Computer Algebra for Lattice Path Combinatorics

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Computer algebra = effective mathematics + algebraic complexity

- effective mathematics: what can be computed?
- algebraic complexity: how fast?

Efficient computer algebra for functional equations

- equations as data structures
- algorithmic proofs of identities
- complexity-driven algorithms

$$\pm \sqrt{5} \text{ as } t^2 - 5 = 0$$

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

$$3^N \text{ in } \tilde{O}(N)$$

Ultimate goals

- automatic computations on functional equations
- computer-driven resolution of difficult problems
Research context

Computer algebra = effective mathematics + algebraic complexity
- effective mathematics: what can be computed?
- algebraic complexity: how fast?

Efficient computer algebra for functional equations
- equations as data structures
  \[ \exp(t) \text{ as } y'(t) = y(t), y(0) = 1 \]
- algorithmic proofs of identities
  \[ \sum_k (-1)^k \binom{2n}{k}^3 = (-1)^n \binom{3n}{n} \binom{2n}{n} \]
- complexity-driven algorithms
  \[ N! = 1 \times 2 \times \cdots \times N \text{ in } \tilde{O}(N) \]

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- automatic computations on functional equations
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**Computer algebra** = effective mathematics + algebraic complexity
- effective mathematics: *what* can be computed?
- algebraic complexity: *how fast?*

**Efficient computer algebra for functional equations**
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  \[ \exp(t) \text{ as } y'(t) = y(t), y(0) = 1 \]
- algorithmic proofs of identities
  \[ \sum_k (-1)^k \binom{2n}{k}^3 = (-1)^n \frac{(3n)_n}{n!} \binom{2n}{n} \]
- complexity-driven algorithms
  \[ N! = 1 \times 2 \times \cdots \times N \text{ in } \tilde{O}(N) \]

**Ultimate goals**
- automatic computations on functional equations
- computer-driven resolution of difficult problems e.g., in combinatorics
An (innocent looking) combinatorial question

Let $\mathcal{S} = \{↑, ←, ↓\}$. A $\mathcal{S}$-walk is a path in $\mathbb{Z}^2$ using only steps from $\mathcal{S}$. Show that, for any integer $n$, the following quantities are equal:

(i) the number $a_n$ of $\mathcal{S}$-walks of length $n$ confined to the upper half plane $\mathbb{Z} \times \mathbb{N}$ that start and end at the origin $(0,0)$;

(ii) the number $b_n$ of $\mathcal{S}$-walks of length $n$ confined to the quarter plane $\mathbb{N}^2$ that start at the origin $(0,0)$ and finish on the diagonal $x = y$. 
An (innocent looking) combinatorial question

Let $\mathcal{S} = \{\uparrow, \leftarrow, \searrow\}$. A $\mathcal{S}$-walk is a path in $\mathbb{Z}^2$ using only steps from $\mathcal{S}$. Show that, for any integer $n$, the following quantities are equal:

(i) the number $a_n$ of $\mathcal{S}$-walks of length $n$ confined to the upper half plane $\mathbb{Z} \times \mathbb{N}$ that start and end at the origin $(0,0)$;

(ii) the number $b_n$ of $\mathcal{S}$-walks of length $n$ confined to the quarter plane $\mathbb{N}^2$ that start at the origin $(0,0)$ and finish on the diagonal $x = y$.

For instance, for $n = 3$, this common value is $a_3 = b_3 = 3$: 

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{example}
\end{figure}
Teaser 1: This problem can be solved using computer algebra!

Teaser 2: The answer has a nice closed form!

\[ a_{3n} = b_{3n} = \frac{(3n)!}{n!^2 \cdot (n + 1)!}, \quad \text{and} \quad a_m = b_m = 0 \quad \text{if 3 does not divide } m. \]

Teaser 3: A certain group attached to the step set \{↑, ←, \} is finite!
Let $\mathcal{S}$ be a subset of $\mathbb{Z}^d$ (step set, or model) and $p_0 \in \mathbb{Z}^d$ (starting point).

Example: $\mathcal{S} = \{(1,0), (-1,0), (1,-1), (-1,1)\}$, $p_0 = (0,0)$
Let $\mathcal{S}$ be a subset of $\mathbb{Z}^d$ (step set, or model) and $p_0 \in \mathbb{Z}^d$ (starting point).

A path (walk) of length $n$ starting at $p_0$ is a sequence $(p_0, p_1, \ldots, p_n)$ of elements in $\mathbb{Z}^d$ such that $p_{i+1} - p_i \in \mathcal{S}$ for all $i$.

Example: $\mathcal{S} = \{(1,0), (-1,0), (1,-1), (-1,1)\}$, $p_0 = (0,0)$
Combinatorial context: lattice paths confined to cones

Let $\mathcal{S}$ be a subset of $\mathbb{Z}^d$ (step set, or model) and $p_0 \in \mathbb{Z}^d$ (starting point).

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Let $\mathcal{C}$ be a cone of $\mathbb{R}^d$ (if $x \in \mathcal{C}$ and $r \geq 0$ then $r \cdot x \in \mathcal{C}$).

Example: $\mathcal{S} = \{(1,0), (-1,0), (1,-1), (-1,1)\}$, $p_0 = (0,0)$ and $\mathcal{C} = \mathbb{R}^2_+$
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Example: $\mathcal{S} = \{(1,0), (-1,0), (1,-1), (-1,1)\}$, $p_0 = (0,0)$ and $\mathcal{C} = \mathbb{R}^2_+$

Questions
- What is the number $a_n$ of $n$-step walks contained in $\mathcal{C}$?
- For $i \in \mathcal{C}$, what is the number $a_{n;i}$ of such walks that end at $i$?
- What about their GF’s $A(t) = \sum_n a_n t^n$ and $A(t;x) = \sum_{n,i} a_{n;i} x^i t^n$?
Why count walks in cones?

Many discrete objects can be encoded in that way:

- discrete mathematics (permutations, trees, words, urns, …)
- statistical physics (Ising model, …)
- probability theory (branching processes, games of chance, …)
- operations research (queueing theory, …)
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- probability theory (branching processes, games of chance, . . .
- operations research (queueing theory, . . .

A history and a survey of lattice path enumeration

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ABSTRACT
In celebration of the Sixth International Conference on Lattice Path Counting and Applications, it is befitting to review the history of lattice path enumeration and to survey how the topic has progressed thus far.

