

About Chapters 4 & 7 of the Yellow Book [1]:

Explicit Criteria for the Finiteness of the Associated Group in the Genus 1 Case.

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Introduction and reminders

- **Piecewise homogeneous random walk with sample paths in \mathbb{Z}_+^2** , the lattice in the positive quarter plane. In the strict interior of \mathbb{Z}_+^2 , the size of the jumps is 1, and $\{p_{ij}, |i|, |j| \leq 1\}$ will denote the generator of the process for this region. Thus a transition $(m, n) \rightarrow (m + i, n + j), m, n > 0$, can take place with probability p_{ij} , and

$$\sum_{|i|, |j| \leq 1} p_{ij} = 1.$$

- No strong assumption about the boundedness of the upward jumps on the axes, neither at $(0, 0)$. In addition, the downward jumps on the x [resp. y] axis are bounded by L [resp. M], where L and M are arbitrary finite integers.
- **Original question :** Find an explicit form for the invariant measure of such process.

The basic functional equation

The invariant measure $\{\pi_{i,j}, i, j \geq 0\}$ satisfies the fundamental bivariate functional equation

$$Q(x, y)\pi(x, y) = q(x, y)\pi(x) + \tilde{q}(x, y)\tilde{\pi}(y) + \pi_0(x, y), \quad (1)$$

where in (1) the unknown functions $\pi(x, y), \pi(x), \tilde{\pi}(y)$ are sought to be analytic in the region $\{(x, y) \in \mathbb{C}^2 : |x| < 1, |y| < 1\}$, and continuous on their respective boundaries.

$$\left\{ \begin{array}{l} \pi(x, y) = \sum_{i, j \geq 1} \pi_{ij} x^{i-1} y^{j-1}, \\ \pi(x) = \sum_{i \geq L} \pi_{i0} x^{i-L}, \quad \tilde{\pi}(y) = \sum_{j \geq M} \pi_{0j} y^{j-M}, \\ Q(x, y) = xy \left[1 - \sum_{i, j \in \mathcal{S}} p_{ij} x^i y^j \right], \quad \sum_{i, j \in \mathcal{S}} p_{ij} = 1, \\ q(x, y) = x^L \left[\sum_{i \geq -L, j \geq 0} p'_{ij} x^i y^j - 1 \right] \equiv x^L (P_{L0}(x, y) - 1), \\ \tilde{q}(x, y) = y^M \left[\sum_{i \geq 0, j \geq -M} p''_{ij} x^i y^j - 1 \right] \equiv y^M (P_{0M}(x, y) - 1), \\ \pi_0(x, y) = \sum_{i=1}^{L-1} \pi_{i0} x^i [P_{i0}(x, y) - 1] + \sum_{j=1}^{M-1} \pi_{0j} y^j [P_{0j}(x, y) - 1] + \pi_{00} (P_{00}(x, y) - 1). \end{array} \right.$$

\mathcal{S} is the set of allowed jumps, and $q, \tilde{q}, q_0, P_{i0}, P_{0j}$, are given probability generating functions supposed to have suitable analytic continuations (as a rule, they are polynomials when the jumps are bounded).

Group and Genus

The function $Q(x, y)$, often referred to as the *kernel* of (1), can be rewritten in the two following equivalent forms

$$Q(x, y) = a(x)y^2 + b(x)y + c(x) = \tilde{a}(y)x^2 + \tilde{b}(y)x + \tilde{c}(y), \quad (2)$$

$$a(x) = p_{1,1}x^2 + p_{0,1}x + p_{-1,1}$$

$$\tilde{a}(y) = p_{1,1}y^2 + p_{1,0}y + p_{1,-1},$$

$$b(x) = p_{1,0}x^2 + (p_{0,0} - 1)x + p_{-1,0}$$

$$\tilde{b}(y) = p_{0,1}y^2 + (p_{0,0} - 1)y + p_{0,-1},$$

$$c(x) = p_{1,-1}x^2 + p_{0,-1}x + p_{-1,-1}$$

$$\tilde{c}(y) = p_{-1,1}y^2 + p_{-1,0}y + p_{-1,-1}.$$

We shall also need the discriminants

$$D(x) \stackrel{\text{def}}{=} b^2(x) - 4a(x)c(x), \quad \tilde{D}(y) \stackrel{\text{def}}{=} \tilde{b}^2(y) - 4\tilde{a}(y)\tilde{c}(y). \quad (3)$$

The polynomials D and \tilde{D} are of degree 4, respectively in x and y .

Let $\mathbb{C}(x)$, $\mathbb{C}(y)$ and $\mathbb{C}(x, y)$ denote the respective fields of rational functions of x, y and (x, y) over \mathbb{C} . Since in general Q is assumed to be irreducible, the quotient field $\mathbb{C}(x, y)$ with respect to Q will be denoted by $\mathbb{C}_Q(x, y)$.

Definition 1. The group of the random walk is the *Galois group* $\mathcal{H} = \langle \xi, \eta \rangle$ of automorphisms of $\mathbb{C}_Q(x, y)$ generated by ξ and η given by

$$\xi(x, y) = \left(x, \frac{c(x)}{y a(x)} \right), \quad \eta(x, y) = \left(\frac{\tilde{c}(y)}{x \tilde{a}(y)}, y \right).$$

► Here ξ and η are involutions, so that $\xi^2 = \eta^2 = I$.

