [B.-Chyzak-van Hoeij-Kauers-Pech, 2015]

Let \mathfrak{S} be one of the step sets 1–19. Then, the generating series $F(t; x, y)$ is expressible using iterated integrals of ${}_2F_1$ expressions.

▷ Proof uses **Creative telescoping**, **ODE factorization**, **ODE solving**:

① If $R = \sum_{\theta} \frac{(-1)^{\theta} \theta(xy)}{J(t; x, y)}$, then $F = [u^{-1}v^{-1}]H$, for $H = \frac{R(t; 1/u, 1/v)}{(1-xu)(1-yv)}$.

② If $P \in \mathbb{Q}(x, y)[t]\langle \partial_t \rangle$ and $U, V \in \mathbb{Q}(x, y, u, v, t)$ such that $L(H) = \partial_u U + \partial_v V$, then $L(F(t; x, y)) = 0$.

Use **creative telescoping** for finding L .

③ **Factor** L as $L_2 \cdot P_1 \cdots P_t$, where L_2 has order 2 and the P_i have order 1.
Solve L_2 in terms of ${}_2F_1$ s and deduce F .

Theorem [B.-Chyzak-van Hoeij-Kauers-Pech, 2015]

Let \mathfrak{S} be one of the step sets 1–19. Then, the generating series $F(t; x, y)$ is expressible using iterated integrals of ${}_2F_1$ expressions.

▷ Proof uses **Creative telescoping**, **ODE factorization**, **ODE solving**:

① If $R = \sum_{\theta} \frac{(-1)^{\theta} \theta(xy)}{J(t; x, y)}$, then $F = [u^{-1}v^{-1}]H$, for $H = \frac{R(t; 1/u, 1/v)}{(1-xu)(1-yv)}$.

Taking algebraic residues commutes with specializing x and y !

② If $P \in \mathbb{Q}(x, y)[t]\langle \partial_t \rangle$ and $U, V \in \mathbb{Q}(x, y, u, v, t)$ such that $L(H) = \partial_u U + \partial_v V$, then $L(F(t; x, y)) = 0$.

Use **creative telescoping** for finding L .

Works in practice with early evaluation $(x, y) = (1, 1)$, but not for symbolic (x, y) .

③ **Factor** L as $L_2 \cdot P_1 \cdots P_t$, where L_2 has order 2 and the P_i have order 1.
Solve L_2 in terms of ${}_2F_1$ s and deduce F .

Theorem [B.-Chyzak-van Hoeij-Kauers-Pech, 2015]

Let \mathfrak{S} be one of the step sets 1–19. Then, the generating series $F(t; x, y)$ is expressible using iterated integrals of ${}_2F_1$ expressions.

▷ Proof uses **Creative telescoping**, **ODE factorization**, **ODE solving**:

① If $R = \sum_{\theta} \frac{(-1)^{\theta} \theta(xy)}{J(t; x, y)}$, then $F = [u^{-1}v^{-1}]H$, for $H = \frac{R(t; 1/u, 1/v)}{(1-xu)(1-yv)}$.

Taking algebraic residues commutes with specializing x and y !

② If $P \in \mathbb{Q}(x, y)[t]\langle \partial_t \rangle$ and $U, V \in \mathbb{Q}(x, y, u, v, t)$ such that $L(H) = \partial_u U + \partial_v V$, then $L(F(t; x, y)) = 0$.

Use **creative telescoping** for finding L .

Works in practice with early evaluation $(x, y) = (1, 1)$, but not for symbolic (x, y) .

Works also for $(0, 0)$, $(x, 0)$, and $(0, y)$!

③ **Factor** L as $L_2 \cdot P_1 \cdots P_t$, where L_2 has order 2 and the P_i have order 1.

Solve L_2 in terms of ${}_2F_1$ s and deduce F .

For $F(t; x, y)$, run whole process for $F(t; 0, 0)$, $F(t; x, 0)$, and $F(t; 0, y)$, then
substitute into Kernel equation!

Creative telescoping allows the uniform treatment of several questions:

- compute differential operators that witness D-finiteness,
- algebraic vs transcendental nature of series,
- asymptotics of coefficients.

