Diagonals and Creative Telescoping
—From 1G to 4G algorithms—

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Creative Telescoping

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{2020\cdot 2021}=?$$

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[50] Develop computer programs for simplifying sums that involve binomial coefficients.

▷ Today: how computer algebra uses Bernoulli's 1682 idea –systematically and algorithmically–, to solve Knuth's 1969 exercise, and more

DIAGONALS

Diagonals of multivariate power series

Definition

If F is a multivariate power series

$$F = \sum_{i_1,\dots,i_n \geq 0} a_{i_1,\dots,i_n} x_1^{i_1} \cdots x_n^{i_n},$$

its diagonal is the univariate power series

$$\mathrm{Diag}(F) \stackrel{def}{=} \sum_i a_{i,\dots,i} t^i.$$

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Example: if $n = 1$, then (trivially)

$$\mathrm{Diag}\,(F) = F(t).$$

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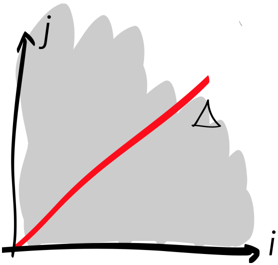

Example: if $n = 2$ and $F = \dfrac{1}{1-x-y} = \sum_{i,j \geq 0} \binom{i+j}{i} x^i y^j$, then

$$\mathrm{Diag}\,(F) = \sum_{n \geq 0} \binom{2n}{n} t^n = 1 + 2\,t + 6\,t^2 + 20\,t^3 + 70\,t^4 + \cdots .$$

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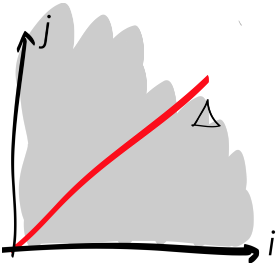

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▷ Diag (F) is not a rational function, even though F is rational.

Pólya's theorem

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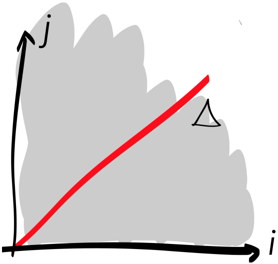

Theorem (Pólya, 1922)

Diagonals of bivariate rational functions are algebraic.

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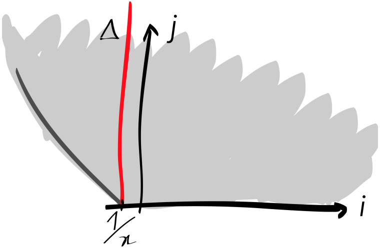


Theorem (Pólya, 1922)

Diagonals of bivariate rational functions are algebraic.

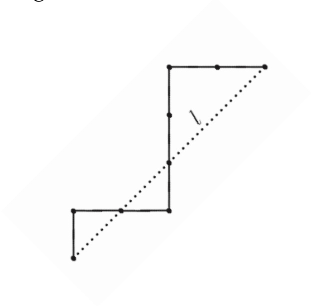
Proof: Since $F\left(x, \frac{t}{x}\right) = \sum_{i,j} a_{i,j} x^{i-j} t^j$ we have that $\mathrm{Diag}(F) = [x^0] F\left(x, \frac{t}{x}\right)$.

Therefore, by Cauchy's integral theorem,

$$\mathrm{Diag}(F) = [x^{-1}] \frac{1}{x} F\left(x, \frac{t}{x}\right) = \frac{1}{2\pi i} \oint_{|x|=\epsilon} F\left(x, \frac{t}{x}\right) \frac{dx}{x}.$$

By the Residue Theorem: last integral is a sum of residues, all algebraic. $\square$

Let B_n be the number of Dyck bridges, i.e. $\{\mathsf{NE}, \mathsf{SE}\}$-walks of length n in $\mathbb{Z}^2$ starting at $(0,0)$ and ending on the horizontal axis.



