Computer Algebra for Lattice Path Combinatorics

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informatics mathematics

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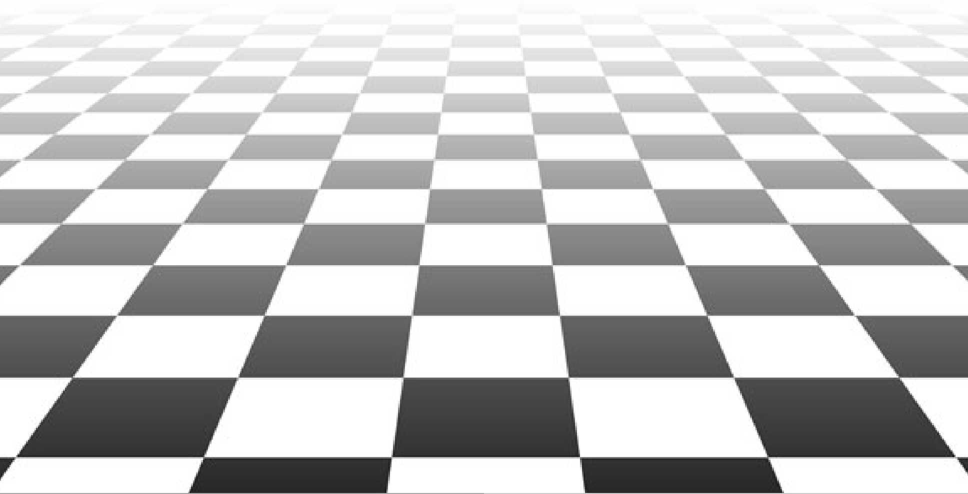
Overview

Part 3: Creative telescoping



DIAGONALS

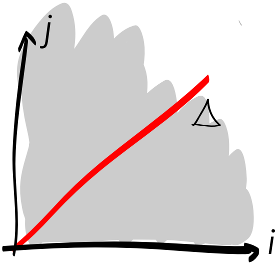
Definition

If F is a formal power series

$$F = \sum_{i_1,\dots,i_n \geq 0} a_{i_1,\dots,i_n} x_1^{i_1} \cdots x_n^{i_n},$$

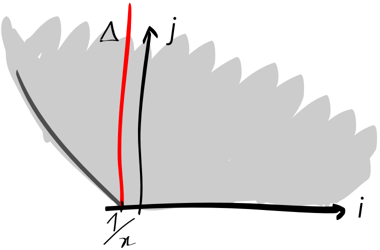
its diagonal is

$$\mathrm{Diag}(F) \stackrel{def}{=} \sum_i a_{i,\dots,i} t^i.$$



Theorem (Pólya, 1922)

Diagonals of bivariate rational functions are algebraic.



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Theorem (Pólya, 1922)

Diagonals of bivariate rational functions are algebraic.

Proof:

$$\mathrm{Diag}\,(F) = [x^{-1}]\frac{1}{x}F\left(x, \frac{t}{x}\right) = \frac{1}{2\pi i}\oint_{|x|=\epsilon} F\left(x, \frac{t}{x}\right)\frac{dx}{x}$$

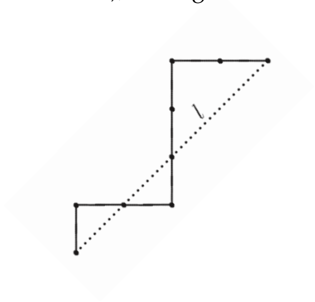
and evaluating the integral by residues concludes (residues are algebraic)

Example: Dyck walks

$$\mathfrak{S} = \{(1,1), (1,-1)\}$$

Let B_n be the number of Dyck bridges (i.e. $\mathfrak{S}$-walks in $\mathbb{Z}^2$ starting at $(0,0)$ and ending on the horizontal axis), of length n

Rotating a Dyck bridge counterclockwise by $\pi/4$



B_n = number of $\{(1,0),(0,1)\}$-walks in $\mathbb{Z}^2$ from $(0,0)$ to (n,n)

$$\Longrightarrow \qquad B(t) = \sum_{n\geq 0} B_n t^n = \operatorname{Diag}\left(\frac{1}{1-x-y}\right)$$

$$B(t) = \frac{1}{2\pi i}\oint_{|x|=\epsilon} \frac{dx}{x-x^2-t} = \left.\frac{1}{1-2x}\right|_{x=\frac{1-\sqrt{1-4t}}{2}} = \frac{1}{\sqrt{1-4t}}$$

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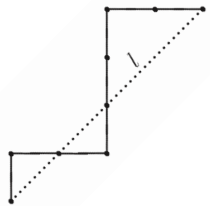
Rotating a Dyck bridge



counterclockwise by $\pi/4$

B_n = number of $\{(1,0),(0,1)\}$-walks in $\mathbb{Z}^2$ from $(0,0)$ to (n,n) $= \binom{2n}{n}$

$$\Longrightarrow \qquad B(t) = \sum_{n\geq 0} B_n t^n = \operatorname{Diag}\left(\frac{1}{1-x-y}\right)$$

$$B(t) = \frac{1}{2\pi i}\oint_{|x|=\epsilon} \frac{dx}{x-x^2-t} = \frac{1}{\sqrt{1-4t}} = \sum_{n\geq 0}\binom{2n}{n} t^n$$

Rothstein-Trager resultant

Let $A, B \in \mathbb{K}[x]$ be such that $\deg(A) < \deg(B)$, with B squarefree.
The rational function $F = A/B$ has simple poles only.

If $F = \sum_i \frac{\gamma_i}{x - \beta_i}$, then the residue γ_i of F at the pole β_i equals $\gamma_i = \frac{A(\beta_i)}{B'(\beta_i)}$.

Theorem. The residues γ_i of F are roots of the resultant

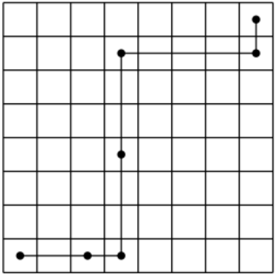
$$R(t) = \mathsf{Res}_x\big(B(x),\, A(x) - t \cdot B'(x)\big).$$

Proof. Poisson's formula: $R(t) = \prod_i \Big(A(\beta_i) - t \cdot B'(\beta_i)\Big)$.