Lemma 2. Let

$$\delta \stackrel{\text{def}}{=} \eta\xi. \tag{4}$$

Then \mathcal{H} has a normal cyclic subgroup $\mathcal{H}_0 = \{\delta^n, n \in \mathbb{Z}\}$, which is finite or infinite, and $\mathcal{H}/\mathcal{H}_0$ is a cyclic group of order 2. ■

- The group \mathcal{H} is finite of order $2n$ if, and only if,

$$\delta^n = I. \tag{5}$$

- The product $\delta = \eta\xi$ is non-commutative, except for $\delta^2 = I$, in which case the group is of order 4.
- We shall write $f_\alpha = \alpha(f)$, for any automorphism $\alpha \in \mathcal{H}$ and any function $f \in \mathbb{C}_Q(x, y)$.
- The fundamental equation (1), together with ξ, η, δ , can be lifted onto the universal covering \mathbb{C} (the finite complex plane).

Let $\{x_\ell\}_{1 \leq \ell \leq 4}$ be the 4 roots of the discriminant $D(x)$ [see equation (3)], which are **the branch points of the Riemann surface** $\mathcal{K} = \{(x, y) \in \mathbb{C}^2 : Q(x, y) = 0\}$.

► Unless otherwise stated, \mathcal{K} is assumed to be of genus 1 (the torus), so that the algebraic curve $Q(x, y) = 0$ admits a **uniformization given in terms of the Weierstrass \wp function and its derivatives, all having periods ω_1, ω_2** . Letting

$$\begin{aligned} D(x) &= b^2(x) - 4a(x)c(x) \stackrel{\text{def}}{=} d_4x^4 + d_3x^3 + d_2x^2 + d_1x + d_0, \\ z &\stackrel{\text{def}}{=} 2a(x)y + b(x), \end{aligned}$$

the 4 real zeros of $D(x)$ are real, x_2 and x_3 being positive, and we have more precisely

$$\begin{cases} -1 \leq x_1 \leq x_2 < 1 < x_3 < x_4, & \text{if } d_4 > 0, \\ x_4 < -1 \leq x_1 \leq x_2 < 1 < x_3, & \text{if } d_4 < 0, \\ x_4 = \infty, & \text{if } d_4 = 0. \end{cases}$$

Then the following formulae hold (see [1], Section 3.3).

1. ► If $d_4 \neq 0$ (4 finite branch points x_1, \dots, x_4) then $D'(x_4) > 0$ and

$$\begin{cases} x(\omega) = x_4 + \frac{D'(x_4)}{\wp(\omega) - \frac{1}{6}D''(x_4)}, \\ z(\omega) = \frac{D'(x_4)\wp'(\omega)}{2\left(\wp(\omega) - \frac{1}{6}D''(x_4)\right)^2}. \end{cases} \quad (6)$$

2. ► If $d_4 = 0$ (3 finite branch points x_1, x_2, x_3 and $x_4 = \infty$) then

$$\begin{cases} x(\omega) = \frac{\wp(\omega) - \frac{d_2}{3}}{d_3}, \\ z(\omega) = -\frac{\wp'(\omega)}{2d_3}. \end{cases}$$

- The following admissible choice of the periods was computed in [1]:

$$\omega_1 = 2i \int_{x_1}^{x_2} \frac{dx}{\sqrt{-D(x)}}, \quad \omega_2 = 2 \int_{x_2}^{x_3} \frac{dx}{\sqrt{D(x)}}, \quad \omega_3 = 2 \int_{X(y_1)}^{x_1} \frac{dx}{\sqrt{D(x)}}.$$

- Note that ω_1 is purely imaginary, while $0 < \omega_3 < \omega_2$.
- It was proved in [1] that the group \mathcal{H} is finite of order $2n$ if and only if

$$n\omega_3 = 0 \pmod{(\omega_1, \omega_2)},$$

or, since ω_3 is real,

$$n\omega_3 = 0 \pmod{(\omega_2)}, \tag{7}$$

where n stands for the **minimal positive integer** with this property.

► On the universal covering \mathbb{C} (the finite complex plane), the automorphisms ξ, η, δ are written ξ^*, η^*, δ^* . They are linear:

$$\xi^*(\omega) = -\omega + \omega_2, \quad \eta^*(\omega) = -\omega + \omega_2 + \omega_3, \quad \delta^*(\omega) = \eta^*\xi^* = \omega + \omega_3. \quad (8)$$

Here $\delta = \eta\xi$ corresponds to $\delta^* = \eta^*\xi^*$. Thus, for any $f(x, y) \in C_Q(x, y)$,

$$\delta(f(x, y)) = f(\delta(x), \delta(y)) = f(x(\delta^*(\omega)), y(\delta^*(\omega))), \quad \omega \in \mathbb{C}.$$

In particular,

$$\begin{cases} \delta(x) = x(\delta^*(\omega)) = x(\omega + \omega_3), \\ \eta(x) = x(\eta^*(\omega)) = x(-\omega + \omega_2 + \omega_3) = x(\omega - \omega_3). \end{cases} \quad (9)$$