Equivalently, B_n = number of $\{\mathsf{N}, \mathsf{E}\}$-walks in $\mathbb{Z}^2$ from $(0,0)$ to (n,n)

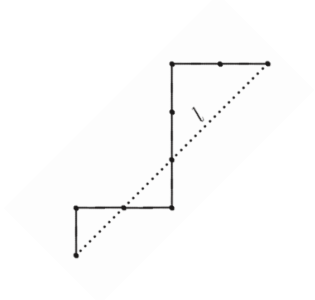
$$\Longrightarrow \qquad B(t) = \sum_{n\geq 0} B_n t^n = \operatorname{Diag}\left(\frac{1}{1-x-y}\right)$$

Then: $$B(t) = \frac{1}{2\pi i}\oint_{|x|=\epsilon} \frac{dx}{x-x^2-t} = \frac{1}{1-2x}\bigg|_{x=\frac{1-\sqrt{1-4t}}{2}} = \frac{1}{\sqrt{1-4t}}$$

Example: Dyck walks

Let B_n be the number of Dyck bridges, i.e. $\{\mathsf{NE}, \mathsf{SE}\}$-walks of length n in $\mathbb{Z}^2$ starting at $(0,0)$ and ending on the horizontal axis.

Rotating a Dyck bridge



counterclockwise by $\pi/4$

Equivalently, B_n = number of $\{\mathsf{N}, \mathsf{E}\}$-walks in $\mathbb{Z}^2$ from $(0,0)$ to (n,n) $= \binom{2n}{n}$

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$$\text{Then:} \quad B(t) = \frac{1}{2\pi i}\oint_{|x|=\epsilon} \frac{dx}{x-x^2-t} = \frac{1}{\sqrt{1-4t}} = \sum_{n\geq 0}\binom{2n}{n} t^n$$

Rothstein-Trager resultant

Let $A, B \in \mathbb{K}[x]$ be such that $\deg(A) < \deg(B)$, with B squarefree.
In particular, the rational function $F = A/B$ has simple poles only.

Lemma. The residue r_i of F at the pole p_i equals $r_i = \dfrac{A(p_i)}{B'(p_i)}$.

Proof. If $F = \sum_i \dfrac{r_i}{x - p_i}$, then $r_i = (F \cdot (x - p_i))_{|x=p_i} = \dfrac{A(x)}{\prod_{j \neq i}(x - p_j)}(p_i)$

Theorem. The residues r_i of F are roots of the resultant

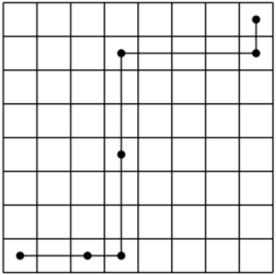
$$R(t) = \mathsf{Res}_x\big(B(x),\, A(x) - t \cdot B'(x)\big).$$

Proof. By Poisson's formula: $R(t) = \prod_i \Big(A(p_i) - t \cdot B'(p_i)\Big).$ □

▷ Introduced by [Rothstein-Trager 1976] for the (indefinite) integration of rational functions.
▷ Generalized by [Bronstein 1992] to multiple poles.

Example: diagonal Rook paths

Question: A chess Rook can move any number of squares horizontally or vertically in one step. How many paths can a Rook take from the lower-left corner square to the upper-right corner square of an $N \times N$ chessboard? Assume that the Rook moves only right or up at each step.



$$1, 2, 14, 106, 838, 6802, 56190, 470010, \ldots$$

Example: diagonal Rook paths

Generating function of the sequence

$$1,\ 2,\ 14,\ 106,\ 838,\ 6802,\ 56190,\ 470010,\ \ldots$$

is

$$\mathsf{Diag}(F) = [x^0]\, F(x, t/x) = \frac{1}{2\pi i} \oint F(x, t/x)\, \frac{dx}{x}, \quad \text{where} \quad F = \frac{1}{1 - \frac{x}{1-x} - \frac{y}{1-y}}.$$

Residue theorem: $\mathsf{Diag}(F)$ is a sum of roots y of the Rothstein-Trager resultant

```
> F:=1/(1-x/(1-x)-y/(1-y)):
> G:=normal(1/x*subs(y=t/x,F)):
> factor(resultant(denom(G),numer(G)-y*diff(denom(G),x),x));
```

$$t^2(1-t)(2y-1)(36ty^2 - 4y^2 + 1 - t)$$

Answer: Generating series of diagonal Rook paths is $\displaystyle \frac{1}{2}\left(1 + \sqrt{\frac{1-t}{1-9t}}\right).$

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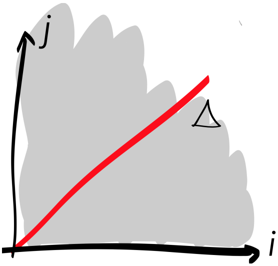
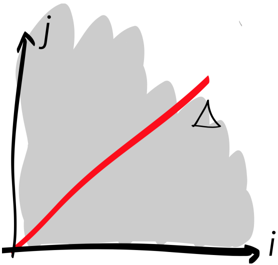


Theorem (Pólya, 1922)