▶ Introduced in computer algebra by [Rothstein-Trager 1976] for the symbolic (indefinite) integration of rational functions.
▶ Generalized by [Bronstein 1992] to multiple poles.

Example: diagonal Rook paths

Question: A chess Rook can move any number of squares horizontally or vertically in one step. How many paths can a Rook take from the lower-left corner square to the upper-right corner square of an $N \times N$ chessboard? Assume that the Rook moves right or up at each step.



$1,\ 2,\ 14,\ 106,\ 838,\ 6802,\ 56190,\ 470010,\ \ldots$

Example: diagonal Rook paths

Generating function of the sequence

$$1,\ 2,\ 14,\ 106,\ 838,\ 6802,\ 56190,\ 470010,\ \ldots$$

is

$$\mathsf{Diag}(F) = [x^0]\, F(x, t/x) = \frac{1}{2\pi i} \oint F(x, t/x)\, \frac{dx}{x}, \quad \text{where} \quad F = \frac{1}{1 - \frac{x}{1-x} - \frac{y}{1-y}}.$$

By the residue theorem, $\mathsf{Diag}(F)$ is a sum of roots $y(t)$ of the Rothstein-Trager resultant

```
> F:=1/(1-x/(1-x)-y/(1-y)):
> G:=normal(1/x*subs(y=t/x,F)):
> factor(resultant(denom(G),numer(G)-y*diff(denom(G),x),x));
```

$$t^2(1-t)(2y-1)(36ty^2 - 4y^2 + 1 - t)$$

Answer: Generating series of diagonal Rook paths is $\frac{1}{2}\left(1 + \sqrt{\frac{1-t}{1-9t}}\right)$.

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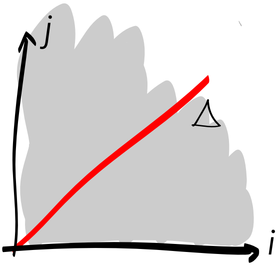

Theorem (Pólya, 1922)

Diagonals of bivariate rational functions are algebraic.[a]

[a] The converse is also true [Furstenberg, 1967]

Pólya's theorem

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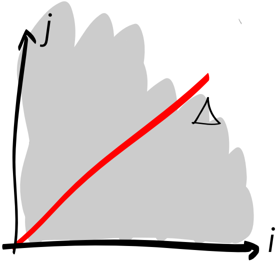

Theorem (Pólya, 1922)

Diagonals of bivariate rational functions are algebraic.

▷ This is false for more than 2 variables. E.g.

$$\mathrm{Diag}\left(\frac{1}{1-x-y-z}\right) = \sum_{n\geq 0} \frac{(3n)!}{n!^3} t^n = {}_2F_1\left(\left.\begin{matrix}\frac{1}{3} & \frac{2}{3} \\ & 1\end{matrix}\right| 27t\right) \quad \text{is transcendental}$$

Definition

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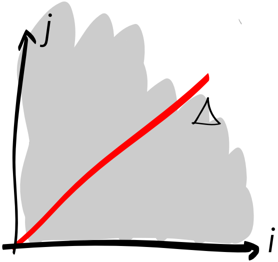

Theorem (Pólya, 1922)

Diagonals of bivariate rational functions are algebraic and thus D-finite.

- ▷ Algebraic equation has exponential size [B., Dumont, Salvy, 2015]
- ▷ Differential equation has polynomial size [B., Chen, Chyzak, Li, 2010]

Lipshitz's theorem

Definition

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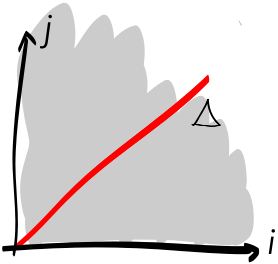

Theorem (Lipshitz, 1988)

Diagonals of rational functions are D-finite.

Example: Diagonal Rook paths on a 3D chessboard

Question [Erickson 2010]
How many ways can a Rook move from $(0,0,0)$ to (N,N,N), where each step is a positive integer multiple of $(1,0,0)$, $(0,1,0)$, or $(0,0,1)$?

1, 6, 222, 9918, 486924, 25267236, 1359631776, 75059524392, ...

Answer [B.-Chyzak-Hoeij-Pech 2011]: GF of 3D diagonal Rook paths is

$$G(t) = 1 + 6 \cdot \int_0^t \frac{{}_2F_1\left(\left.\begin{matrix}1/3 & 2/3\\ & 2\end{matrix}\right| \frac{27x(2-3x)}{(1-4x)^3}\right)}{(1-4x)(1-64x)}\,dx$$

Proof of Lipshitz's theorem on the 3D Rooks example

Problem: Show that $\operatorname{Diag}(F)$ is D-finite, where $F(x,y,z)$ is

$$\left(1-\sum_{n\geq 1} x^n - \sum_{n\geq 1} y^n - \sum_{n\geq 1} z^n\right)^{-1} = \frac{(1-x)(1-y)(1-z)}{1-2(x+y+z)+3(xy+yz+zx)-4xyz}$$

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Idea: If one is able to find a nonzero differential operator of the form

$$L(t,\partial_t,\partial_x,\partial_y) = P(t,\partial_t) + (\text{ higher-order terms in } \partial_x \text{ and } \partial_y \text{ })$$

that annihilates $G = \frac{1}{xy}\cdot F\left(x,\frac{y}{x},\frac{t}{y}\right)$, then $P(t,\partial_t)$ annihilates $\mathrm{Diag}(F)$.

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Proof:

1. $\mathrm{Diag}(F) = [x^0 y^0]\, F\left(x, \frac{y}{x}, \frac{t}{y}\right)$
2. $0 = L(G) = P(G) + \partial_x(\cdot) + \partial_y(\cdot)$
3. $0 = [x^{-1}y^{-1}]L(G) = [x^{-1}y^{-1}]P(G) = P([x^{-1}y^{-1}]G) = P(\mathrm{Diag}(F))$

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▷ Remaining task: Show that such an L does exist.