\mathcal{H} is generated by the elements ξ and η , and we can define the homomorphism

$$h(R(x, y)) \stackrel{\text{def}}{=} R(h(x), h(y)), \quad \forall h \in \mathcal{H}, \quad \forall R \in \mathbb{C}_Q(x, y).$$

► For any $R \in \mathbb{C}_Q(x, y)$, the following equivalences hold:

$$\begin{cases} \xi(R) = R & \iff R \in \mathbb{C}(x), \\ \eta(R) = R & \iff R \in \mathbb{C}(y), \end{cases} \quad (10)$$

so that $\mathbb{C}(x)$ (resp. $\mathbb{C}(y)$) is the set of elements of $\mathbb{C}_Q(x, y)$ invariant with respect to ξ (resp. η). Indeed, R has the general form

$$R(x, y) = A(x) + B(x)y \pmod{Q(x, y)},$$

where $A(x)$ and $B(x)$ are elements of $\mathbb{C}(x)$. [Hint: $\xi(R) = R$ and $\xi(y) \neq y$, so that necessarily $B(x) \equiv 0$].

Introduce the matrix

$$\mathbb{P} = \begin{pmatrix} p_{11} & p_{10} & p_{1,-1} \\ p_{01} & p_{00} - 1 & p_{0,-1} \\ p_{-1,1} & p_{-1,0} & p_{-1,-1} \end{pmatrix}, \quad (11)$$

and let $\vec{C}_1, \vec{C}_2, \vec{C}_3$ (resp. $\vec{D}_1, \vec{D}_2, \vec{D}_3$) denote the column vectors of \mathbb{P} (resp. of \mathbb{P}^T , the transpose matrix of \mathbb{P}).

Proposition 3. *Assume there exists a positive integer s such that*

$$\delta^s(x) = x. \quad (12)$$

Then $\delta^s = I$ and the group is of order $2s$, where s stands for the smallest integer with property (12).

► *Sketch of proof.* Each of the three following permutations

$$x \iff y, \quad \delta \iff \delta^{-1}, \quad \mathbb{P} \iff \mathbb{P}^T,$$

implies the two other ones.

Hence, for any integer $k \geq 1$, the quantity $\rho(x, y, k) \stackrel{\text{def}}{=} \delta^k(x) \cdot \delta^{-k}(y)$ remains invariant by permuting \mathbb{P} with \mathbb{P}^T .

► Assume first $s = 2m$. Then (12) becomes $\delta^m(x) = \delta^{-m}(x)$, and

$$\rho(x, y, m) = \delta^{-m}(x) \cdot \delta^{-m}(y) = \delta^m(x) \cdot \delta^m(y),$$

where the second equality is obtained by replacing \mathbb{P} by \mathbb{P}^T . Then, comparing with the definition of $\rho(x, y, m)$, we get $\delta^m(y) = \delta^{-m}(y)$, which yields in turn $\delta^s(y) = y$, whence $\delta^s = I$.

► If s is odd, say $s = 2m + 1$, the argument works in exactly the same way. In this case

$$\rho(x, y, m) = \delta^{-(m+1)}(x) \cdot \delta^{-m}(y) = \delta^m(x) \cdot \delta^{m+1}(y),$$

(by exchanging again \mathbb{P} with \mathbb{P}^T), which implies $\delta^{m+1}(y) = \delta^{-m}(y)$, that is $\delta^s(y) = y$, QED. ■

Corollary 4.

1. If there exists an integer s such that $\delta^s(x) = r(x)$, where $r(x)$ represents a rational fraction of x , then $\delta^{2s}(x) = x$ and the group is of **order $4s$** .
2. If there exists an integer s such that $\delta^s(x) = t(y)$, where $t(y)$ represents a rational fraction of y , then $\delta^{2s-1}(x) = x$ and the group is of **order $4s - 2$** .

In both cases, s stands for the **smallest integer** with the corresponding property.

► *Proof.* Note first the identities $\xi\delta^s\xi = \delta^{-s}$ and $\eta\delta^s\xi = \delta^{-s+1}$.

So, the following chain of equalities holds.

$$\delta^s(x) = r(x) \implies \xi\delta^s\xi(x) = \delta^s(x) \iff \delta^{-s}(x) = \delta^s(x) \iff \delta^{2s}(x) = x.$$

Similarly

$$\delta^s(x) = t(y) \implies \eta\delta^s\xi(x) = \delta^s(x) \iff \delta^{-s+1}(x) = \delta^s(x) \iff \delta^{2s-1}(x) = x.$$

► In both cases, the conclusion follows from Proposition 3, in which s is replaced respectively by $2s$ and $2s - 1$. ■