We start the history with early games of chance specifically the ruin problem which later appears as the ballot problem. We discuss André's Reflection Principle and its misnomer, its relation with the method of images and possible origins from physics and Brownian motion, and the earliest evidence of lattice path techniques and solutions. In the survey, we give representative articles on lattice path enumeration found in the literature in the last 35 years by the lattice, step set, boundary, characteristics counted, and solution method. Some of this work appears in the author's 2005 dissertation.
An old topic: ballot problem [Bertrand, 1887]

Suppose that candidates $A$ and $B$ are running in an election. If $a$ votes are cast for $A$ and $b$ votes are cast for $B$, where $a > b$, then the probability that $A$ stays ahead of $B$ throughout the counting of the ballots is $(a - b)/(a + b)$.

Lattice path reformulation: find the number of paths that start at the origin and never touch the $x$-axis, consisting of $a$ upsteps $\uparrow$ and $b$ downsteps $\downarrow$.

Reflection principle [Aebly, 1923]: paths in $\mathbb{N}^2$ from $(1, 1)$ to $T(a + b, a - b)$ that do touch the $x$-axis are in bijection with paths in $\mathbb{Z}^2$ from $(1, -1)$ to $T$.

Answer: $(\text{paths in } \mathbb{Z}^2 \text{ from } (1, 1) \text{ to } T) - (\text{paths in } \mathbb{Z}^2 \text{ from } (1, -1) \text{ to } T)$

$$\left(\begin{array}{c} a + b - 1 \\ a - 1 \end{array}\right) - \left(\begin{array}{c} a + b - 1 \\ b - 1 \end{array}\right) = \frac{a - b}{a + b} \left(\begin{array}{c} a + b \\ a \end{array}\right)$$
An old topic: ballot problem [Bertrand, 1887]

Suppose that candidates $A$ and $B$ are running in an election. If $a$ votes are cast for $A$ and $b$ votes are cast for $B$, where $a > b$, then the probability that $A$ stays ahead of $B$ throughout the counting of the ballots is $(a - b)/(a + b)$.

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Answer: when $a = n + 1$ and $b = n$, this is the Catalan number

$$C_n = \frac{1}{2n+1} \binom{2n+1}{n+1} = \frac{1}{n+1} \binom{2n}{n}$$
...but still a topical issue
A (very) basic cone: the full space

Rational series [folklore]

If $\mathcal{G} \subset \mathbb{Z}^d$ is finite and $\mathcal{C} = \mathbb{R}^d$, then

$$a_n = |\mathcal{G}|^n, \text{ i.e. } A(t) = \sum_{n\geq 0} a_n t^n = \frac{1}{1 - |\mathcal{G}| t}.$$ 

More generally:

$$A(t; x) = \sum_{n,i} a_{n;i} x^i t^n = \frac{1}{1 - t \sum_{s\in\mathcal{G}} x^s}.$$
Also well-known: a (rational) half-space

Algebraic series [Bousquet-Mélou, Petkovšek, 2000]

If $\mathcal{S} \subset \mathbb{Z}^d$ is finite and $\mathcal{C}$ is a rational half-space, then $A(t; x)$ is algebraic, given by an explicit system of polynomial equations.

Example: For Dyck paths (ballot problem), $A(t; 1) = \sum_{n \geq 0} C_n t^n = \frac{1 - \sqrt{1 - 4t}}{2t}$
The “next” case: intersection of two half-spaces

\[ f(i, j; n) = \begin{cases} 
0 & \text{if } i < 0 \text{ or } j < 0 \text{ or } n < 0, \\
\sum_{i', j' \in S} f(i - i', j - j'; n - 1) & \text{otherwise}. 
\end{cases} \]
The “next” case: intersection of two half-spaces
Approach: Experimental Mathematics using Computer Algebra
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Algorithmes Efficaces en Calcul Formel

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Romain Lebreton
Grégoire Lecerf
Bruno Salvy
Éric Schost
Lattice walks with small steps in the quarter plane

- From now on: we focus on **nearest-neighbor walks in the quarter plane**, i.e. walks in $\mathbb{N}^2$ starting at $(0, 0)$ and using steps in a fixed subset $\mathcal{S}$ of

  \[ \{\searrow, \leftarrow, \nwarrow, \uparrow, \rightarrow, \nearrow, \downarrow\}. \]

- Example with $n = 45, i = 14, j = 2$ for:

  \[ \mathcal{S} = \]

  ![Diagram of lattice walks](image-url)
Lattice walks with small steps in the quarter plane

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$$\{\searrow, \leftarrow, \nwarrow, \uparrow, \nearrow, \rightarrow, \swarrow, \downarrow\}.$$ 

▷ Example with $n = 45$, $i = 14$, $j = 2$ for:

▷ Counting sequence: $f_{n;i,j} =$ number of walks of length $n$ ending at $(i, j)$.
Lattice walks with small steps in the quarter plane

▷ From now on: we focus on nearest-neighbor walks in the quarter plane, i.e. walks in \( \mathbb{N}^2 \) starting at \((0, 0)\) and using steps in a fixed subset \( \mathcal{S} \) of \( \{↙, ←, ↖, ↑, →, ↘, ↓\} \).

▷ Example with \( n = 45, i = 14, j = 2 \) for:

\[
\mathcal{S} = \begin{array}{c}
\uparrow & \downarrow \\
\downarrow & \uparrow \\
\end{array}
\]

▷ Counting sequence: \( f_{n;i,j} \) = number of walks of length \( n \) ending at \((i,j)\).

▷ Specializations:
  
  - \( f_{n;0,0} \) = number of walks of length \( n \) returning to origin ("excursions");
  
  - \( f_n = \sum_{i,j \geq 0} f_{n;i,j} \) = number of walks with prescribed length \( n \).
Complete generating function:

\[ F(t; x, y) = \sum_{n=0}^{\infty} \left( \sum_{i,j=0}^{\infty} f_{n;i,j} x^i y^j \right) t^n \in \mathbb{Q}[x, y][[t]]. \]
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Specializations:

- GF of excursions: \( F(t; 0, 0); \)
- GF of walks: \( F(t; 1, 1) = \sum_{n \geq 0} f_n t^n; \)
- GF of horizontal returns: \( F(t; 1, 0); \)
- GF of diagonal returns: \( "F(t; 0, \infty)" := [x^0] F(t; x, 1/x). \)
Generating functions and combinatorial problems

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Combinatorial questions:

Given \( \mathcal{S} \), what can be said about \( F(t; x, y) \), resp. \( f_{n;i,j} \), and their variants?