Proof of Lipshitz's theorem on the 3D Rooks example

Counting argument: By Leibniz's rule, the $\binom{N+4}{4}$ rational functions

$$t^i \partial_t^j \partial_x^k \partial_y^\ell (G), \quad 0 \le i+j+k+\ell \le N$$

are contained in the $\mathbb{Q}$-vector space of dimension $\le 18(N+1)^3$ spanned by

$$\frac{t^i x^j y^k}{\text{denom}(G)^{N+1}}, \quad 0 \le i \le 2N+1,\ 0 \le j \le 3N+2,\ 0 \le k \le 3N+2.$$

▷ If $\binom{N+4}{4} > 18(N+1)^3$, then there exists $L(t, \partial_t, \partial_x, \partial_y)$ (resp. $P(t, \partial_t)$) of total degree at most N, such that $LG = 0$ (resp. $P(\text{Diag}(F)) = 0$).

▷ $N = 425$ is the smallest integer satisfying $\binom{N+4}{4} > 18(N+1)^3$

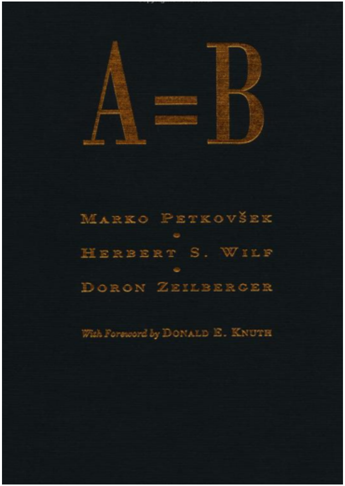
▷ Finding the operator P by Lipshitz' argument would require solving a linear system with 1,391,641,251 unknowns and 1,391,557,968 equations!

▷ A better solution is provided by creative telescoping.

CREATIVE TELESCOPING

Creative Telescoping

General framework in computer algebra –initiated by Zeilberger in the '90s– for proving identities on multiple integrals and sums with parameters.



▷ JNCF 2011, Frédéric Chyzak: "Le télescopage créatif pour l'intégration et la sommation paramétrées"

Examples I: hypergeometric summation

- $$\sum_{k\in\mathbb{Z}} (-1)^k \binom{a+b}{a+k}\binom{a+c}{c+k}\binom{b+c}{b+k} = \frac{(a+b+c)!}{a!b!c!}$$ [Dixon 1891]

- $A_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2$ satisfies the recurrence [Apéry 1978]:

$$(n+1)^3 A_{n+1} = (34n^3 + 51n^2 + 27n + 5) A_n - n^3 A_{n-1}.$$

(*Neither Cohen nor I had been able to prove this in the intervening two months* [Van der Poorten 1979])

- $$\sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 = \sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k} \sum_{j=0}^{k} \binom{k}{j}^3$$ [Strehl 1992]

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$$(n+1)^3 A_{n+1} = (34n^3+51n^2+27n+5)A_n - n^3 A_{n-1}.$$

(*The specific problem was mentioned to Don Zagier, who solved it with irritating speed* [Van der Poorten 1979])

- $$\sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 = \sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k}\sum_{j=0}^{k}\binom{k}{j}^3$$ [Strehl 1992]

Examples II: Integrals and Diagonals

- $$\int_0^1 \frac{\cos(zu)}{\sqrt{1-u^2}}\,du = \int_1^{+\infty} \frac{\sin(zu)}{\sqrt{u^2-1}}\,du = \frac{\pi}{2} J_0(z);$$

- $$\int_0^{+\infty} x J_1(ax) I_1(ax) Y_0(x) K_0(x)\,dx = -\frac{\ln(1-a^4)}{2\pi a^2}$$ [Glasser-Montaldi 1994];

- $$\frac{1}{2\pi i} \oint \frac{(1+2xy+4y^2)\exp\left(\frac{4x^2y^2}{1+4y^2}\right)}{y^{n+1}(1+4y^2)^{\frac{3}{2}}}\,dy = \frac{H_n(x)}{\lfloor n/2 \rfloor !}$$ [Doetsch 1930];

- $$\operatorname{Diag} \frac{1}{(1-x-y)(1-z-u)-xyzu} = \sum_{n\geq 0} A_n t^n$$ [Straub 2014].

Summation by Creative Telescoping

$$I_n := \sum_{k=0}^{n} \binom{n}{k} = 2^n.$$

IF one knows Pascal's triangle:

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1} = 2\binom{n}{k} + \binom{n}{k-1} - \binom{n}{k},$$

then summing over k gives

$$I_{n+1} = 2I_n.$$

The initial condition $I_0 = 1$ concludes the proof.

Creative Telescoping for Sums

$$F_n = \sum_k u_{n,k} = ?$$

IF one knows $P(n, S_n)$ and $R(n, k, S_n, S_k)$ such that

$$(P(n, S_n) + \Delta_k R(n, k, S_n, S_k)) \cdot u_{n,k} = 0$$

(where Δ_k is the difference operator, $\Delta_k \cdot v_{n,k} = v_{n,k+1} - v_{n,k}$), then the sum "telescopes", leading to

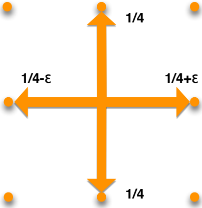
$$P(n, S_n) \cdot F_n = 0.$$

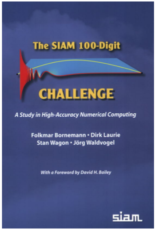
Zeilberger's Algorithm [1990]

Input: a hypergeometric term $u_{n,k}$, i.e., $u_{n+1,k}/u_{n,k}$ and $u_{n,k+1}/u_{n,k}$ rational functions in n and k;
Output:

- a linear recurrence (P) satisfied by $F_n = \sum_k u_{n,k}$
- a certificate (Q), such that checking the result is easy from $P(n, S_n) \cdot u_{n,k} = \Delta_k Q \cdot u_{n,k}$.