BACK TO THE EXERCISE

-Solution-

The exercise

Let $\mathfrak{S} = \{\uparrow, \leftarrow, \searrow\}$. A \mathfrak{S} -walk is a path in \mathbb{Z}^2 using only steps from \mathfrak{S} . Show that, for any integer n , the following quantities are equal:

- (i) the number a_n of \mathfrak{S} -walks of length n confined to the upper half plane $\mathbb{Z} \times \mathbb{N}$ that start and end at the origin $(0,0)$;
- (ii) the number b_n of \mathfrak{S} -walks of length n confined to the quarter plane \mathbb{N}^2 that start at the origin $(0,0)$ and finish on the diagonal $x = y$.

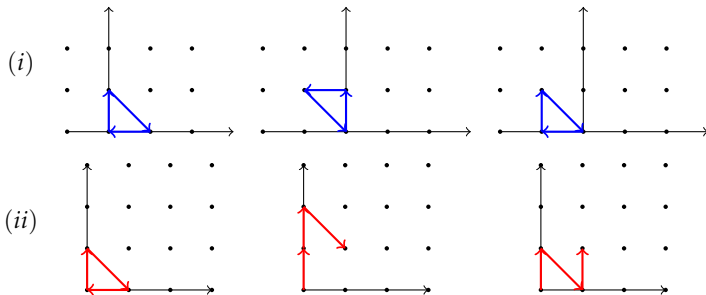
The exercise

Let $\mathfrak{S} = \{\uparrow, \leftarrow, \searrow\}$. A \mathfrak{S} -walk is a path in \mathbb{Z}^2 using only steps from \mathfrak{S} . Show that, for any integer n , the following quantities are equal:

(i) the number a_n of \mathfrak{S} -walks of length n confined to the upper half plane $\mathbb{Z} \times \mathbb{N}$ that start and end at the origin $(0,0)$;

(ii) the number b_n of \mathfrak{S} -walks of length n confined to the quarter plane \mathbb{N}^2 that start at the origin $(0,0)$ and finish on the diagonal $x = y$.

For instance, for $n = 3$, this common value is $a_3 = b_3 = 3$:



A recurrence relation for $\{\uparrow, \leftarrow, \searrow\}$ -walks in $\mathbb{Z} \times \mathbb{N}$

$h(n; i, j)$ = nb. of $\{\uparrow, \leftarrow, \searrow\}$ -walks in $\mathbb{Z} \times \mathbb{N}$ of length n from $(0, 0)$ to (i, j)

The numbers $h(n; i, j)$ satisfy

$$h(n; i, j) = \begin{cases} 0 & \text{if } j < 0 \text{ or } n < 0, \\ \mathbb{1}_{i=j=0} & \text{if } n = 0, \\ \sum_{(i', j') \in \mathfrak{S}} h(n-1; i-i', j-j') & \text{otherwise.} \end{cases}$$

```
> h:=proc(n,i,j)
  option remember;
  if j<0 or n<0 then 0
  elif n=0 then if i=0 and j=0 then 1 else 0 fi
  else h(n-1,i,j-1)+h(n-1,i+1,j)+h(n-1,i-1,j+1) fi
end:

> A:=series(add(h(n,0,0)*t^n,n=0..12),t,12);
```

$$A = 1 + 3t^3 + 30t^6 + 420t^9 + O(t^{12})$$

A recurrence relation for $\{\uparrow, \leftarrow, \searrow\}$ -walks in \mathbb{N}^2

$q(n; i, j)$ = nb. of $\{\uparrow, \leftarrow, \searrow\}$ -walks in \mathbb{N}^2 of length n from $(0, 0)$ to (i, j)

The numbers $q(n; i, j)$ satisfy

$$q(n; i, j) = \begin{cases} 0 & \text{if } i < 0 \text{ or } j < 0 \text{ or } n < 0, \\ \mathbb{1}_{i=j=0} & \text{if } n = 0, \\ \sum_{(i', j') \in \mathfrak{S}} q(n-1; i-i', j-j') & \text{otherwise.} \end{cases}$$

```
> q:=proc(n,i,j)
  option remember;
  if i<0 or j<0 or n<0 then 0
  elif n=0 then if i=0 and j=0 then 1 else 0 fi
  else q(n-1,i,j-1)+q(n-1,i+1,j)+q(n-1,i-1,j+1) fi
end:

> B:=series(add(add(q(n,k,k),k=0..n)*t^n,n=0..12),t,12);
```