- **Structure** of \( F \): algebraic? transcendental? solution of ODE?
- **Explicit form** of \( F \)? of \( f_{n;i,j} \)?
- **Asymptotics** of \( f_{n;0,0} \) of \( f_n \)?
Generating functions and combinatorial problems

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- Asymptotics of \( f_{n;0,0} \)? of \( f_n \)?

Our goal: Use computer algebra to give computational answers.
Small-step models of interest

Among the $2^8$ step sets $\mathcal{S} \subseteq \{-1, 0, 1\}^2 \setminus \{(0,0)\}$, some are:

- trivial,
- simple,
- intrinsic to the half plane,
- symmetrical.

One is left with 79 interesting distinct models. Is any further classification possible?
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One is left with **79 interesting distinct models**.

*Is any further classification possible?*
The 79 models

Non-singular

Singular
The 79 models

Non-singular

Singular
Two important models: **Kreweras** and **Gessel** walks

\[ \mathcal{G} = \{\downarrow, \leftarrow, \uparrow\} \quad F_{\mathcal{G}}(t; x, y) \equiv K(t; x, y) \]

\[ \mathcal{G} = \{\uparrow, \swarrow, \leftarrow, \rightarrow\} \quad F_{\mathcal{G}}(t; x, y) \equiv G(t; x, y) \]

Example: A Kreweras excursion.
“Special” models

Dyck:

Motzkin:

Pólya:

Kreweras:

Gessel:

Gouyou-Beauchamps:

King walks:

Tandem walks:
Algebraic reformulation: solving a functional equation

Generating function: \( G(t; x, y) = \sum_{n=0}^{\infty} \sum_{i=0}^{n} \sum_{j=0}^{n} g_{n;i,j} t^n x^i y^j \in Q[x, y][[t]] \)

"Kernel equation":

\[
G(t; x, y) = 1 + t \left( xy + x + \frac{1}{xy} + \frac{1}{x} \right) G(t; x, y) \\
- t \left( \frac{1}{x} + \frac{1}{xy} \right) G(t; 0, y) - t \frac{1}{xy} (G(t; x, 0) - G(t; 0, 0))
\]
Algebraic reformulation: solving a functional equation

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Task: Solve this functional equation!
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\]

Task: For the other models – solve 78 similar equations!
Classification of univariate power series

- **algebraic**
- **D-finite**
- **hypergeometric**

**D-finite power series**

$S(t) = \sum_{n=0}^{\infty} s_n t^n \in \mathbb{Q}[[t]]$

- $\triangledown$ **algebraic** if $P(t, S(t)) = 0$ for some $P(x, y) \in \mathbb{Z}[x, y]\{0\}$;
- $\triangledown$ **D-finite** if $c_r(t) S(r)(t) + \cdots + c_0(t) S(t) = 0$ for some $c_i \in \mathbb{Z}[t]$, not all zero;
- $\triangledown$ **hypergeometric** if $s_{n+1}s_n \in \mathbb{Q}(n)$.
Classification of univariate power series

D-finite power series

algebraic

hypergeometric

\[ S(t) = \sum_{n=0}^{\infty} s_n t^n \in \mathbb{Q}[[t]] \] is

▷ algebraic if \( P(t, S(t)) = 0 \) for some \( P(x, y) \in \mathbb{Z}[x, y] \setminus \{0\}; \]
Classification of univariate power series

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- **hypergeometric** if \( \frac{s_{n+1}}{s_n} \in \mathbb{Q}(n) \). E.g.,

\[
\ln(1 - t); \quad \frac{\arcsin(\sqrt{t})}{\sqrt{t}}; \quad (1 - t)^\alpha, \alpha \in \mathbb{Q}
\]
Classification of univariate power series

\[ S(t) = \sum_{n=0}^{\infty} s_nt^n \in \mathbb{Q}[[t]] \text{ is } \]

\begin{itemize}
  \item \textit{algebraic} if \( P(t, S(t)) = 0 \) for some \( P(x, y) \in \mathbb{Z}[x, y] \setminus \{0\} \);
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  \item \textit{hypergeometric} if \( \frac{s_{n+1}}{s_n} \in \mathbb{Q}(n) \). E.g.,
\end{itemize}

\[ _2F_1\left( \begin{array}{c} a \ b \\ c \end{array} \bigg| t \right) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n} \frac{t^n}{n!}, \quad (a)_n = a(a+1) \cdots (a+n-1). \]
Classification of univariate power series

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- **algebraic** if \( P(t, S(t)) = 0 \) for some \( P(x, y) \in \mathbb{Z}[x, y] \setminus \{0\} \);

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\[
\binom{a}{b} \binom{c}{d} \binom{e}{t} = \sum_{n=0}^{\infty} \frac{(a)_n}{(d)_n} \frac{(b)n}{(e)n} \frac{(c)n}{n!} t^n, \quad (a)_n = a(a+1) \cdots (a+n-1).
\]
Classification of univariate power series

D-finite power series

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- algebraic if \( P(t, S(t)) = 0 \) for some \( P(x, y) \in \mathbb{Z}[x, y] \setminus \{0\} \);
- D-finite if \( c_r(t) S^{(r)}(t) + \cdots + c_0(t) S(t) = 0 \) for some \( c_i \in \mathbb{Z}[t] \), not all zero;
- hypergeometric if \( \frac{s_{n+1}}{s_n} \in \mathbb{Q}(n) \).

Theorem [Schwarz, 1873; Beukers, Heckman, 1989]

Characterization of \{ hypergeometric \} \( \cap \) \{ algebraic \}.
A multivariate power series $S \in \mathbb{Q}[[x, y, t]]$ is called algebraic if it is the root of a polynomial $P \in \mathbb{Q}[x, y, t, T]$. D-finite series are those that satisfy a system of linear partial differential equations with polynomial coefficients.
S ∈ ℚ[[x, y, t]] is **algebraic** if it is the root of a polynomial \( P ∈ ℚ[ x, y, t, T] \).