Example: SIAM flea





$$U_{n,k} := \binom{2n}{2k}\binom{2k}{k}\binom{2n-2k}{n-k}\left(\frac{1}{4}+c\right)^k\left(\frac{1}{4}-c\right)^k\frac{1}{4^{2n-2k}},$$

$$p_n = \sum_{k=0}^{n} U_{n,k} \quad = \text{probability of return to } (0,0) \text{ at step } 2n.$$

```
> p:=SumTools[Hypergeometric][Zeilberger](U,n,k,Sn);
```

$$[\left(4\,n^2+16\,n+16\right)Sn^2+\left(-4\,n^2+32\,c^2n^2+96\,c^2n-12\,n+72\,c^2-9\right)Sn$$
$$+128\,c^4n+64\,c^4n^2+48\,c^4,\ \text{...(BIG certificate)...}]$$

$$I(t) = \oint_\gamma H(t,x)\,dx =?$$

IF one knows $P(t,\partial_t)$ and $R(t,x,\partial_t,\partial_x)$ such that

$$\left(P(t,\partial_t) + \partial_x R(t,x,\partial_t,\partial_x)\right)\cdot H(t,x) = 0,$$

then the integral "telescopes", leading to

$$P(t,\partial_t)\cdot I(t) = 0.$$

Example: diagonal Rook paths again

Generating function of the sequence

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By creative telescoping, $\mathsf{Diag}(F)$ satisfies the differential equation

```
> F:=1/(1-x/(1-x)-y/(1-y)):
> G:=normal(1/x*subs(y=t/x,F)):
> Zeilberger(G, t, x, Dt)[1];
```

$$(9t^2-10t+1)\partial_t^2 + (18t-14)\partial_t$$

Answer: Generating series of diagonal Rook paths is $\frac{1}{2}\left(1+\sqrt{\frac{1-t}{1-9t}}\right)$.

CT for Multiple rational integrals

Problem:

$$\left.\begin{array}{rl} \mathbf{x} = x_1, \ldots, x_n & \text{— integration variables} \\ t & \text{— parameter} \\ H(t, \mathbf{x}) & \text{— rational function} \\ \gamma & \text{— } n\text{-cycle in } \mathbb{C}^n \end{array}\right\} \oint_\gamma H(t, \mathbf{x}) d\mathbf{x}$$

Principle of creative telescoping

$$\underbrace{\sum_{k=0}^{r} c_k(t) \frac{\partial^k H}{\partial t^k} = \overbrace{\sum_{i=1}^{n} \frac{\partial A_i}{\partial x_i}}^{\text{certificate}}}_{\text{telescopic relation}} \implies \overbrace{\left(\sum_{k=0}^{r} c_k(t) \partial_t^k \right)}^{\text{telescoper}} \cdot \oint_\gamma H d\mathbf{x} = 0$$

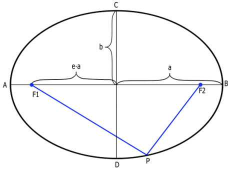
Task:

1. find the $c_k(t)$ which satisfy a telescopic relation,
2. ideally, without computing the certificate (A_i).

Example: Perimeter of an ellipse

Perimeter of an ellipse of eccentricity e, semi-major axis 1 [Euler, 1733]

$$p(e) = 4\int_0^1 \sqrt{\frac{1-e^2x^2}{1-x^2}}\,\mathrm{d}x = 2\pi - \frac{\pi}{2}e^2 - \frac{3\pi}{32}e^4 - \cdots$$



Creative telescoping finds the telescopic relation

$$\left((e-e^3)\partial_e^2 + (1-e^2)\partial_e + e\right)\cdot\left(\frac{1}{1-\frac{1-e^2x^2}{(1-x^2)y^2}}\right) =$$

$$\partial_x\left(-\frac{e\left(-1-x+x^2+x^3\right)y^2\left(-3+2x+y^2+x^2\left(-2+3e^2-y^2\right)\right)}{\left(-1+y^2+x^2(e^2-y^2)\right)^2}\right)$$

$$+\,\partial_y\left(\frac{2e\left(-1+e^2\right)x\left(1+x^3\right)y^3}{\left(-1+y^2+x^2(e^2-y^2)\right)^2}\right)$$

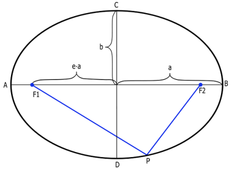
Thus $(e-e^3)p'' + (1-e^2)p' + ep = 0.$

▷ Note the size of the certificate (compared to that of the telescoper).

Example: Perimeter of an ellipse

Perimeter of an ellipse of eccentricity e, semi-major axis 1 [Euler, 1733]

$$p(e) = 4\int_0^1 \sqrt{\frac{1-e^2x^2}{1-x^2}}\,\mathrm{d}x = 4\oint \frac{\mathrm{d}x\mathrm{d}y}{1-\frac{1-e^2x^2}{(1-x^2)y^2}},$$



Creative telescoping finds the telescopic relation

$$\left((e-e^3)\partial_e^2 + (1-e^2)\partial_e + e\right)\cdot\left(\frac{1}{1-\frac{1-e^2x^2}{(1-x^2)y^2}}\right) =$$

$$\partial_x\left(-\frac{e\left(-1-x+x^2+x^3\right)y^2\left(-3+2x+y^2+x^2\left(-2+3e^2-y^2\right)\right)}{\left(-1+y^2+x^2(e^2-y^2)\right)^2}\right)$$

$$+\,\partial_y\left(\frac{2e\left(-1+e^2\right)x\left(1+x^3\right)y^3}{\left(-1+y^2+x^2(e^2-y^2)\right)^2}\right)$$

Thus $(e-e^3)p'' + (1-e^2)p' + ep = 0$.

▷ Note the size of the certificate (compared to that of the telescoper).

Example: 3D rook paths [B.-Chyzak-Hoeij-Pech 2011]

Task: Given G in $\mathbb{Q}(t,x,y)$, construct a linear differential operator $P(t,\partial_t)$, and two rational functions R and S in $\mathbb{Q}(t,x,y)$ such that

$$P(G) = \frac{\partial R}{\partial x} + \frac{\partial S}{\partial y}.$$

Solution: Creative telescoping!

```
> G:=subs(y=y/x,z=t/y,1/(1-x/(1-x)-y/(1-y)-z/(1-z)))/y/x:
> P,R,S:=op(op(Mgfun:-creative_telescoping(G,t::diff,[x::diff,y::diff]))):
> P;
```

$$\begin{aligned} P = \quad & t(t-1)(64t-1)(3t-2)(6t+1)\partial_t^3 \\ & +(4608t^4-6372t^3+813t^2+514t-4)\partial_t^2 \\ & +4(576t^3-801t^2-108t+74)\partial_t \end{aligned}$$

▷ The whole computation takes <10 seconds on a personal laptop.

▷ Proves a recurrence conjectured by [Erickson 2010]

Brief review on CT algorithms

General-purpose creative telescoping algorithms:

- using linear algebra [Lipshitz, 1988];
- using non-commutative Gröbner bases:
  - and elimination [Takayama, 1990; Zeilberger, 1990];
  - and rational resolution of differential equations [Chyzak, 2000];
  - and very efficient heuristics [Koutschan, 2009, 2010].