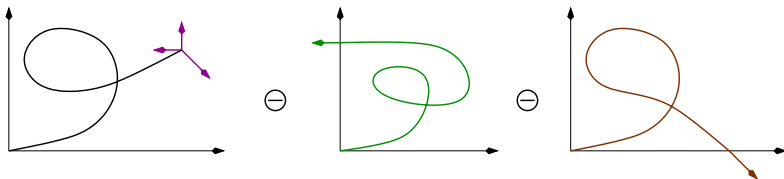
$$B = 1 + 3t^3 + 30t^6 + 420t^9 + O(t^{12})$$

A functional equation for $\{\uparrow, \leftarrow, \searrow\}$ -walks in \mathbb{N}^2

Generating function: $Q(t; x, y) = \sum_{n=0}^{\infty} \sum_{i=0}^n \sum_{j=0}^n q(n; i, j) t^n x^i y^j \in \mathbb{Q}[x, y][[t]]$

Kernel equation ($\bar{x} = 1/x$, $\bar{y} = 1/y$):

$$Q(t; x, y) \equiv Q(x, y) = 1 + t(y + \bar{x} + x\bar{y})Q(x, y) - t\bar{x}Q(0, y) - tx\bar{y}Q(x, 0)$$



A functional equation for $\{\uparrow, \leftarrow, \searrow\}$ -walks in \mathbb{N}^2

Generating function: $Q(t; x, y) = \sum_{n=0}^{\infty} \sum_{i=0}^n \sum_{j=0}^n q(n; i, j) t^n x^i y^j \in \mathbb{Q}[x, y][[t]]$

Kernel equation ($\bar{x} = 1/x$, $\bar{y} = 1/y$):

$$Q(t; x, y) \equiv Q(x, y) = 1 + t(y + \bar{x} + x\bar{y})Q(x, y) - t\bar{x}Q(0, y) - tx\bar{y}Q(x, 0)$$

or

$$(1 - t(y + \bar{x} + x\bar{y}))Q(x, y) = 1 - t\bar{x}Q(0, y) - tx\bar{y}Q(x, 0),$$

A functional equation for $\{\uparrow, \leftarrow, \searrow\}$ -walks in \mathbb{N}^2

Generating function: $Q(t; x, y) = \sum_{n=0}^{\infty} \sum_{i=0}^n \sum_{j=0}^n q(n; i, j) t^n x^i y^j \in \mathbb{Q}[x, y][[t]]$

Kernel equation ($\bar{x} = 1/x$, $\bar{y} = 1/y$):

$$Q(t; x, y) \equiv Q(x, y) = 1 + t(y + \bar{x} + x\bar{y})Q(x, y) - t\bar{x}Q(0, y) - tx\bar{y}Q(x, 0)$$

or

$$(1 - t(y + \bar{x} + x\bar{y}))Q(x, y) = 1 - t\bar{x}Q(0, y) - tx\bar{y}Q(x, 0),$$

or

$$(1 - t(y + \bar{x} + x\bar{y}))xyQ(x, y) = xy - tyQ(0, y) - tx^2Q(x, 0)$$

Task (Q): Find $B(t) = [x^0] Q(x, \bar{x})$

... and a functional equation for $\{\uparrow, \leftarrow, \searrow\}$ -walks in $\mathbb{Z} \times \mathbb{N}$

Generating function: $H(t; x, y) = \sum_{n=0}^{\infty} \sum_{i=-n}^n \sum_{j=0}^{\infty} h(n; i, j) t^n x^i y^j \in \mathbb{Q}[x, \bar{x}, y][[t]]$

Kernel equation ($\bar{x} = 1/x$, $\bar{y} = 1/y$):

$$H(t; x, y) \equiv H(x, y) = 1 + t(y + \bar{x} + x\bar{y})H(x, y) - tx\bar{y}H(x, 0)$$

or

$$(1 - t(y + \bar{x} + x\bar{y}))H(x, y) = 1 - tx\bar{y}H(x, 0),$$

... and a functional equation for $\{\uparrow, \leftarrow, \searrow\}$ -walks in $\mathbb{Z} \times \mathbb{N}$