S ∈ ℚ[[x, y, t]] is **D-finite** if it satisfies a system of linear partial differential equations with polynomial coefficients

\[
\sum_i a_i(t, x, y) \frac{\partial^i S}{\partial x^i} = 0, \quad \sum_i b_i(t, x, y) \frac{\partial^i S}{\partial y^i} = 0, \quad \sum_i c_i(t, x, y) \frac{\partial^i S}{\partial t^i} = 0.
\]
Main results (I): algebraicity of Gessel walks

**Theorem** [Kreweras, 1965; 100 pages long combinatorial proof!]

\[ K(t; 0, 0) = 3F_2 \left( \begin{array}{ccc} 1/3 & 2/3 & 1 \\ 3/2 & 2 \\ & & & & 27t^3 \end{array} \right) = \sum_{n=0}^{\infty} \frac{4^n(3^n)}{(n+1)(2n+1)} t^{3n}. \]

**Theorem** [Kauers, Koutschan, Zeilberger, 2009: former Gessel’s conj. 1]

\[ G(t; 0, 0) = 3F_2 \left( \begin{array}{ccc} 5/6 & 1/2 & 1 \\ 5/3 & 2 \\ & & & & 16t^2 \end{array} \right) = \sum_{n=0}^{\infty} \frac{(5/6)_n(1/2)_n}{(5/3)_n(2)_n} (4t)^{2n}. \]

**Question:** What about the structure of \( K(t; x, y) \) and \( G(t; x, y) \)?
Main results (I): algebraicity of Gessel walks

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▷ Computer-driven discovery and proof.
▷ **Guess’n’Prove method**, using **Hermite-Padé approximants**

\[ \text{Minimal polynomial } P(x, y, t, G(t; x, y)) = 0 \text{ has } > 10^{11} \text{ terms; } \approx 30 \text{ Gb (!)} \]
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▷ Computer-driven discovery and proof.
▷ **Guess’n’Prove** method, using Hermite-Padé approximants†

▷ Recent (human) proofs [B., Kurkova, Raschel, 2013; Bousquet-Mélou, 2015]

\[ \text{Minimal polynomial } P(x, y, t, G(t; x, y)) = 0 \text{ has } > 10^{11} \text{ terms; } \approx 30 \text{ Gb} (!) \]
Main results (II): Explicit form for $G(t; x, y)$

**Theorem [B., Kauers, van Hoeij, 2010]**

Let $V = 1 + 4t^2 + 36t^4 + 396t^6 + \cdots$ be a root of

$$(V - 1)(1 + 3/V)^3 = (16t)^2,$$

let $U = 1 + 2t^2 + 16t^4 + 2xt^5 + 2(x^2 + 83)t^6 + \cdots$ be a root of

$$x(V - 1)(V + 1)U^3 - 2V(3x + 5xV - 8Vt)U^2
- xV(V^2 - 24V - 9)U + 2V^2(xV - 9x - 8Vt) = 0,$$

let $W = t^2 + (y + 8)t^4 + 2(y^2 + 8y + 41)t^6 + \cdots$ be a root of

$$y(1 - V)W^3 + y(V + 3)W^2 - (V + 3)W + V - 1 = 0.$$

Then $G(t; x, y)$ is equal to

$$\frac{64(U(V+1)-2V)V^{3/2}}{x(U^2-V(U^2-8U+9-V))^2} - \frac{y(W-1)^4(1-Wy)V^{-3/2}}{t(y+1)(1-W)(W^2y+1)^2} - \frac{1}{tx(y+1)}.$$

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\]

▷ Computer-driven discovery and proof.
▷ Proof uses guessed minimal polynomials for \( G(t; x, 0) \) and \( G(t; 0, y) \).
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▷ Computer-driven discovery and proof.
▷ Proof uses guessed minimal polynomials for $G(t;x,0)$ and $G(t;0,y)$.
▷ Recent (human) proofs [B., Kurkova, Raschel, 2013; Bousquet-Mélou, 2015]
What is “scientific method”? Philosophers and non-philosophers have discussed this question and have not yet finished discussing it. Yet as a first introduction it can be described in three syllables:

Guess and test.

Mathematicians too follow this advice in their research although they sometimes refuse to confess it. They have, however, something which the other scientists cannot really have. For mathematicians the advice is

First guess, then prove.
A typical *guess’n’prove* algorithmic proof

**Theorem**

\[
g(t) := G(\sqrt{t}; 0, 0) = \sum_{n=0}^{\infty} \frac{(5/6)_n (1/2)_n}{(5/3)_n (2)_n} (16t)^n \text{ is algebraic.}
\]
A typical *guess’n’prove* algorithmic proof

**Theorem**

\[ g(t) := G(\sqrt{t}; 0, 0) = \sum_{n=0}^{\infty} \frac{(5/6)^n (1/2)^n (16t)^n}{(5/3)^n (2)^n} \] is algebraic.

**Proof:** First **guess** a polynomial \( P(t, T) \) in \( \mathbb{Q}[t, T] \), then **prove** that \( P \) admits the power series \( g(t) = \sum_{n=0}^{\infty} g_n t^n \) as a root.
A typical guess’n’prove algorithmic proof

Theorem

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1. Find \( P \) such that \( P(t, g(t)) = 0 \mod t^{100} \) by (structured) linear algebra.
A typical *guess’n’prove* algorithmic proof

**Theorem**

\[
g(t) := G(\sqrt{t}; 0, 0) = \sum_{n=0}^{\infty} \frac{(5/6)^n(1/2)^n}{(5/3)^n(2)^n} (16t)^n \quad \text{is algebraic.}
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**Proof:** First **guess** a polynomial \( P(t, T) \) in \( \mathbb{Q}[t, T] \), then **prove** that \( P \) admits the power series \( g(t) = \sum_{n=0}^{\infty} g_n t^n \) as a root.

1. Find \( P \) such that \( P(t, g(t)) = 0 \mod t^{100} \) by (structured) linear algebra.

2. **Implicit function theorem:** \( \exists! \) root \( r(t) \in \mathbb{Q}[[t]] \) of \( P \).
A typical *guess’n’prove* algorithmic proof

**Theorem**

\[ g(t) := G(\sqrt{t}; 0, 0) = \sum_{n=0}^{\infty} \frac{(5/6)^n (1/2)^n}{(5/3)^n (2)^n} (16t)^n \]

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1. Find \( P \) such that \( P(t, g(t)) = 0 \mod t^{100} \) by (structured) linear algebra.