▷ Drawbacks: Bad or unknown complexity; all compute certificates.

Rational case:

- univariate integrals [B., Chen, Chyzak, Li, 2010];
- double integrals [Chen, Kauers, Singer, 2012];
- multiple integrals [B., Lairez, Salvy, 2013], [Lairez 2015].

Problem: Given $H = P/Q \in \mathbb{K}(t,x)$ compute $\oint_\gamma H(t,x) \mathrm{d}x$

Hermite reduction: H can be written in reduced form

$$H = \partial_x(g) + \frac{a}{Q^\star},$$

where $Q^\star$ is the squarefree part of Q and $\deg_x(a) < d^\star := \deg_x(Q^\star)$.

Algorithm [B., Chen, Chyzak, Li, 2010]

(1) For $i = 0, 1, \ldots, d^\star$ compute Hermite reduction of $\partial_t^i(H)$:

$$\partial_t^i(H) = \partial_x(g_i) + \frac{a_i}{Q^\star}, \quad \deg_x(a_i) < d^\star$$

(2) Find the first linear relation over $\mathbb{K}(t)$ of the form $\sum_{k=0}^r c_k a_k = 0$.

▷ $L = \sum_{k=0}^r c_k \partial_t^k$ is a telescoper (and $\sum_{k=0}^r c_k g_k$ the corresponding certificate).

4G Creative Telescoping: multivariate case

$H = \frac{P}{Q}$ — a rational function in t and $\mathbf{x} = x_1, \ldots, x_n$

$d_{\mathbf{x}}$ — the degree of Q w.r.t. $\mathbf{x}$

d_t — $\max(\deg_t P, \deg_t Q)$

Theorem ([B., Lairez, Salvy, 2013])

A telescoper for H can be computed using $\tilde{\mathcal{O}}(e^{3n} d_{\mathbf{x}}^{8n} d_t)$ *operations.*

The minimal telescoper has order $\leq d_{\mathbf{x}}^n$ *and degree* $\mathcal{O}(e^n d_{\mathbf{x}}^{3n} d_t)$.

These size bounds are generically reached.

- First polynomial time algorithm for rational creative telescoping.
- It avoids the costly computation of certificates.
- Generically, certificates have size $\Omega(d_{\mathbf{x}}^{n^2/2})$.
- General-purpose algorithms have doubly-exponential complexity.
- Applies to diagonals: $\mathrm{Diag}(F)(t) = \frac{1}{(2\pi i)^n} \oint F\left(\frac{t}{x_1 \ldots x_n}, x_1, \ldots, x_n\right)$.

Main ingredients of the integration algorithm

Griffiths–Dwork method for the generic case
Linear reduction classical in algebraic geometry;
Generalization of Hermite's reduction.

Fast linear algebra on polynomial matrices
Macaulay matrices encoding Gröbner bases computations;
Sophisticated algorithms due to Villard, Storjohann, Zhou, etc.

Deformation technique for the general case
Input perturbation using a new free variable.

▷ Recent, highly non-trivial, extension by [Lairez, 2015] tremendously improves the efficiency of the algorithm in [B., Lairez, Salvy, 2013]

WALKS IN THE QUARTER PLANE

The 19 D-finite cases with nonzero orbit sum

	OEIS	$\mathfrak{S}$	Pol size	ODE size		OEIS	$\mathfrak{S}$	Pol size	ODE size
1	A005566		—	3, 4	13	A151275		—	5, 24
2	A018224		—	3, 5	14	A151314		—	5, 24
3	A151312		—	3, 8	15	A151255		—	4, 16
4	A151331		—	3, 6	16	A151287		—	5, 19
5	A151266		—	5, 16	17	A001006		2, 2	2, 3
6	A151307		—	5, 20	18	A129400		2, 2	2, 3
7	A151291		—	5, 15	19	A005558		—	3, 5
8	A151326		—	5, 18					
9	A151302		—	5, 24	20	A151265		6, 8	4, 9
10	A151329		—	5, 24	21	A151278		6, 8	4, 12
11	A151261		—	4, 15	22	A151323		4, 4	2, 3
12	A151297		—	5, 18	23	A060900		8, 9	3, 5

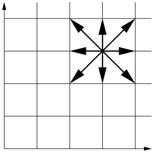
Equation sizes = {order, degree}@(algeq, diffeq)

Task: Prove Cases 1–19 in the tables [B. & Kauers 2009] for $F(t;1,1)$

	OEIS	$\mathfrak{S}$	alg?	asympt		OEIS	$\mathfrak{S}$	alg?	asympt
1	A005566		N	$\frac{4}{\pi}\frac{4^n}{n}$	13	A151275		N	$\frac{12\sqrt{30}}{\pi}\frac{(2\sqrt{6})^n}{n^2}$
2	A018224		N	$\frac{2}{\pi}\frac{4^n}{n}$	14	A151314		N	$\frac{\sqrt{6}\lambda\mu C^{5/2}}{5\pi}\frac{(2C)^n}{n^2}$
3	A151312		N	$\frac{\sqrt{6}}{\pi}\frac{6^n}{n}$	15	A151255		N	$\frac{24\sqrt{2}}{\pi}\frac{(2\sqrt{2})^n}{n^2}$
4	A151331		N	$\frac{8}{3\pi}\frac{8^n}{n}$	16	A151287		N	$\frac{2\sqrt{2}A^{7/2}}{\pi}\frac{(2A)^n}{n^2}$
5	A151266		N	$\frac{1}{2}\sqrt{\frac{3}{\pi}}\frac{3^n}{n^{1/2}}$	17	A001006		Y	$\frac{3}{2}\sqrt{\frac{3}{\pi}}\frac{3^n}{n^{3/2}}$
6	A151307		N	$\frac{1}{2}\sqrt{\frac{5}{2\pi}}\frac{5^n}{n^{1/2}}$	18	A129400		Y	$\frac{3}{2}\sqrt{\frac{3}{\pi}}\frac{6^n}{n^{3/2}}$
7	A151291		N	$\frac{4}{3\sqrt{\pi}}\frac{4^n}{n^{1/2}}$	19	A005558		N	$\frac{8}{\pi}\frac{4^n}{n^2}$
8	A151326		N	$\frac{2}{\sqrt{3\pi}}\frac{6^n}{n^{1/2}}$					
9	A151302		N	$\frac{1}{3}\sqrt{\frac{5}{2\pi}}\frac{5^n}{n^{1/2}}$	20	A151265		Y	$\frac{2\sqrt{2}}{\Gamma(1/4)}\frac{3^n}{n^{3/4}}$
10	A151329		N	$\frac{1}{3}\sqrt{\frac{7}{3\pi}}\frac{7^n}{n^{1/2}}$	21	A151278		Y	$\frac{3\sqrt{3}}{\sqrt{2}\Gamma(1/4)}\frac{3^n}{n^{3/4}}$
11	A151261		N	$\frac{12\sqrt{3}}{\pi}\frac{(2\sqrt{3})^n}{n^2}$	22	A151323		Y	$\frac{\sqrt{2}3^{3/4}}{\Gamma(1/4)}\frac{6^n}{n^{3/4}}$
12	A151297		N	$\frac{\sqrt{3}B^{7/2}}{2\pi}\frac{(2B)^n}{n^2}$	23	A060900		Y	$\frac{4\sqrt{3}}{3\Gamma(1/3)}\frac{4^n}{n^{2/3}}$