Generating function: $H(t; x, y) = \sum_{n=0}^{\infty} \sum_{i=-n}^n \sum_{j=0}^{\infty} h(n; i, j) t^n x^i y^j \in \mathbb{Q}[x, \bar{x}, y][[t]]$

Kernel equation ($\bar{x} = 1/x$, $\bar{y} = 1/y$):

$$H(t; x, y) \equiv H(x, y) = 1 + t(y + \bar{x} + x\bar{y})H(x, y) - tx\bar{y}H(x, 0)$$

or

$$(1 - t(y + \bar{x} + x\bar{y}))H(x, y) = 1 - tx\bar{y}H(x, 0),$$

or

$$(1 - t(y + \bar{x} + x\bar{y}))yH(x, y) = y - txH(x, 0)$$

Task (H): Find $A(t) = [x^0]H(x, 0)$

- The kernel equation reads (with $K(x, y) = 1 - t(y + \bar{x} + x\bar{y})$):

$$K(x, y)yH(x, y) = y - txH(x, 0)$$

- The kernel equation reads (with $K(x, y) = 1 - t(y + \bar{x} + x\bar{y})$):

$$K(x, y)yH(x, y) = y - txH(x, 0)$$

- Let

$$y_0 = \frac{x - t - \sqrt{(t - x)^2 - 4t^2x^3}}{2tx} = xt + t^2 + (x^2 + \bar{x})t^3 + (3x + \bar{x}^2)t^4 + \dots$$

be the (unique) root in $\mathbb{Q}[x, \bar{x}][[t]]$ of $K(x, y_0) = 0$.

- The kernel equation reads (with $K(x, y) = 1 - t(y + \bar{x} + x\bar{y})$):

$$K(x, y)yH(x, y) = y - txH(x, 0)$$

- Let

$$y_0 = \frac{x - t - \sqrt{(t - x)^2 - 4t^2x^3}}{2tx} = xt + t^2 + (x^2 + \bar{x})t^3 + (3x + \bar{x}^2)t^4 + \dots$$

be the (unique) root in $\mathbb{Q}[x, \bar{x}][[t]]$ of $K(x, y_0) = 0$.

- Then

$$0 = K(x, y_0)yH(x, y_0) = y_0 - txH(x, 0),$$

thus

$$H(x, 0) = \frac{y_0}{tx} \quad \text{and} \quad A(t) = \left[x^0 \right] \frac{y_0}{tx}.$$

The kernel method for $\mathbb{Z} \times \mathbb{N}$

- The kernel equation reads (with $K(x, y) = 1 - t(y + \bar{x} + x\bar{y})$):

$$K(x, y)yH(x, y) = y - txH(x, 0)$$

- Let

$$y_0 = \frac{x - t - \sqrt{(t-x)^2 - 4t^2x^3}}{2tx} = xt + t^2 + (x^2 + \bar{x})t^3 + (3x + \bar{x}^2)t^4 + \dots$$

be the (unique) root in $\mathbb{Q}[x, \bar{x}][[t]]$ of $K(x, y_0) = 0$.

- Then

$$0 = K(x, y_0)yH(x, y_0) = y_0 - txH(x, 0),$$

thus

$$H(x, 0) = \frac{y_0}{tx} \quad \text{and} \quad A(t) = \left[x^0 \right] \frac{y_0}{tx}.$$

- **Creative telescoping** then proves:

$$(27t^4 - t)A''(t) + (108t^3 - 4)A'(t) + 54t^2A(t) = 0.$$

> Zeilberger(1/x * sqrt((t-x)^2 - 4*t^2*x^3)/(2*t^2*x^2), t, x, Dt);

The group of the model $\{\uparrow, \leftarrow, \searrow\}$

Step set $\mathfrak{S} = \{(-1, 0), (0, 1), (1, -1)\}$, with **characteristic polynomial**

$$\chi(x, y) = \frac{1}{x} + y + x \cdot \frac{1}{y} = \bar{x} + y + x\bar{y}$$

Step set $\mathfrak{S} = \{(-1, 0), (0, 1), (1, -1)\}$, with **characteristic polynomial**

$$\chi(x, y) = \frac{1}{x} + y + x \cdot \frac{1}{y} = \bar{x} + y + x\bar{y}$$

$\chi(x, y)$ is left unchanged by the rational transformations

$$\Phi : (x, y) \mapsto (\bar{x}y, y) \quad \text{and} \quad \Psi : (x, y) \mapsto (x, x\bar{y}).$$