Diagonals of bivariate rational functions are algebraic.[a]

[a] The converse is also true [Furstenberg, 1967]



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▷ This is false for more than 2 variables. E.g.

$$\mathrm{Diag}\left(\frac{1}{1-x-y-z}\right) = \sum_{n\geq 0} \frac{(3n)!}{n!^3} t^n = {}_2F_1\left(\begin{matrix}\frac{1}{3} & \frac{2}{3} \\ & 1\end{matrix} \middle| 27t\right) \quad \text{is transcendental}$$

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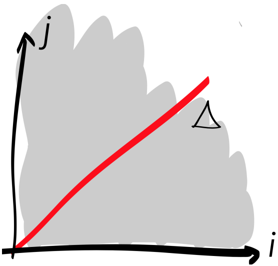

Theorem (Pólya, 1922)

Diagonals of bivariate rational functions are algebraic and thus D-finite.

- ▷ Algebraic equation has exponential size [B., Dumont, Salvy, 2015]
- ▷ Differential equation has polynomial size [B., Chen, Chyzak, Li, 2010]

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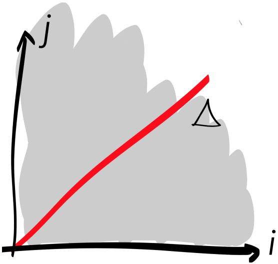

Theorem (Lipshitz, 1988)

Diagonals of multivariate rational functions are D-finite.

Example: Diagonal Rook paths on a 3D chessboard

Question [Erickson 2010]
How many ways can a Rook move from $(0,0,0)$ to (N,N,N), where each step is a positive integer multiple of $(1,0,0), (0,1,0)$, or $(0,0,1)$?

1, 6, 222, 9918, 486924, 25267236, 1359631776, 75059524392, ...

Answer [B., Chyzak, van Hoeij, Pech, 2011]: GF of 3D diagonal Rook paths is

$$G(t) = 1 + 6 \cdot \int_0^t \frac{{}_2F_1\left(\begin{matrix} 1/3 \quad 2/3 \\ 2 \end{matrix} \,\middle|\, \frac{27x(2-3x)}{(1-4x)^3}\right)}{(1-4x)(1-64x)}\, dx$$

Proof of Lipshitz's theorem on the 3D Rooks example

Problem: Show that $\mathrm{Diag}(F)$ is D-finite, where $F(x,y,z)$ is

$$\left(1 - \sum_{n\geq 1} x^n - \sum_{n\geq 1} y^n - \sum_{n\geq 1} z^n\right)^{-1} = \frac{(1-x)(1-y)(1-z)}{1-2(x+y+z)+3(xy+yz+zx)-4xyz}$$

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Idea: If one is able to find a nonzero differential operator of the form

$$L(t,\partial_t,\partial_x,\partial_y) = P(t,\partial_t) + (\text{ higher-order terms in } \partial_x \text{ and } \partial_y\ )$$

that annihilates $G = \frac{1}{xy}\cdot F\left(x,\frac{y}{x},\frac{t}{y}\right)$, then $P(t,\partial_t)$ annihilates $\mathrm{Diag}(F)$.

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Proof:

1. $\operatorname{Diag}(F) = [x^0 y^0]\, F\left(x, \frac{y}{x}, \frac{t}{y}\right)$
2. $0 = L(G) = P(G) + \partial_x(\cdot) + \partial_y(\cdot)$
3. $0 = [x^{-1}y^{-1}]L(G) = [x^{-1}y^{-1}]P(G) = P([x^{-1}y^{-1}]G) = P(\operatorname{Diag}(F))$

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▷ Remaining task: Show that such an L does exist.

Proof of Lipshitz's theorem on the 3D Rooks example

Counting argument: By Leibniz's rule, the $\binom{N+4}{4}$ rational functions

$$t^i \partial_t^j \partial_x^k \partial_y^\ell (G), \quad 0 \le i+j+k+\ell \le N$$

are contained in the $\mathbb{Q}$-vector space of dimension $\le 18(N+1)^3$ spanned by

$$\frac{t^i x^j y^k}{\operatorname{denom}(G)^{N+1}}, \quad 0 \le i \le 2N+1,\ 0 \le j \le 3N+2,\ 0 \le k \le 3N+2.$$

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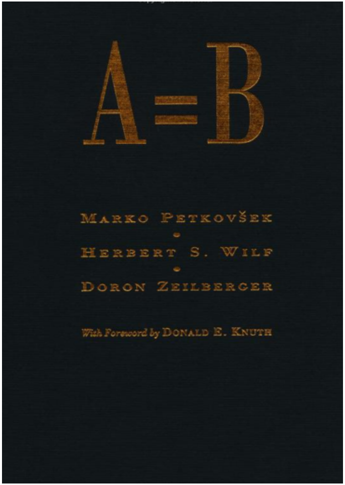
▷ Finding the operator P by Lipshitz' argument would require solving a linear system with 1,391,641,251 unknowns and 1,391,557,968 equations (!)