2. Implicit function theorem: \( \exists! \) root \( r(t) \in \mathbb{Q}[[t]] \) of \( P \).

3. \( r(t) = \sum_{n=0}^{\infty} r_n t^n \) being algebraic, it is D-finite, and so is \( (r_n) \):

   \[ (n + 2)(3n + 5)r_{n+1} - 4(6n + 5)(2n + 1)r_n = 0, \quad r_0 = 1 \]

\[ \Rightarrow \text{solution } r_n = \frac{(5/6)^n (1/2)^n}{(5/3)^n (2)^n} 16^n = g_n, \text{ thus } g(t) = r(t) \text{ is algebraic.} \]
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Equation sizes = (order, degree)

▷ Computerized discovery: enumeration + guessing [B., Kauers, 2009]
▷ 1–22: Confirmed by human proofs in [Bousquet-Mélou, Mishna, 2010]
▷ 23: Confirmed by a human proof in [B., Kurkova, Raschel, 2013]
Main results (III): Models with D-Finite $F(t; 1, 1)$

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$A = 1 + \sqrt{2}, \quad B = 1 + \sqrt{3}, \quad C = 1 + \sqrt{6}, \quad \lambda = 7 + 3\sqrt{6}, \quad \mu = \sqrt{\frac{4\sqrt{6} - 1}{19}}$

▶ Computerized discovery: conv. acc. + LLL/PSLQ [B., Kauers, 2009]
▶ Confirmed by human proofs using ACSV in [Melnzer, Wilson, 2015]
The group of a model: the simple walk case

The characteristic polynomial \( \chi_\mathcal{G} := x + \frac{1}{x} + y + \frac{1}{y} \)
The characteristic polynomial \( \chi_S := x + \frac{1}{x} + y + \frac{1}{y} \) is left invariant under
\[
\psi(x, y) = \left( x, \frac{1}{y} \right), \quad \phi(x, y) = \left( \frac{1}{x}, y \right),
\]
The group of a model: the simple walk case

The characteristic polynomial \( \chi_S := x + \frac{1}{x} + y + \frac{1}{y} \) is left invariant under

\[
\psi(x, y) = \left( x, \frac{1}{y} \right), \quad \phi(x, y) = \left( \frac{1}{x}, y \right),
\]

and thus under any element of the group

\[
\langle \psi, \phi \rangle = \left\{ (x, y), \left( x, \frac{1}{y} \right), \left( \frac{1}{x}, \frac{1}{y} \right), \left( \frac{1}{x}, y \right) \right\}.
\]
The polynomial \( \chi_{\mathcal{G}} := \sum_{(i,j) \in \mathcal{G}} x^i y^j = \sum_{i=-1}^{1} B_i(y) x^i = \sum_{j=-1}^{1} A_j(x) y^j \)
The polynomial $\chi_S := \sum_{(i,j) \in S} x^i y^j = \sum_{i=-1}^{1} B_i(y) x^i = \sum_{j=-1}^{1} A_j(x) y^j$ is left invariant under

$$\psi(x, y) = \left(x, \frac{A_{-1}(x) y}{A_{+1}(x)} \right), \quad \phi(x, y) = \left(\frac{B_{-1}(y) x}{B_{+1}(y)} y, y \right),$$

where $\psi$ and $\phi$ are transformations that leave the polynomial invariant.
The group of a model: the general case

The polynomial

\[ \chi_S := \sum_{(i,j) \in S} x^i y^j = \sum_{i=-1}^{1} B_i(y) x^i = \sum_{j=-1}^{1} A_j(x) y^j \]

is left invariant under

\[ \psi(x, y) = \left( x, \frac{A_{-1}(x)}{A_{+1}(x)} \frac{1}{y} \right), \quad \phi(x, y) = \left( \frac{B_{-1}(y)}{B_{+1}(y)} \frac{1}{x}, y \right), \]

and thus under any element of the group

\[ G_S := \langle \psi, \phi \rangle. \]
Examples of groups

Order 4,

Order ∞
Examples of groups

Order 4,

order 6,
Examples of groups

Order 4, order 6, order 8,
Examples of groups

Order 4,

order 6,

order 8,

order $\infty$. 
An important concept: the orbit sum (OS)

When $G_S$ is finite, the orbit sum of $S$ is the polynomial in $\mathbb{Q}[x, x^{-1}, y, y^{-1}]$:

$$OS_S := \sum_{\theta \in G_S} (-1)^\theta \theta(xy)$$

- E.g., for the simple walk, with $G_S = \{(x, y), (x, \frac{1}{y}), (\frac{1}{x}, \frac{1}{y}), (\frac{1}{x}, y)\}$:

  $$OS = x \cdot y - \frac{1}{x} \cdot y + \frac{1}{x} \cdot \frac{1}{y} - x \cdot \frac{1}{y}$$

- For 4 models, the orbit sum is zero:

  E.g., for the Kreweras model:

  $$OS = x \cdot y - \frac{1}{xy} \cdot y + \frac{1}{xy} \cdot x - y \cdot x + y \cdot \frac{1}{xy} - x \cdot \frac{1}{xy} = 0$$
The 79 models: finite and infinite groups

79 models
The 79 models: finite and infinite groups

- 23 admit a finite group
  - [Mishna’07]

- 56 have an infinite group
  - [Bousquet-Mélou, Mishna’10]

- 79 models

- 19 transcendental (OS \( \not= 0 \))
  - [Gessel, Zeilberger’92]
  - [Bousquet-Mélou’02]

- 4 algebraic (OS \( = 0 \))
  - (3 Kreweras-type + Gessel)
  - [BMM’10] + [B., Kauers’10]

- \( \rightarrow \) all non-D-finite

- [Mishna, Rechnitzer’07] and [Melczer, Mishna’13] for 5 singular models

- [Kurkova, Raschel’13] and [B., Raschel, Salvy’13] for all others

Alin Bostan

Computer Algebra for Lattice Path Combinatorics
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The kernel $J = 1 - t \cdot \sum_{(i,j) \in S} x^i y^j = 1 - t \left( x + \frac{1}{x} + y + \frac{1}{y} \right)$ is invariant under the change of $(x, y)$ into, respectively:

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The kernel $J = 1 - t \cdot \sum_{(i,j) \in \mathcal{G}} x^i y^j = 1 - t \left( x + \frac{1}{x} + y + \frac{1}{y} \right)$ is invariant under the change of $(x, y)$ into, respectively:

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Kernel equation:

\[J(t; x, y)xyF(t; x, y) = xy - txF(t; x, 0) - tyF(t; 0, y)\]
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\[
J(t; x, y)\frac{1}{x} \frac{1}{y} F(t; \frac{1}{x}, \frac{1}{y}) = \frac{1}{x} \frac{1}{y} - t\frac{1}{x}F(t; \frac{1}{x}, 0) - t\frac{1}{y}F(t; 0, \frac{1}{y})
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Kernel equation:

\[
J(t; x, y) y F(t; x, y) = xy - t x F(t; x, 0) - t y F(t; 0, y)
\]

\[
- J(t; x, y) \frac{1}{x} y F(t; \frac{1}{x}, y) = - \frac{1}{x} y + t \frac{1}{x} F(t; \frac{1}{x}, 0) + t y F(t; 0, y)
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$J(t; x, y)xy F(t; x, y) = xy - txF(t; x, 0) - tyF(t; 0, y)$

$- J(t; x, y) \frac{1}{x} y F(t; \frac{1}{x}, y) = - \frac{1}{x} y + t \frac{1}{x} F(t; \frac{1}{x}, 0) + tyF(t; 0, y)$

$J(t; x, y) \frac{1}{x} \frac{1}{y} F(t; \frac{1}{x}, \frac{1}{y}) = \frac{1}{x} \frac{1}{y} - t \frac{1}{x} F(t; \frac{1}{x}, 0) - t \frac{1}{y} F(t; 0, \frac{1}{y})$

$- J(t; x, y) x \frac{1}{y} F(t; x, \frac{1}{y}) = - x \frac{1}{y} + txF(t; x, 0) + t \frac{1}{y} F(t; 0, \frac{1}{y})$

Summing up yields the orbit equation:

$$\sum_{\theta \in G} (-1)^{\theta} \theta (xy F(t; x, y)) = \frac{xy - \frac{1}{x} y + \frac{1}{x} \frac{1}{y} - x \frac{1}{y}}{J(t; x, y)}$$
The kernel \( J = 1 - t \cdot \sum_{(i,j) \in \mathcal{S}} x^i y^j = 1 - t \left( x + \frac{1}{x} + y + \frac{1}{y} \right) \) is invariant under the change of \((x, y)\) into, respectively:

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\[
J(t; x, y)xyF(t; x, y) = xy - txF(t; x, 0) - tyF(t; 0, y) \\
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\]

Taking positive parts yields:

\[
\left[ x > y \right] \sum_{\theta \in G} (-1)^\theta \theta (xy F(t; x, y)) = [x > y] \frac{xy - \frac{1}{x}y + \frac{1}{x}\frac{1}{y} - x\frac{1}{y}}{J(t; x, y)}
\]
The kernel $J = 1 - t \cdot \sum_{(i,j) \in S} x^i y^j = 1 - t \left( x + \frac{1}{x} + y + \frac{1}{y} \right)$ is invariant under the change of $(x, y)$ into, respectively:

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$$J(t; x, y)\frac{1}{x} \frac{1}{y} F(t; \frac{1}{x}, \frac{1}{y}) = \frac{1}{x} \frac{1}{y} - t\frac{1}{x}F(t; \frac{1}{x}, 0) - t\frac{1}{y} F(t; 0, \frac{1}{y})$$

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Summing up and taking positive parts yields:

$$xy F(t; x, y) = [x^y] \frac{xy - \frac{1}{x}y + \frac{1}{x} \frac{1}{y} - x \frac{1}{y}}{J(t; x, y)}$$
The kernel $J = 1 - t \cdot \sum_{(i,j) \in \mathcal{S}} x^i y^j = 1 - t \left( x + \frac{1}{x} + y + \frac{1}{y} \right)$ is invariant under the change of $(x, y)$ into, respectively:

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$$GF = \text{PosPart} \left( \frac{\text{OS}}{\text{kernel}} \right)$$
The kernel $J = 1 - t \cdot \sum_{(i,j) \in S} x^i y^j = 1 - t \left(x + \frac{1}{x} + y + \frac{1}{y}\right)$ is invariant under the change of $(x, y)$ into, respectively:

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J(t; x, y) \, x y F(t; x, y) = \ xy - t x F(t; x, 0) - t y F(t; 0, y)
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\[
GF = \text{PosPart} \left( \frac{OS}{\ker} \right) = \text{D-finite} \ [\text{Lipshitz, 1988}]
\]

\(\triangleright\) Argument works if \(OS \neq 0\): algebraic version of the reflection principle
The kernel \( J = 1 - t \cdot \sum_{(i,j) \in \mathcal{S}} x^i y^j = 1 - t \left( x + \frac{1}{x} + y + \frac{1}{y} \right) \) is invariant under the change of \((x,y)\) into, respectively:

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\]

▷ Creative Telescoping finds a differential equation for PosPart(OS/ker)
Theorem [B., Chyzak, van Hoeij, Kauers, Pech, 2016]

Let $\mathcal{G}$ be one of the 19 models with finite group $\mathcal{G}_\mathcal{S}$, and non-zero orbit sum. Then

- $F_{\mathcal{G}}$ is expressible using iterated integrals of $2F_1$ expressions.
- Among the $19 \times 4$ specializations of $F_{\mathcal{G}}(t; x, y)$ at $(x, y) \in \{0, 1\}^2$, only 4 are algebraic: for $\mathcal{G} = \begin{array}{c} \_ \\ \_ \end{array}$ at $(1, 1)$, and $\mathcal{G} = \begin{array}{cc} \_ & \_ \\ \_ & \_ \end{array}$ at $(1, 0), (0, 1), (1, 1)$
Main results (IV): explicit expressions for models 1–19

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Example (King walks in the quarter plane, A025595)

\[
F_{\ddag \downarrow}(t; 1, 1) = \frac{1}{t} \int_0^t \frac{1}{(1 + 4x)^3} \cdot 2F_1 \left( \begin{array}{c} 3 \ 3 \\ 2 \ 2 \end{array} \left| \frac{16x(1 + x)}{(1 + 4x)^2} \right. \right) dx
\]

\[
= 1 + 3t + 18t^2 + 105t^3 + 684t^4 + 4550t^5 + 31340t^6 + 219555t^7 + \cdots
\]
Main results (IV): explicit expressions for models 1–19