Lemma 5. *On the algebraic curve $\{Q(x, y) = 0\}$, the following general relations hold:*

$$\begin{cases} \eta(x) &= \frac{xv(y) - u(y)}{xw(y) - v(y)}, \\ \xi(y) &= \frac{y\tilde{v}(x) - \tilde{u}(x)}{y\tilde{w}(x) - \tilde{v}(x)}, \end{cases} \quad (13)$$

where u, v, w, h (resp. $\tilde{u}, \tilde{v}, \tilde{w}, \tilde{h}$) are polynomials of degree ≤ 2 . In particular, there exist affine solutions

$$(u(y), v(y), w(y))^T = \vec{A}y + \vec{B}, \quad (\tilde{u}(x), \tilde{v}(x), \tilde{w}(x))^T = \vec{E}x + \vec{F}, \quad (14)$$

with column vectors

$$\vec{A} = (u_0, v_0, w_0)^T, \quad \vec{B} = (u_1, v_1, w_1)^T, \quad \vec{E} = (\tilde{u}_0, \tilde{v}_0, \tilde{w}_0)^T, \quad \vec{F} = (\tilde{u}_1, \tilde{v}_1, \tilde{w}_1)^T.$$

$$\begin{cases} \vec{A} &= (\alpha\vec{C}_2 + \beta\vec{C}_1) \times \vec{C}_3, \\ \vec{B} &= \vec{C}_1 \times (\alpha\vec{C}_3 + \beta\vec{C}_2), \\ \vec{E} &= (\tilde{\alpha}\vec{D}_2 + \tilde{\beta}\vec{D}_1) \times \vec{D}_3, \\ \vec{F} &= \vec{D}_1 \times (\tilde{\alpha}\vec{D}_3 + \tilde{\beta}\vec{D}_2), \end{cases} \quad (15)$$

where $\alpha, \tilde{\alpha}, \beta, \tilde{\beta}$ are arbitrary complex constants, and the operator “ \times ” stands for the cross vector product. In addition, when \mathbb{P} is of rank 3, none of the vectors $\vec{A}, \vec{B}, \vec{E}, \vec{F}$ vanish. Choosing in (15) $\alpha = \tilde{\alpha} = 0, \beta = \tilde{\beta} = 1$, gives

$$\begin{cases} u(y) = y\Delta_{13} - \Delta_{12}, & \tilde{u}(x) = x\Delta_{31} - \Delta_{21}, \\ v(y) = y\Delta_{23} - \Delta_{22}, & \tilde{v}(x) = x\Delta_{32} - \Delta_{22}, \\ w(y) = y\Delta_{33} - \Delta_{32}, & \tilde{w}(x) = x\Delta_{33} - \Delta_{23}, \end{cases} \quad (16)$$

where Δ_{ij} denotes the cofactor of the $(i, j)^{th}$ entry of the matrix \mathbb{P} given in (11). ■

Lemma 6. *Let γ be an endomorphism defined on the algebraic surface \mathcal{K} , which is assumed to be invariant on the field $\mathbb{C}(x)$ of rational functions of x , and such that*

$$\gamma(y) = \frac{yf(x) - e(x)}{yg(x) + h(x)}, \quad (17)$$

where e, f, g, h are polynomials of degree 1 in x . (Note that this is always possible, as shown in Lemma 5.) Then, for γ to be an involution, the condition $f(x) + h(x) \equiv 0$ is necessary and sufficient.

Remark. The result of Lemma 6 does not hold if polynomials e, f, h are not of degree 1. For instance, one can check directly that, if g or e are taken to be of degree 2, then any involution γ necessarily has the form

$$\gamma(y) = \frac{(h - b)y - c}{ay + h}.$$

Groups of Order 4

Proposition 7. *The group \mathcal{H} is of order 4 if, and only if,*

$$\begin{vmatrix} p_{11} & p_{10} & p_{1,-1} \\ p_{01} & p_{00} - 1 & p_{0,-1} \\ p_{-1,1} & p_{-1,0} & p_{-1,-1} \end{vmatrix} = 0, \quad (18)$$

and this is the only case where the matrix \mathbb{P} has rank 2.

Proof. The equality $\delta^2 = I$ can be rewritten as $\xi\eta = \eta\xi$, which by Proposition 3 is, for instance, equivalent to

$$\xi\eta(x) = \eta(x),$$

where we have used $\xi(x) = x$. So, $\eta(x)$ is left invariant by ξ , which implies

$$\eta(x) \in \mathbb{C}(x).$$

► Finally, η is both an involution and a conformal automorphism on $\mathbb{C}(x)$. Consequently, η is a **fractional linear transform** of the form

$$\eta(x) = \frac{rx + s}{tx - r},$$

where all coefficients belong to \mathbb{C} . The following chain of equivalences hold.

$$\begin{aligned} \eta(x) = \frac{rx + s}{tx - r} &\Leftrightarrow tx \cdot \eta(x) = r(x + \eta(x)) + s \\ &\Leftrightarrow 1, x + \eta(x), x \cdot \eta(x) \text{ are linearly dependent on } \mathbb{C} \\ &\Leftrightarrow 1, -\frac{\tilde{b}(y)}{\tilde{a}(y)}, \frac{\tilde{c}(y)}{\tilde{a}(y)} \text{ are linearly dependent on } \mathbb{C} \\ &\Leftrightarrow \tilde{a}(y), \tilde{b}(y), \tilde{c}(y) \text{ are also linearly dependent on } \mathbb{C}, \end{aligned}$$

where equation (2) has been used in the form

$$Q(x, y) = \tilde{a}(y)x^2 + \tilde{b}(y)x + \tilde{c}(y).$$



Groups of Order 6

Proposition 8. \mathcal{H} is of order 6 if, and only if,

$$\begin{vmatrix} \Delta_{11} & \Delta_{21} & \Delta_{12} & \Delta_{22} \\ \Delta_{12} & \Delta_{22} & \Delta_{13} & \Delta_{23} \\ \Delta_{21} & \Delta_{31} & \Delta_{22} & \Delta_{32} \\ \Delta_{22} & \Delta_{32} & \Delta_{23} & \Delta_{33} \end{vmatrix} = 0, \quad (19)$$

where the Δ_{ij} 's have been given in Lemma 5.