Step set $\mathfrak{S} = \{(-1, 0), (0, 1), (1, -1)\}$, with **characteristic polynomial**

$$\chi(x, y) = \frac{1}{x} + y + x \cdot \frac{1}{y} = \bar{x} + y + x\bar{y}$$

$\chi(x, y)$ is left unchanged by the rational transformations

$$\Phi : (x, y) \mapsto (\bar{x}y, y) \quad \text{and} \quad \Psi : (x, y) \mapsto (x, x\bar{y}).$$

Φ and Ψ are involutions, and generate a finite dihedral group D_3 of order 6:

$$\begin{array}{ccccc}
 & & \Psi & & \\
 & & \text{—————} & & \\
 & \Phi & (\bar{x}y, y) & \text{—————} & (\bar{x}y, \bar{x}) & \Phi & \\
 (x, y) & \text{———} & & & & \text{———} & (\bar{y}, \bar{x}) \\
 & \Psi & (x, x\bar{y}) & \text{—————} & (\bar{y}, x\bar{y}) & \Psi & \\
 & & & \Phi & & &
 \end{array}$$

- **Orbit equation:**

$$\begin{aligned} xyQ(x, y) - \bar{x}y^2Q(\bar{x}y, y) + \bar{x}^2yQ(\bar{x}y, \bar{x}) \\ - \bar{x}\bar{y}Q(\bar{y}, \bar{x}) + x\bar{y}^2Q(\bar{y}, x\bar{y}) - x^2\bar{y}Q(x, x\bar{y}) = \\ \frac{xy - \bar{x}y^2 + \bar{x}^2y - \bar{x}\bar{y} + x\bar{y}^2 - x^2\bar{y}}{1 - t(y + \bar{x} + x\bar{y})} \end{aligned}$$

- **Orbit equation:**

$$\begin{aligned} xyQ(x, y) - \bar{x}y^2Q(\bar{x}y, y) + \bar{x}^2yQ(\bar{x}y, \bar{x}) \\ - \bar{x}\bar{y}Q(\bar{y}, \bar{x}) + x\bar{y}^2Q(\bar{y}, x\bar{y}) - x^2\bar{y}Q(x, x\bar{y}) = \\ \frac{xy - \bar{x}y^2 + \bar{x}^2y - \bar{x}\bar{y} + x\bar{y}^2 - x^2\bar{y}}{1 - t(y + \bar{x} + x\bar{y})} \end{aligned}$$

- **Corollary** [Bousquet-Mélou & Mishna, 2010]:

$$xyQ(x, y) = [x^{>0}y^{>0}] \frac{xy - \bar{x}y^2 + \bar{x}^2y - \bar{x}\bar{y} + x\bar{y}^2 - x^2\bar{y}}{1 - t(y + \bar{x} + x\bar{y})}$$

- **Orbit equation:**

$$\begin{aligned} xyQ(x, y) - \bar{x}y^2Q(\bar{x}y, y) + \bar{x}^2yQ(\bar{x}y, \bar{x}) \\ - \bar{x}\bar{y}Q(\bar{y}, \bar{x}) + x\bar{y}^2Q(\bar{y}, x\bar{y}) - x^2\bar{y}Q(x, x\bar{y}) = \\ \frac{xy - \bar{x}y^2 + \bar{x}^2y - \bar{x}\bar{y} + x\bar{y}^2 - x^2\bar{y}}{1 - t(y + \bar{x} + x\bar{y})} \end{aligned}$$

- **Corollary [Bousquet-Mélou & Mishna, 2010]:**

$$xyQ(x, y) = [x^{>0}y^{>0}] \frac{xy - \bar{x}y^2 + \bar{x}^2y - \bar{x}\bar{y} + x\bar{y}^2 - x^2\bar{y}}{1 - t(y + \bar{x} + x\bar{y})}$$

- **Corollary [B.-Chyzak-van Hoeij-Kauers-Pech, 2015]:**

$$B(t) = [z^0]Q(z, \bar{z}) = [u^{-1}v^{-1}z^{-1}] \frac{\bar{u}\bar{v} - u\bar{v}^2 + u^2\bar{v} - uv + \bar{u}v^2 - \bar{u}^2v}{z(1 - zu)(1 - v\bar{z})(1 - t(\bar{v} + u + \bar{u}v))}$$