Theorem [B., Chyzak, van Hoeij, Kauers, Pech, 2016]
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Example (King walks in the quarter plane, A025595)

\[
F_{\bigtriangleup}(t; 1, 1) = \frac{1}{t} \int_0^t \frac{1}{(1 + 4x)^3} \cdot _2F_1\left(\frac{3}{2}, \frac{3}{2}; \frac{16x(1 + x)}{(1 + 4x)^2}\right) \, dx
\]

\[
= 1 + 3t + 18t^2 + 105t^3 + 684t^4 + 4550t^5 + 31340t^6 + 219555t^7 + \cdots
\]

- Computer-driven discovery and proof; no human proof yet.
- Proof uses creative telescoping, ODE factorization, ODE solving.
Hypergeometric Series Occurring in Explicit Expressions for $F(t; x, y)$

<table>
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<th>$\mathcal{S}$</th>
<th>occurring $\text{2F}_1$</th>
<th>$w$</th>
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</tr>
</tbody>
</table>

▷ All related to the complete elliptic integrals $\int_0^{\pi/2} (1 - k^2 \sin^2 \theta)^{\pm \frac{1}{2}} d\theta$
Main results (V): non-D-finiteness for models with an infinite group

**Theorem [B., Raschel, Salvy, 2013]**

Let $\mathcal{S}$ be one of the 51 non-singular models with infinite group $G_{\mathcal{S}}$. Then $F_{\mathcal{S}}(t;0,0)$, and in particular $F_{\mathcal{S}}(t;x,y)$, are non-D-finite.
Main results (V): non-D-finiteness for models with an infinite group

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- [Bernardi, Bousquet-Mélou, Raschel, 2016] For 9 of these 51 models, $F_\mathcal{G}(t;x,y)$ is nevertheless D-algebraic!
- [Dreyfus, Hardouin, Roques, Singer, 2017]: hypertranscendence of the remaining 42 models.
The 56 models with infinite group

In blue, non-singular models, solved by [B., Raschel, Salvy, 2013]
In red, singular models, solved by [Melmzer, Mishna, 2013]
Example: the scarecrows

[B., Raschel, Salvy, 2013]: $F_\square(t;0,0)$ is not D-finite for the models

\[ F_\square(t;0,0) = \text{not D-finite} \]

For the 1st and the 3rd, the excursions sequence $[t^n] F_\square(t;0,0)$

\[ 1, 0, 0, 2, 4, 8, 28, 108, 372, \ldots \]

is $\sim K \cdot 5^n \cdot n^{-\alpha}$, with $\alpha = 1 + \pi / \arccos(1/4) = 3.383396 \ldots$

[Denisov, Wachtel, 2013]

The irrationality of $\alpha$ prevents $F_\square(t;0,0)$ from being D-finite.

[Katz, 1970; Chudnovsky, 1985; André, 1989]
Summary: Classification of 2D non-singular walks

**The Main Theorem**  Let $\mathcal{S}$ be one of the 74 non-singular models. The following assertions are equivalent:

1. The full generating function $F_{\mathcal{S}}(t; x, y)$ is D-finite
2. the excursions generating function $F_{\mathcal{S}}(t; 0, 0)$ is D-finite
3. the excursions sequence $[t^n] F_{\mathcal{S}}(t; 0, 0)$ is $\sim K \cdot \rho^n \cdot n^\alpha$, with $\alpha \in \mathbb{Q}$
4. the group $\mathcal{G}_{\mathcal{S}}$ is finite (and $|\mathcal{G}_{\mathcal{S}}| = 2 \cdot \min\{\ell \in \mathbb{N}^\ast \mid \frac{\ell}{\alpha+1} \in \mathbb{Z}\}$)
5. the step set $\mathcal{S}$ has either an axial symmetry, or zero drift and cardinality different from 5.

Moreover, under (1)–(5), $F_{\mathcal{S}}(t; x, y)$ is algebraic if and only if the model $\mathcal{S}$ has positive covariance $\sum_{(i,j) \in \mathcal{S}} ij - \sum_{(i,j) \in \mathcal{S}} i \cdot \sum_{(i,j) \in \mathcal{S}} j > 0$, and iff it has OS $= 0$. In this case, $F_{\mathcal{S}}(t; x, y)$ is expressible using nested radicals. If not, $F_{\mathcal{S}}(t; x, y)$ is expressible using iterated integrals of 2 expressions.
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If not, $F_{\mathcal{G}}(t; x, y)$ is expressible using iterated integrals of $2F_1$ expressions.
Summary: Walks with small steps in $\mathbb{N}^2$

quadrant models $\mathcal{G}$: 79

$|\mathcal{G}| < \infty$: 23

- orbit sum $\neq 0$: 19
  - Kernel method + CT: D-finite

- orbit sum $= 0$: 4
  - Guess’n’Prove: algebraic

$|\mathcal{G}| = \infty$: 56

asymptotics + GB: non-D-finite
Extensions: Walks in $\mathbb{N}^2$ with small repeated steps

2D quadrant models: 527

$|G_S| < \infty$: 118

orbit sum $\neq 0$: 95

kernel method: 94

D-finite

CA: $\leftrightarrow$

D-finite

$|G_S| = \infty?: 409$

orbit sum $= 0$: 23

non-D-finite?

22: reducible to Kreweras/Gessel

CA: $\leftrightarrow$

D-finite

[Alin Bostan, Bousquet-Mélou, Kauers, Melczer, 2015]

1457 D-finite, 79 algebraic, 3 pearls:

[Du, Hou, Wang, 2015]: proofs that groups are infinite in the 409 cases, and GF are non-D-finite in 366 cases.

[Kauers, Yatchak, 2015]: extension to $4^8 = 65536$ models with mult. $\leq 3$. 
Extensions: Walks in $\mathbb{N}^2$ with small repeated steps

2D quadrant models: 527

$|G_\mathcal{G}| < \infty$: 118
- orbit sum $\neq 0$: 95
  - kernel method: 94
    - D-finite
  - CA: D-finite

$|G_\mathcal{G}| = \infty$: 409
- orbit sum $= 0$: 23
  - reducible to Kreweras/Gessel
  - CA: D-finite
  - CA: algebraic
  - non-D-finite?