Sketch of proof. In this case $(\xi\eta)^3 = I$, which is equivalent to

$$\eta\xi\eta = \xi\eta\xi. \quad (20)$$

Applying (20) for instance to x , we get

$$\xi\eta(x) = \eta\xi\eta(x),$$

which shows that $\xi\eta(x)$ is invariant with respect to η .

Similarly, $\eta\xi(y)$ is invariant with respect to ξ . Hence (20) is plainly equivalent to

$$\begin{cases} \xi\eta(x) = P(y), \\ \eta\xi(y) = R(x), \end{cases}$$

where P and R are rational. Then

$$y = R(\xi\eta(x)) = R \circ P(y),$$

or, equivalently,

$$R \circ P = I, \tag{21}$$

so that P and R are fractional linear transforms.

Thus (21) yields the relation

$$\xi(y) = \frac{p\eta(x) + q}{r\eta(x) + s}. \quad (22)$$

Hence, there is a **linear dependence on \mathbb{C} between the 4 elements $1, \xi(y), \eta(x), \xi(y)\eta(x)$** , with 4 unknown constants (in fact only 3 by homogeneity). Starting from equation (13), we choose $\eta(x)$ by means of (16),

$$\eta(x) = \frac{y(x\Delta_{23} - \Delta_{13}) - x\Delta_{22} + \Delta_{12}}{y(x\Delta_{33} - \Delta_{23}) - x\Delta_{32} + \Delta_{22}}. \quad (23)$$

Instantiating now (23) in (22), we obtain

$$\xi(y) = \frac{y[p(x\Delta_{23} - \Delta_{13}) + q(x\Delta_{33} - \Delta_{23})] + p(\Delta_{12} - x\Delta_{22}) + q(\Delta_{22} - x\Delta_{32})}{y[r(x\Delta_{23} - \Delta_{13}) + s(x\Delta_{33} - \Delta_{23})] + r(\Delta_{12} - x\Delta_{22}) + s(\Delta_{22} - x\Delta_{32})}. \quad (24)$$

Then, according to (17),

$$\xi(y) = \frac{yf(x) - e(x)}{yg(x) + h(x)},$$

where

$$\begin{cases} e(x) &= p(x\Delta_{22} - \Delta_{12}) + q(x\Delta_{32} - \Delta_{22}), \\ f(x) &= p(x\Delta_{23} - \Delta_{13}) + q(x\Delta_{33} - \Delta_{23}), \\ g(x) &= r(x\Delta_{23} - \Delta_{13}) + s(x\Delta_{33} - \Delta_{23}), \\ h(x) &= r(\Delta_{12} - x\Delta_{22}) + s(\Delta_{22} - x\Delta_{32}), \end{cases} \quad (25)$$

and we can compare system (25) with the solution presented in equation (15) of Lemma 5.

► The final step is to analyze the feasibility of a global **linear system formed of 8 equations with 6 unknown variables.** ■

Criterion for Groups of Order $4m$

Proposition 9. *The group \mathcal{H} is of order $4m$ if and only if the Weierstrass \wp function with periods (ω_1, ω_2) satisfies the equation*

$$\wp(m\omega_3) = \wp(\omega_2/2). \quad (26)$$

► *Proof.* Recalling that $\delta = \eta\xi$, we have here $\delta^{2m} = I$, that is

$$(\xi\eta)^m = (\eta\xi)^m. \quad (27)$$

By applying equation (27) at x (or even at an arbitrary element of $\mathbb{C}(x)$), and replacing $\xi(x)$ by x in the right-hand side, we obtain

$$\xi\delta^m(x) = \delta^m(x),$$

showing that **the involution $\delta^m(x)$ is invariant with respect to ξ .**

Hence $\delta^m(x)$ is an element of $\mathbb{C}(x)$, so that

$$\delta^m(x) = F(x) = \frac{xf - e}{xg - f}, \quad (28)$$

where $F(x)$ is a simple **fractional linear transform**, with constants e, f, g , to be determined. Hence, equation (28) implies the existence of a linear dependence between the functions

$$x.\delta^m(x), x + \delta^m(x), \mathbf{1}. \quad (29)$$

Lemma 10. *For the group to be of order $4m$, a necessary and sufficient condition is that the three functions*

$$x(\omega - m\omega_3/2).x(\omega + m\omega_3/2), x(\omega - m\omega_3/2) + x(\omega + m\omega_3/2), \mathbf{1}, \quad (30)$$

be linearly dependent, $\forall \omega \in \mathbb{C}$. ■

Recall that $\wp'^2 = 4\wp^3 - g_2\wp - g_3$, and let, for arbitrary u, v ,

$$A(u, v) \stackrel{\text{def}}{=} \wp(u+v) + \wp(u-v), \quad B(u, v) \stackrel{\text{def}}{=} \wp(u+v)\wp(u-v)$$