$$A = 1+\sqrt{2},\ B = 1+\sqrt{3},\ C = 1+\sqrt{6},\ \lambda = 7+3\sqrt{6},\ \mu = \sqrt{\frac{4\sqrt{6}-1}{19}}$$

The group of a model



The polynomial $\chi_{\mathfrak{S}} := \sum_{(i,j)\in\mathfrak{S}} x^i y^j = \sum_{i=-1}^{1} B_i(y) x^i = \sum_{j=-1}^{1} A_j(x) y^j$

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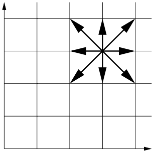

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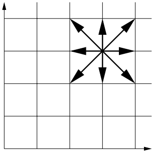
$$\psi(x,y) = \left(x, \frac{A_{-1}(x)}{A_{+1}(x)} \frac{1}{y}\right), \qquad \phi(x,y) = \left(\frac{B_{-1}(y)}{B_{+1}(y)} \frac{1}{x}, y\right),$$



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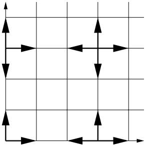
and thus under any element of the group

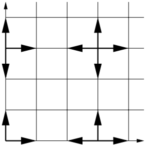
$$\mathcal{G}_{\mathfrak{S}} := \langle \psi, \phi \rangle.$$

D-Finiteness via the finite group [Bousquet-Mélou & Mishna 2010]



$J = 1 - t \cdot \sum_{(i,j)\in\mathfrak{S}} x^i y^j = 1 - t\left(x + \frac{1}{x} + y + \frac{1}{y}\right)$ is invariant under the change of (x, y) into, respectively:

$$\left(\tfrac{1}{x}, y\right), \left(\tfrac{1}{x}, \tfrac{1}{y}\right), \left(x, \tfrac{1}{y}\right).$$



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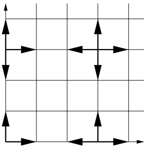
Kernel equation:

$$J(t; x, y)xyF(t; x, y) = xy - txF(t; x, 0) - tyF(t; 0, y)$$



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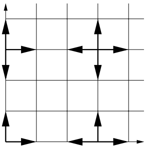

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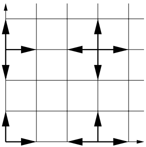

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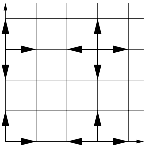

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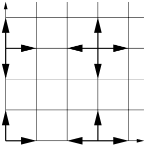
\end{aligned}$$

Summing up yields the orbit equation:

$$\sum_{\theta\in\mathcal{G}} (-1)^{\theta}\theta\big(xy\,F(t;x,y)\big) = \frac{xy - \frac{1}{x}y + \frac{1}{x}\frac{1}{y} - x\frac{1}{y}}{J(t;x,y)}$$



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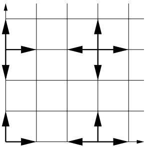
\end{aligned}$$

Taking positive parts yields:

$$[x^> y^>] \sum_{\theta\in\mathcal{G}} (-1)^\theta \theta\big(xy\,F(t;x,y)\big) = [x^> y^>] \frac{xy - \frac{1}{x}y + \frac{1}{x}\frac{1}{y} - x\frac{1}{y}}{J(t;x,y)}$$



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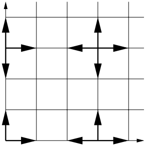

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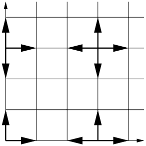
\end{aligned}$$

$$\mathsf{GF} \;=\; \mathsf{PosPart}\left(\frac{\mathsf{OS}}{\mathsf{kernel}}\right)$$



$J = 1 - t \cdot \sum_{(i,j)\in\mathfrak{S}} x^i y^j = 1 - t\left(x + \frac{1}{x} + y + \frac{1}{y}\right)$ is invariant under the change of (x, y) into, respectively:

$$\left(\tfrac{1}{x}, y\right), \left(\tfrac{1}{x}, \tfrac{1}{y}\right), \left(x, \tfrac{1}{y}\right).$$

Kernel equation:

$$\begin{aligned}
J(t;x,y)xyF(t;x,y) &= xy - txF(t;x,0) - tyF(t;0,y) \\
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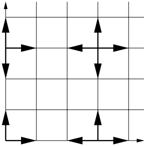
$$\text{GF} = \text{PosPart}\left(\frac{\text{OS}}{\text{ker}}\right) = \text{D-finite [Lipshitz'88]}$$

D-Finiteness via the finite group [Bousquet-Mélou & Mishna 2010]



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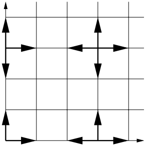
▷ This is an algebraic version of the reflection principle



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▷ Creative telescoping finds a differential equation for PosPart(OS/ker)

Cases 1–19 are D-Finite

Theorem [Bousquet-Mélou & Mishna, 2010]

Let $\mathfrak{S}$ be one of the step sets 1–19. Then, the invariant group $\mathcal{G}$ is finite and:

$$xyt\,F(t;x,y) = [x^{>}y^{>}]\,\frac{\sum_{\theta\in\mathcal{G}}(-1)^{\theta}\theta(xy)}{J(t;x,y)}.$$

In particular, $F(t;x,y)$ is D-finite.