▷ A better solution is provided by creative telescoping.

CREATIVE TELESCOPING

General framework in computer algebra –initiated by Zeilberger in the '90s– for proving identities on multiple integrals and sums with parameters.



Examples I: hypergeometric summation

- $\displaystyle\sum_{k\in\mathbb{Z}} (-1)^k \binom{a+b}{a+k}\binom{b+c}{b+k}\binom{c+a}{c+k} = \frac{(a+b+c)!}{a!b!c!}$ [Dixon 1903]

- $\displaystyle A_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2$ satisfies the recurrence [Apéry 1978]:

$$(n+1)^3 A_{n+1} = (34n^3 + 51n^2 + 27n + 5) A_n - n^3 A_{n-1}.$$

(*Neither Cohen nor I had been able to prove this in the intervening two months* [Van der Poorten 1979])

- $\displaystyle\sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 = \sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k} \sum_{j=0}^{k} \binom{k}{j}^3$ [Strehl 1992]

Examples I: hypergeometric summation

- $\displaystyle\sum_{k\in\mathbb{Z}} (-1)^k \binom{a+b}{a+k}\binom{b+c}{b+k}\binom{c+a}{c+k} = \frac{(a+b+c)!}{a!b!c!}$ [Dixon 1903]

- $\displaystyle A_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2$ satisfies the recurrence [Apéry 1978]:

$$(n+1)^3 A_{n+1} = (34n^3 + 51n^2 + 27n + 5) A_n - n^3 A_{n-1}.$$

(*The specific problem was mentioned to Don Zagier, who solved it with irritating speed* [Van der Poorten 1979])

- $\displaystyle\sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 = \sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k} \sum_{j=0}^{k} \binom{k}{j}^3$ [Strehl 1992]

Examples II: Integrals and Diagonals

- $$\int_0^1 \frac{\cos(zu)}{\sqrt{1-u^2}}\, du = \int_1^{+\infty} \frac{\sin(zu)}{\sqrt{u^2-1}}\, du = \frac{\pi}{2} J_0(z)$$

- $$\frac{1}{2\pi i} \oint \frac{(1+2xy+4y^2)\exp\left(\frac{4x^2y^2}{1+4y^2}\right)}{y^{n+1}(1+4y^2)^{\frac{3}{2}}}\, dy = \frac{H_n(x)}{\lfloor n/2 \rfloor !}$$ [Doetsch 1930]

- $$\int_0^{+\infty} x J_1(ax) I_1(ax) Y_0(x) K_0(x)\, dx = -\frac{\ln(1-a^4)}{2\pi a^2}$$ [Glasser-Montaldi'94]

- $$\operatorname{Diag} \frac{1}{(1-x-y)(1-z-u)-xyzu} = \sum_{n \geq 0} A_n t^n$$ [Straub 2014].

Summation by Creative Telescoping

$$I_n := \sum_{k=0}^{n} \binom{n}{k} = 2^n.$$

Principle: **IF** one knows Pascal's triangle:

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1} = 2\binom{n}{k} + \binom{n}{k-1} - \binom{n}{k},$$

then summing over k telescopes and yields

$$I_{n+1} = 2I_n.$$

The initial condition $I_0 = 1$ concludes the proof.

$$F_n = \sum_k u_{n,k} =?$$

IF one knows $P(n, S_n)$ (telescoper) and $R(n, k, S_n, S_k)$ (certificate) such that

$$(P(n, S_n) + \Delta_k R(n, k, S_n, S_k)) \cdot u_{n,k} = 0$$

(where Δ_k is the difference operator, $\Delta_k \cdot v_{n,k} = v_{n,k+1} - v_{n,k}$),
then the sum "telescopes", leading to

$$P(n, S_n) \cdot F_n = 0.$$

Zeilberger's Algorithm [1990]

Input: a hypergeometric term $u_{n,k}$, i.e., $\frac{u_{n+1,k}}{u_{n,k}}$ and $\frac{u_{n,k+1}}{u_{n,k}}$ are in $\mathbb{Q}(n,k)$

Output:

- a linear recurrence, called telescoper, (P) satisfied by $F_n = \sum_k u_{n,k}$
- a certificate (Q), for checking the result: $P(n, S_n) \cdot u_{n,k} = \Delta_k Q \cdot u_{n,k}$.