Then, setting for now $X \stackrel{\text{def}}{=} \wp(u)$, $Y \stackrel{\text{def}}{=} \wp(v)$, we have

Lemma 11.

$$A(u, v) = \frac{(X + Y)(4XY - g_2) - 2g_3}{2(X - Y)^2}, \quad (31)$$

$$B(u, v) = \frac{(XY)^2 + \frac{g_2}{2}XY + g_3(X + Y) + \frac{g_2^2}{16}}{(X - Y)^2}. \quad (32)$$

Let

$$S(u, v) \stackrel{\text{def}}{=} x(u + v) + x(u - v), \quad P(u, v) \stackrel{\text{def}}{=} x(u + v)x(u - v). \quad (33)$$

Since

$$x(\omega) = p + \frac{q}{\wp(\omega) - r}, \quad (34)$$

where p, q, r are known constants [see equation (6)], we have

$$\begin{cases} S = \frac{2pB + (q - 2pr)A + 2r(pr - q)}{B - rA + r^2}, \\ P = \frac{p^2B + p(q - pr)A + (pr - q)^2}{B - rA + r^2}. \end{cases} \quad (35)$$

► Taking $u = \omega, v = m\omega_3/2$, the claims involving (29) and (30) are merely equivalent to the existence of a non-trivial linear relation

$$eS + fP + g = 0, \quad \forall X \in \mathbb{C}. \quad (36)$$

► In other words $S, P, 1$, considered as functions of X , are linearly dependent.

With $Y = \wp(m\omega_3/2)$, the independence condition reads

$$\Delta(Y) \stackrel{\text{def}}{=} \begin{vmatrix} 4Y & 4Y^2 - g_2 & -(g_2Y + 2g_3) \\ 2Y^2 & g_2Y + 2g_3 & 2g_3Y + g_2^2/8 \\ 1 & -2Y & Y^2 \end{vmatrix} = 0, \quad (37)$$

which yields exactly (26), by using the factorization of $\Delta(Y)$ as the product of 3 polynomials of degree 2 in Y . . .

However, the computation of $\wp(m\omega_3/2)$, via the recursive relationship

$$\wp((l+1)\omega_3/2) + \wp((l-1)\omega_3/2) = \frac{(\wp(l\omega_3/2) + \wp(\omega_3/2))(4\wp(l\omega_3/2)\wp(\omega_3/2) - g_2) - 2g_3}{2(\wp(l\omega_3/2) - \wp(\omega_3/2))^2},$$

is hardly exploitable. . .

► **Case $m = 2k$.** Applying the operator δ^{-k} in (29) amounts to saying that

$$\delta^k(x).\xi\delta^k(x), \delta^k(x) + \xi\delta^k(x), \mathbf{1}, \quad (38)$$

are linearly dependent. But $\delta^k(x).\xi\delta^k(x)$ and $\delta^k(x) + \xi\delta^k(x)$ are elements of $\mathbb{C}(x)$, and by (34), (31), (32), they are in fact ratios of **polynomials of degree 2 in x with the same denominator.**

In addition, letting $\zeta_j(x) \stackrel{\text{def}}{=} \delta^j(x) + \xi\delta^j(x)$, the following recursive scheme holds.

$$\begin{cases} \zeta_0(x) &= 2x, \quad \zeta_1(x) = \delta^{-1}(x) + \delta(x), \\ \zeta_j(x) &= \zeta_{j-1}(\zeta_1(x)) - \zeta_{j-2}(x), \quad \forall j \geq 2. \end{cases} \quad (39)$$

► **Case $m = 2k - 1$.** Upon applying here the operator δ^{-k+1} in (29) and using the identity $\delta^{-k+1}(x) = \eta\delta^k(x)$, we obtain that

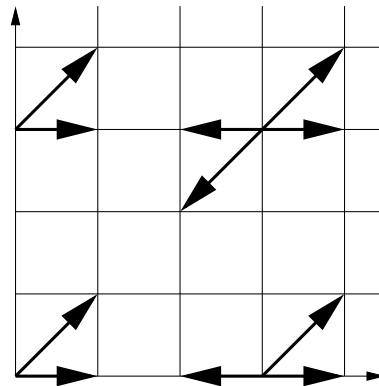
$$\delta^k(x).\eta\delta^k(x), \delta^k(x) + \eta\delta^k(x), \mathbf{1} \quad (40)$$

are linearly dependent. Moreover, $\delta^k(x).\eta\delta^k(x)$ and $\delta^k(x) + \eta\delta^k(x)$ are elements of $\mathbb{C}(y)$, namely ratios of **polynomials of degree 2 in y with the same denominator.**