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Proof: Use [Lipshitz'88] for positive parts of rational functions:
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▷ Constructive proof, but it leads to a highly inefficient algorithm to get an ODE for $F(t;x,y)$.

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TOO LARGE TO BE MERELY WRITTEN!

Explicit Expressions for the Cases 1–19

Theorem [B.-Chyzak-van Hoeij-Kauers-Pech, 2015]

Let $\mathfrak{S}$ be one of the step sets 1–19. Then, the generating series $F(t; x, y)$ is expressible using iterated integrals of ${}_2F_1$ expressions.

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Let $\mathfrak{S}$ be one of the step sets 1–19. Then, the generating series $F(t;1,1)$ is expressible using iterated integrals of ${}_2F_1$ expressions.

Example: King walks in the quarter plane (A025595)

$$F(t;1,1) = \frac{1}{t}\int_0^t \frac{1}{(1+4x)^3} \cdot {}_2F_1\left(\begin{matrix}\frac{3}{2}\ \frac{3}{2}\\ 2\end{matrix}\,\middle|\, \frac{16x(1+x)}{(1+4x)^2}\right) dx$$

$$= 1 + 3t + 18t^2 + 105t^3 + 684t^4 + 4550t^5 + 31340t^6 + 219555t^7 + \cdots$$

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▷ Proof uses Creative telescoping, ODE factorization, ODE solving:

1. If $R = \sum_\theta \frac{(-1)^\theta \, \theta(xy)}{J(t;x,y)}$, then $F = [u^{-1}v^{-1}]H$, for $H = \frac{R(t;1/u,1/v)}{(1-xu)(1-yv)}$.

2. If $P \in \mathbb{Q}(x,y)[t]\langle\partial_t\rangle$ and $U, V \in \mathbb{Q}(x,y,u,v,t)$ such that $L(H) = \partial_u U + \partial_v V$, then $L(F(t;x,y)) = 0$.
   Use creative telescoping for finding L.

3. Factor L as $L_2 \cdot P_1 \cdots P_t$, where L_2 has order 2 and the P_i have order 1.
   Solve L_2 in terms of ${}_2F_1$s and deduce F.

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   Use creative telescoping for finding L.
   Works in practice with early evaluation $(x,y) = (1,1)$, but not for symbolic (x,y).
   Works also for $(0,0)$, $(x,0)$, and $(0,y)$!
3. Factor L as $L_2 \cdot P_1 \cdots P_t$, where L_2 has order 2 and the P_i have order 1.
   Solve L_2 in terms of ${}_2F_1$s and deduce F.

For $F(t;x,y)$, run whole process for $F(t;0,0)$, $F(t;x,0)$, and $F(t;0,y)$, then substitute into Kernel equation!

Summary – the 19 cases with finite group

Creative telescoping allows the uniform treatment of several questions:

- compute differential operators that witness D-finiteness,
- algebraic vs transcendental nature of series,
- asymptotics of coefficients.

BACK TO THE EXERCISE

–Solution–

The exercise

Let $\mathfrak{S} = \{\uparrow, \leftarrow, \searrow\}$. A $\mathfrak{S}$-walk is a path in $\mathbb{Z}^2$ using only steps from $\mathfrak{S}$. Show that, for any integer n, the following quantities are equal:

(i) the number a_n of $\mathfrak{S}$-walks of length n confined to the upper half plane $\mathbb{Z} \times \mathbb{N}$ that start and end at the origin $(0,0)$;

(ii) the number b_n of $\mathfrak{S}$-walks of length n confined to the quarter plane $\mathbb{N}^2$ that start at the origin $(0,0)$ and finish on the diagonal $x = y$.

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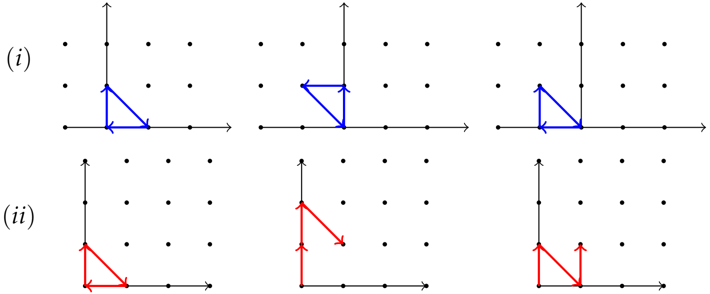
For instance, for $n = 3$, this common value is $a_3 = b_3 = 3$:



A recurrence relation for $\{\uparrow, \leftarrow, \searrow\}$-walks in $\mathbb{Z} \times \mathbb{N}$

$h(n;i,j)$ = nb. of $\{\uparrow, \leftarrow, \searrow\}$-walks in $\mathbb{Z} \times \mathbb{N}$ of length n from $(0,0)$ to (i,j)

The numbers $h(n;i,j)$ satisfy

$$h(n;i,j) = \begin{cases} 0 & \text{if } j < 0 \text{ or } n < 0, \\ \mathbb{1}_{i=j=0} & \text{if } n = 0, \\ \sum_{(i',j') \in \mathfrak{S}} h(n-1; i-i', j-j') & \text{otherwise.} \end{cases}$$

```
> h:=proc(n,i,j)
  option remember;
    if j<0 or n<0 then 0
    elif n=0 then if i=0 and j=0 then 1 else 0 fi
    else h(n-1,i,j-1)+h(n-1,i+1,j)+h(n-1,i-1,j+1) fi
  end:
```

```
> A:=series(add(h(n,0,0)*t^n,n=0..12),t,12);
```

$$A = 1 + 3t^3 + 30t^6 + 420t^9 + O(t^{12})$$

A recurrence relation for $\{\uparrow, \leftarrow, \searrow\}$-walks in $\mathbb{N}^2$

$q(n;i,j)$ = nb. of $\{\uparrow, \leftarrow, \searrow\}$-walks in $\mathbb{N}^2$ of length n from $(0,0)$ to (i,j)