Solving the kernel equation for \mathbb{N}^2

- **Orbit equation:**

$$\begin{aligned} xyQ(x, y) - \bar{x}y^2Q(\bar{x}y, y) + \bar{x}^2yQ(\bar{x}y, \bar{x}) \\ - \bar{x}\bar{y}Q(\bar{y}, \bar{x}) + x\bar{y}^2Q(\bar{y}, x\bar{y}) - x^2\bar{y}Q(x, x\bar{y}) = \\ \frac{xy - \bar{x}y^2 + \bar{x}^2y - \bar{x}\bar{y} + x\bar{y}^2 - x^2\bar{y}}{1 - t(y + \bar{x} + x\bar{y})} \end{aligned}$$

- **Corollary [Bousquet-Mélou & Mishna, 2010]:**

$$xyQ(x, y) = [x^{>0}y^{>0}] \frac{xy - \bar{x}y^2 + \bar{x}^2y - \bar{x}\bar{y} + x\bar{y}^2 - x^2\bar{y}}{1 - t(y + \bar{x} + x\bar{y})}$$

- **Corollary [B.-Chyzak-van Hoeij-Kauers-Pech, 2015]:**

$$B(t) = [z^0]Q(z, \bar{z}) = [u^{-1}v^{-1}z^{-1}] \frac{\bar{u}\bar{v} - u\bar{v}^2 + u^2\bar{v} - uv + \bar{u}v^2 - \bar{u}^2v}{z(1 - zu)(1 - v\bar{z})(1 - t(\bar{v} + u + \bar{u}v))}$$

- **Multivariate Creative Telescoping** gives a differential equation for $B(t)$:

$$(27t^4 - t)B''(t) + (108t^3 - 4)B'(t) + 54t^2B(t) = 0.$$

We have proved that $A(t)$ and $B(t)$ are both solutions of

$$(27t^4 - t)y''(t) + (108t^3 - 4)y'(t) + 54t^2y(t) = 0.$$

Solving this equation proves:

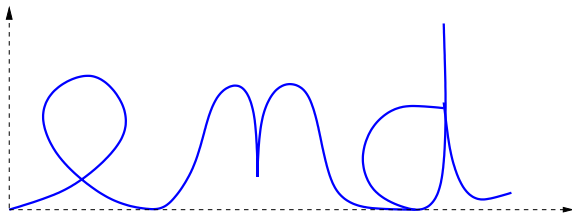
$$A(t) = B(t) = {}_2F_1\left(\begin{matrix} 1/3 & 2/3 \\ 2 \end{matrix} \middle| 27t^3\right) = \sum_{n=0}^{\infty} \frac{(3n)!}{n!^3} \frac{t^{3n}}{n+1}.$$

Thus the two sequences are equal to

$$a_{3n} = b_{3n} = \frac{(3n)!}{n!^2 \cdot (n+1)!}, \quad \text{and} \quad a_m = b_m = 0 \quad \text{if } 3 \text{ does not divide } m.$$

- Automatic classification of restricted lattice walks, with M. Kauers. *Proceedings FPSAC*, 2009.
- The complete generating function for Gessel walks is algebraic, with M. Kauers. *Proceedings of the American Mathematical Society*, 2010.
- Explicit formula for the generating series of diagonal 3D Rook paths, with F. Chyzak, M. van Hoeij and L. Pech. *Séminaire Lotharingien de Combinatoire*, 2011.
- Non-D-finite excursions in the quarter plane, with K. Raschel and B. Salvy. *Journal of Combinatorial Theory A*, 2013.
- On 3-dimensional lattice walks confined to the positive octant, with M. Bousquet-Mélou, M. Kauers and S. Melczer. *Annals of Comb.*, 2016.
- A human proof of Gessel's lattice path conjecture, with I. Kurkova, K. Raschel, *Transactions of the American Mathematical Society*, 2017.
- Hypergeometric expressions for generating functions of walks with small steps in the quarter plane, with F. Chyzak, M. van Hoeij, M. Kauers and L. Pech, *European Journal of Combinatorics*, 2017.
- *Computer Algebra for Lattice Path Combinatorics*, preprint, 2017.

This is the



Thanks for your attention!