Proposition 12. *The group \mathcal{H} is of order 8 if, and only if, the third order determinant*

$$\begin{vmatrix} 2\Delta_{22}\Delta_{32} & -2(\Delta_{22}^2 + \Delta_{12}\Delta_{32} - \Delta_{21}\Delta_{23}) & 2\Delta_{12}\Delta_{22} \\ -(\Delta_{21}\Delta_{33} + \Delta_{31}\Delta_{23}) & +\Delta_{11}\Delta_{33} + \Delta_{31}\Delta_{13} & -(\Delta_{11}\Delta_{23} + \Delta_{21}\Delta_{13}) \\ \Delta_{32}^2 - \Delta_{31}\Delta_{33} & -2\Delta_{32}\Delta_{22} + \Delta_{31}\Delta_{23} + \Delta_{21}\Delta_{33} & \Delta_{22}^2 - \Delta_{21}\Delta_{23} \\ \Delta_{22}^2 - \Delta_{21}\Delta_{23} & -2\Delta_{22}\Delta_{12} + \Delta_{11}\Delta_{23} + \Delta_{13}\Delta_{21} & \Delta_{12}^2 - \Delta_{11}\Delta_{13} \end{vmatrix}$$

is equal to zero, where Δ_{ij} denotes the cofactor of the $(i, j)^{th}$ entry of the matrix \mathbb{P} given in (11). ■



Example: *Gessel's walk.*

Criterion for Groups of Order $4m - 2$

Here

$$\delta^{2m-1} = I, \quad (41)$$

which by Proposition 3 and Corollary 4 is equivalent to

$$\eta(\delta^m(x)) = \delta^m(x),$$

that is

$$\delta^m(x) = G(y) \in \mathbb{C}(y). \quad (42)$$

Similarly, upon applying (41) to y , we get

$$\delta^{-m}(y) = \delta^{m-1}(y) = \xi(\delta^{-m}(y)),$$

whence

$$\delta^{-m}(y) = F(x) \in \mathbb{C}(x).$$

Applying now δ^{-m} to both members of (42) yields

$$x = \delta^{-m}(G(y)) = G(\delta^{-m}(y)) = G \circ F(x),$$

which shows that $G \circ F = I$, and hence G and F are **simple fractional linear transforms**.

► Setting for instance

$$G(y) = -\frac{py + q}{ry + s},$$

where p, q, r, s are arbitrary complex constants, the problem is to achieve the linear relation

$$r y \delta^m(x) + s \delta^m(x) + py + q = 0 \quad \text{mod } Q(x, y), \quad (43)$$

which is necessary and sufficient for the group to be of order $4m - 2$.

Final results

For the group to be finite, there is a **unique condition tantamount to the cancellation of a determinant**, the elements of which are intricate functions of the coefficients of the transition matrix \mathbb{P} , but nonetheless recursively computable.

- ▶ The determinant is of order 3, for groups of order $4m$, $m \geq 1$.
- ▶ The determinant is of order 4, for groups of order $4m - 2$, $m \geq 1$.
- ▶ The condition depends on the entries of the matrix \mathbb{P} in a polynomial way, as shown in the next three theorems.

Theorem 13. For any integer $s \geq 1$, we have

$$\delta^s(x) = \frac{yU_s(x) + V_s(x)}{W_s(x)} \pmod{Q(x, y)}, \quad (44)$$

where U_s, V_s, W_s are second degree polynomials.

Theorem 14. The finiteness of the group is always equivalent to the cancellation of a single constant, which depends on the entries of \mathbb{P} in a polynomial way. In other words, the group is finite if and only if the non-negative (p_{ij}) 's belong to the intersection of some algebraic hypersurface with the hyperplane $\sum_{\mathcal{S}} p_{ij} = 1$.

Theorem 15. For the group \mathcal{H} to be finite, the necessary and sufficient condition is $\det(\Omega) = 0$, where Ω is a matrix of order 3 (resp. 4) when the group is of order $4m$ (resp. $4m+2$).

► Sketch of proofs.

Let

$$\delta^s(x) - \xi \delta^s(x) \stackrel{\text{def}}{=} 2H, \quad X = \wp(\omega), \quad Y = \wp(s\omega_3).$$

Then

$$\delta^s(x) = \frac{S(\omega, s\omega_3)}{2} + H,$$

with $S(\omega, s\omega_3)$ given by (35).

$$H = \frac{q[\wp(s\omega_3 - \omega) - \wp(s\omega_3 + \omega)]}{2[\wp(s\omega_3 + \omega) - r][\wp(s\omega_3 - \omega) - r]} = \frac{q \wp'(\omega) \wp'(s\omega_3)}{D(X, Y)} = \frac{2q^2 \wp'(s\omega_3)[2a(x)y + b(x)]}{(x - p)^2 D(X, Y)}$$

where, by using (31), (32), (34), (35),

$$D(X, Y) = 2(X - Y)^2(B - rA + r^2)$$

is a second degree polynomial in X and Y .

On the other hand, by construction, we can a priori write

$$\delta^s(x) = M_s(x)y + N_s(x) \quad \text{mod } Q(x, y), \quad (45)$$

where M_s and N_s are rational fractions whose numerators and denominators are **polynomials** of (a priori) unknown degrees, but with coefficients given in terms of polynomials of the entries of \mathbb{P} . The decomposition (45) is unique, so that, comparing with (44), we have

$$\begin{cases} M_s(x) = \frac{4q^2 \wp'(s\omega_3)a(x)}{W_s(x)}, \\ N_s(x) = \frac{V_s(x)}{W_s(x)}, \end{cases}$$

where V_s, W_s are the second degree polynomials given by Theorem 13.