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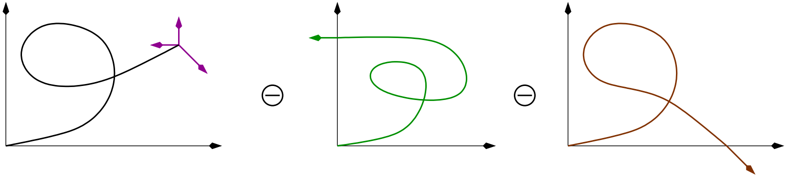
A functional equation for $\{\uparrow, \leftarrow, \searrow\}$-walks in $\mathbb{N}^2$

Generating function: $Q(t;x,y) = \sum_{n=0}^{\infty} \sum_{i=0}^{n} \sum_{j=0}^{n} q(n;i,j) t^n x^i y^j \in \mathbb{Q}[x,y][[t]]$

Kernel equation ($\bar{x} = 1/x,\ \bar{y} = 1/y$):

$$Q(t;x,y) \equiv Q(x,y) = 1 + t(y + \bar{x} + x\bar{y})Q(x,y) - t\bar{x}Q(0,y) - tx\bar{y}Q(x,0)$$



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Task (Q): Find $B(t) = [x^0]\, Q(x,\bar{x})$

... and a functional equation for $\{\uparrow, \leftarrow, \searrow\}$-walks in $\mathbb{Z} \times \mathbb{N}$

Generating function: $H(t;x,y) = \sum_{n=0}^{\infty} \sum_{i=-n}^{n} \sum_{j=0}^{\infty} h(n;i,j) t^n x^i y^j \in \mathbb{Q}[x,\bar{x},y][[t]]$

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Task (H): Find $A(t) = [x^0]\, H(x,0)$

The kernel method for $\mathbb{Z} \times \mathbb{N}$

- The kernel equation reads (with $K(x,y) = 1 - t(y + \bar{x} + x\bar{y})$):

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$$y_0 = \frac{x - t - \sqrt{(t-x)^2 - 4t^2x^3}}{2tx} \qquad = xt + t^2 + (x^2 + \bar{x})t^3 + (3x + \bar{x}^2)t^4 + \cdots$$

be the (unique) root in $\mathbb{Q}[x,\bar{x}][[t]]$ of $K(x, y_0) = 0$.

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thus

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$$H(x,0) = \frac{y_0}{tx} \quad \text{and} \quad A(t) = \left[x^0\right] \frac{y_0}{tx}.$$

- Creative telescoping then proves:

$$(27t^4 - t)A''(t) + (108t^3 - 4)A'(t) + 54t^2 A(t) = 0.$$

```
> Zeilberger(1/x * sqrt((t-x)^2 - 4*t^2*x^3)/(2*t^2*x^2), t, x, Dt);
```

The group of the model $\{\uparrow, \leftarrow, \searrow\}$

Step set $\mathfrak{S} = \{(-1,0),(0,1),(1,-1)\}$, with characteristic polynomial

$$\chi(x,y) = \frac{1}{x} + y + x \cdot \frac{1}{y} = \bar{x} + y + x\bar{y}$$

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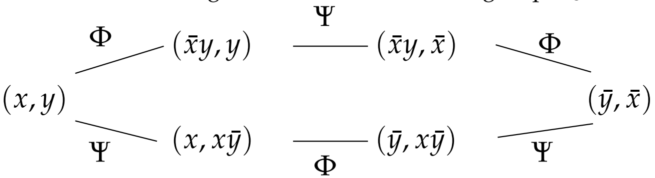
Φ and Ψ are involutions, and generate a finite dihedral group D_3 of order 6:



Solving the kernel equation for $\mathbb{N}^2$

- Orbit equation:

$$xyQ(x,y) - \bar{x}y^2Q(\bar{x}y,y) + \bar{x}^2yQ(\bar{x}y,\bar{x}) - \bar{x}\bar{y}Q(\bar{y},\bar{x}) + x\bar{y}^2Q(\bar{y},x\bar{y}) - x^2\bar{y}Q(x,x\bar{y}) = \frac{xy - \bar{x}y^2 + \bar{x}^2y - \bar{x}\bar{y} + x\bar{y}^2 - x^2\bar{y}}{1 - t(y + \bar{x} + x\bar{y})}$$

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- Corollary [Bousquet-Mélou & Mishna, 2010]:

$$xyQ(x,y) = [x^{>0}y^{>0}]\ \frac{xy - \bar{x}y^2 + \bar{x}^2y - \bar{x}\bar{y} + x\bar{y}^2 - x^2\bar{y}}{1 - t(y + \bar{x} + x\bar{y})}$$

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- Corollary [B.-Chyzak-van Hoeij-Kauers-Pech, 2015]:

$$B(t) = [z^0]Q(z,\bar{z}) = [u^{-1}v^{-1}z^{-1}]\ \frac{\bar{u}\bar{v} - u\bar{v}^2 + u^2\bar{v} - uv + \bar{u}v^2 - \bar{u}^2v}{z(1-zu)(1-v\bar{z})(1-t(\bar{v}+u+\bar{u}v))}$$

Solving the kernel equation for $\mathbb{N}^2$

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- Multivariate Creative Telescoping gives a differential equation for $B(t)$:

$$(27t^4 - t)B''(t) + (108t^3 - 4)B'(t) + 54t^2B(t) = 0.$$

Conclusion

We have proved that $A(t)$ and $B(t)$ are both solutions of

$$(27t^4 - t)y''(t) + (108t^3 - 4)y'(t) + 54t^2 y(t) = 0.$$

Solving this equation proves:

$$A(t) = B(t) = {}_2F_1\left(\begin{matrix}1/3 & 2/3\\ & 2\end{matrix}\,\middle|\, 27\,t^3\right) = \sum_{n=0}^{\infty} \frac{(3n)!}{n!^3} \frac{t^{3n}}{n+1}.$$

Thus the two sequences are equal to

$$a_{3n} = b_{3n} = \frac{(3n)!}{n!^2 \cdot (n+1)!}, \qquad \text{and} \quad a_m = b_m = 0 \quad \text{if 3 does not divide } m.$$

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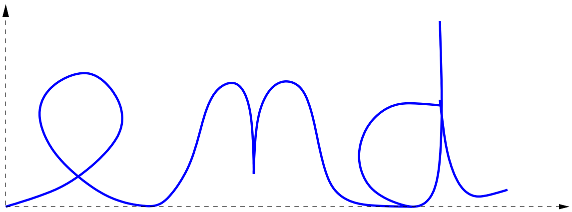

Thanks for your attention!