[B., Bousquet-Mélou, Kauers, Melczer, 2015]

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  - 1457 D-finite, 79 algebraic, 3 pearls:
A pearl among models in $\mathbb{N}^2$ with small but repeated steps

**Theorem** [B., Bousquet-Mélou, Kauers, Melczer, 2015]

Let $e_n = \# \left\{ \text{walks of length } n \text{ in } \mathbb{N}^2 \text{ from } (0,0) \text{ to } (0,0) \right\}$

$$(e_n)_{n \geq 0} = (1, 0, 3, 0, 26, 0, 323, 0, 4830, 0, 80910, \ldots)$$

Then

$$e_{2n} = \frac{6(6n + 1)!(2n + 1)!}{(3n)!(4n + 3)!(n + 1)!}.$$  

▷ Current proof is computer-driven.
▷ Open problem: find a *human proof*.  

Alin Bostan  
Computer Algebra for Lattice Path Combinatorics
Extensions: Walks in $\mathbb{N}^2$ with large steps

quadrant models with steps in $\{-2, -1, 0, 1\}^2$: 13 110

| orbit $| < \infty$: 240 | orbit $| = \infty$: 12 870 |
| OS $\neq 0$: 431 | OS $= 0$: 9 |
| D-finite | D-finite? |
| $\alpha$ rational: 16 | $\alpha$ irrational: 12 854 |
| non-D-finite? | non-D-finite |

[B., Bousquet-Mélou, Melczer, 2017]

- Example: For the model

$$xyF(t; x, y) = [x^0 y^0] \frac{(x - 2x^{-2})(y - (x - x^{-2})y^{-1})}{1 - t(xy^{-1} + y + x^{-2}y^{-1})}$$
Two pearls among the 9 difficult models with large steps

**Conjecture 1 [B., Bousquet-Mélou, Melczer, 2017]**

For the model \( \downarrow \rightarrow \), writing \( \phi(t) = \frac{108 t(1+4t)^2}{(12t-1)^3} \), then \( F(t^{1/2};0,0) \) is equal to

\[
\frac{1}{3t} - \sqrt{\frac{1-12t}{6t}} \left( \binom{2}{F_1} \left( \binom{1}{6} \binom{1}{3} \phi(t) \right) + 2F_1 \left( \binom{-1}{6} \binom{2}{3} \phi(t) \right) \right).
\]

**Conjecture 2 [B., Bousquet-Mélou, Melczer, 2017]**

For the model \( \rightarrow \downarrow \), \( F(t;0,0) \) is equal to

\[
\frac{(1 - 24 U + 120 U^2 - 144 U^3)}{(1 - 3 U)(1 - 2 U)^{3/2}(1 - 6 U)^{9/2}} \left( 1 - 4 U \right),
\]

where \( U = t^4 + 53 t^8 + 4363 t^{12} + \cdots \) is the unique series in \( \mathbb{Q}[[t]] \) satisfying

\[
U \left( 1 - 2 U \right)^3 \left( 1 - 3 U \right)^3 \left( 1 - 6 U \right)^9 = t^4 \left( 1 - 4 U \right)^4.
\]
Extensions: Walks with small steps in $\mathbb{N}^3$

$2^{3^3} - 1 \approx 67$ million models, of which $\approx 11$ million inherently 3D

3D octant models $\mathcal{G}$ with $\leq 6$ steps: 20804

- $|\mathcal{G}| < \infty$: 170
- Orbit sum $\neq 0$: 108
  - Kernel method: D-finite
- Orbit sum $= 0$: 62
  - 2D-reducible: 43
  - Not 2D-reducible: 19
    - D-finite
    - Non-D-finite?

- $|\mathcal{G}| = \infty$?: 20634
  - Non-D-finite?

[B., Bousquet-Mélou, Kauers, Melczer, 2015]

▷ Open question: are there non-D-finite models with a finite group?
Extensions: Walks with small steps in $\mathbb{N}^3$

$2^{3^3} - 1 \approx 67$ million models, of which $\approx 11$ million inherently 3D

3D octant models $\mathcal{S}$ with $\leq 6$ steps: 20804

$|\mathcal{G}_\mathcal{S}| < \infty$: 170

orbit sum $\neq 0$: 108

kernel method: D-finite

orbit sum $= 0$: 62

2D-reducible: 43

not 2D-reducible: 19

$|\mathcal{G}_\mathcal{S}| = \infty$?: 20634

non-D-finite?

D-finite

non-D-finite?

[B., Bousquet-Mélou, Kauers, Melczer, 2015]

▷ Open question: are there non-D-finite models with a finite group?

▷ [Du, Hou, Wang, 2015]: proofs that groups are infinite in the 20634 cases

▷ [Bacher, Kauers, Yatchak, 2016]: extension to all 3D models; 170 models found with $|\mathcal{G}_\mathcal{S}| < \infty$ and orbit sum 0 (instead of 19)
19 mysterious 3D-models
Open question: 3D Kreheras

Two different computations suggest:

\[ k_{4n} \approx C \cdot \frac{256^n}{n^{3.3257570041744\ldots}} , \]

so excursions are very probably transcendental (and even non-D-finite)
Conclusion

- Computer algebra may solve difficult combinatorial problems
- Classification of $F(t; x, y)$ fully completed for 2D small step walks
- Robust algorithmic methods, based on efficient algorithms:
  - Guess’n’Prove
  - Creative Telescoping
- Brute-force and/or use of naive algorithms = hopeless.
  E.g. size of algebraic equations for $G(t; x, y) \approx 30$Gb.
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Computer algebra may solve difficult combinatorial problems

Classification of $F(t; x, y)$ fully completed for 2D small step walks

Robust algorithmic methods, based on efficient algorithms:
  - Guess’n’Prove
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Lack of “purely human” proofs for some results.

Open: is $F(t; 1, 1)$ non-D-finite for all 56 models with infinite group?

Many beautiful open questions for 2D models with repeated or large steps, and in dimension $> 2$. 
## Future reasearch

**Fundamental computer algebra**
- **structured** power series and matrices
- **basic operations** on operators (mod $p$)
- **factorization** of operators (mod $p$)
- Hermite-Padé approximants
- $\times$ and Hadamard $\odot$
- $p$-curvature

**Computer algebra for functional equations**
- **minimality** of operators (order vs. total size)
- faster guessing
- faster **Creative Telescoping**
- desingularisation
- structured and certified
- 4G, reduction-based

**Applications**
- **Combinatorics**
  - lattice walks
  - algorithmic hyper-transcendence
  - other classes of combinatorial objects
  - solving discrete PDEs
  - symmetries, various groups
  - diff. Galois and Tutte invariants
  - urns, maps
- **Number theory**
  - transcedence of values of E- and G-functions


Thanks for your attention!