By homogeneity, we can always rewrite

$$M_s(x) = \frac{A_s a(x)}{F_s(x)},$$

where A_s, K_s are real constants with

$$A_s = K_s[4q^2 \wp'(s\omega_3)], \quad F_s(x) = K_s W_s(x).$$

► By Corollary 4, the group is of order $4s$ if and only if $M_s \equiv 0$, that is $A_s = 0$, where now A_s depends only on the entries of \mathbb{P} in a complicated polynomial form. It is also equivalent to $\wp'(s\omega_3) = 0$.

► When the group is of order $4s + 2$, exchange the role of x and y by uniformizing $y(\omega)$. Then, *mutatis mutandis*, this yields

$$\delta^s(x) = \frac{\tilde{A}_s \tilde{a}(y)x + \tilde{V}_s(y)}{\tilde{F}_s(y)} \quad \text{mod } Q(x, y),$$

where \tilde{F}_s, \tilde{V}_s are second degree polynomials. . . ■

About the genus 0 case

Here the Riemann surface $\mathcal{K} = \{(x, y) \in \mathbb{C}^2 : Q(x, y) = 0\}$ is of **genus 0 (the Riemann Sphere)**, thus admitting a uniformization in terms of **simple rational functions**.

For all non-singular random walks, \mathbf{S} has genus 0 if, and only if, one of the following relations holds:

$$M_x = M_y = 0, \quad (46)$$

$$p_{10} = p_{11} = p_{01} = 0, \quad (47)$$

$$p_{10} = p_{1,-1} = p_{0,-1} = 0, \quad (48)$$

$$p_{-1,0} = p_{-1,-1} = p_{0,-1} = 0, \quad (49)$$

$$p_{01} = p_{-1,0} = p_{-1,1} = 0. \quad (50)$$

Define the drift $\vec{\mathbf{M}} = (\sum i p_{ij}, \sum j p_{ij})$ and $\theta = \arccos(-r)$, where r denotes the **correlation coefficient**.

Theorem. [1, 3]

(a) When $\vec{M} = 0$, *the group \mathcal{H} is finite if and only if θ/π is rational*, in which case its order is equal to

$$2 \inf\{\ell \in \mathbb{Z}_+^* : \ell\theta/\pi \in \mathbb{Z}\}.$$

(b) When $\vec{M} \neq 0$, *the order of \mathcal{H} is always infinite* in the four remaining cases ■

Sketch of proof of part (a). *The main idea consists in working by continuity from the genus 1 case!* Now letting $\vec{M} \rightarrow 0$, so that $x_2, x_3 \rightarrow 1$, we have

$$\left\{ \begin{array}{l} \omega_1 \rightarrow i\infty, \\ \omega_2 \rightarrow \alpha_2 = \frac{\pi}{[C(x_4 - 1)(1 - x_1)]^{1/2}}, \\ \omega_3 \rightarrow \alpha_3 = \int_{X_0(y_1)}^{x_1} \frac{dx}{(1 - x)[C(x - x_1)(x - x_4)]^{1/2}}. \\ \frac{\theta}{\pi} = \lim_{\vec{M} \rightarrow 0} \frac{\omega_2}{\omega_3} = \frac{\alpha_2}{\alpha_3}. \end{array} \right. \quad (51)$$

► **Sketch of proof of part (b).**

By symmetry, it suffices to consider the case $p_{10} = p_{1,-1} = p_{0,-1} = 0$. Then

$$\begin{cases} \omega_1 \rightarrow i\alpha_1, \text{ with } \alpha_1 \in (0, \infty), \\ \omega_2 \rightarrow \infty, \\ \omega_3 \rightarrow \alpha_3 \in (0, \infty). \end{cases}$$

Hence, the limit group can be interpreted as the group of transformations

$$\langle \omega \mapsto -\omega, \omega \mapsto -\omega + \alpha_3 \rangle$$

on $\mathbb{C}/(\alpha_1\mathbb{Z})$. This group is obviously infinite, and so is \mathcal{H} .

As for combinatorics..?

Let $f(i, j, k)$ denote the number of paths starting from $(0, 0)$ and ending at (i, j) at time k (or after k steps). Then the corresponding CGF

$$F(x, y, z) = \sum_{i, j, k \geq 0} f(i, j, k) x^i y^j z^k \quad (52)$$

satisfies the functional equation

$$K(x, y, z)F(x, y, z) = c(x)F(x, 0, z) + \tilde{c}(y)F(0, y, z) + c_0(x, y, z), \quad (53)$$

where

$$K(x, y, z) = xy \left[\sum_{(i, j) \in \mathcal{S}} x^i y^j - 1/z \right].$$

► The same machinery applies (the algebra is even simpler), but note here that the group depends on the parameter z .

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- [2] **G. FAYOLLE AND R. IASNOGOROSKI.** Random Walks in the Quarter Plane: Advances in Explicit Criteria for the Finiteness of the Associated Group in the Genus 1 Case. *Markov Processes and Related Fields*, 21(4), Dec 2015.
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Thank you for your attention!

But, what to do now ?

The trick will be to avoid the pitfalls, seize the opportunities, and get back home by six o'clock.

[Woody Allen, *My Speech to the Graduates*, Side Effects, 1980].