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```
> T := binomial(n,k);
> Zpair:=SumTools[Hypergeometric][Zeilberger](T,n,k,Sn):
> tel:=Zpair[1];
```

$$S_n - 2$$

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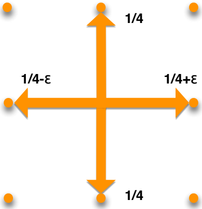
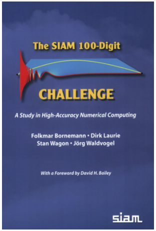
▷ Can check using the certificate:

```
> cert:=Zpair[2];
> iszero:=(subs(n=n+1,T) - 2*T) - (subs(k=k+1,cert) - cert);
> simplify(convert(%,GAMMA));
```

$$0$$

Example: from the SIAM challenges





$$U_{n,k} := \binom{2n}{2k}\binom{2k}{k}\binom{2n-2k}{n-k}\left(\frac{1}{4}+c\right)^k\left(\frac{1}{4}-c\right)^k\frac{1}{4^{2n-2k}},$$

$$p_n = \sum_{k=0}^{n} U_{n,k} \quad = \text{probability of return to } (0,0) \text{ at step } 2n.$$

```
> p:=SumTools[Hypergeometric][Zeilberger](U,n,k,Sn);
```

$$[\left(4\,n^2+16\,n+16\right)Sn^2+\left(-4\,n^2+32\,c^2n^2+96\,c^2n-12\,n+72\,c^2-9\right)Sn$$
$$+128\,c^4n+64\,c^4n^2+48\,c^4, \text{ ...(BIG certificate)...}]$$

$$I(t) = \oint_{\gamma} H(t,x)\, dx = ?$$

IF one knows $P(t, \partial_t)$ (telescoper) and $Q(t, x, \partial_t, \partial_x)$ (certificate) such that

$$\left(P(t, \partial_t) + \partial_x Q(t, x, \partial_t, \partial_x)\right) \cdot H(t, x) = 0,$$

then the integral "telescopes", leading to

$$P(t, \partial_t) \cdot I(t) = 0.$$

The Almkvist-Zeilberger Algorithm [1990]

Input: a hyperexponential function $H(t,x)$, i.e., $\frac{\frac{\partial H}{\partial t}}{H}$ and $\frac{\frac{\partial H}{\partial x}}{H}$ are in $\mathbb{Q}(t,x)$
Output:

- a linear differential operator $P(t,\partial_t)$ satisfied by $I(t) = \oint_\gamma H(t,x)\,dx$
- a $G(t,x) \in \mathbb{Q}(t,x)$ such that $P(t,\partial_t) \cdot H(t,x) = \frac{\partial}{\partial x}\Big(G(t,x) \cdot H(t,x)\Big)$.

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Algorithm: Write $\mathbb{L} = \mathbb{Q}(t)$. For $r = 0,1,2,\ldots$ do

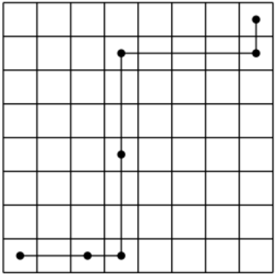
1. compute $a(x) := \frac{\frac{\partial H}{\partial x}}{H} \in \mathbb{L}(x)$ and $b_k(x) := \frac{\frac{\partial^k H}{\partial t^k}}{H} \in \mathbb{L}(x)$ for $k = 0,\ldots,r$
2. decide whether the inhomogeneous parametrized LDE

$$\frac{\partial G}{\partial x} + a(x) \cdot G = \sum_{k=0}^{r} c_k \cdot b_k(x)$$

   admits a rational solution $G \in \mathbb{L}(x)$, for some $c_0,\ldots,c_r \in \mathbb{L}$ not all zero
3. if so, then return $P := \sum_{k=0}^{r} c_k \partial_t^k$ and G; else increase r by 1 and repeat

Example: Diagonal Rook paths

Question: A chess Rook can move any number of squares horizontally or vertically in one step. How many paths can a Rook take from the lower-left corner square to the upper-right corner square of an $N \times N$ chessboard? Assume that the Rook moves only right or up at each step.



$$(r_n)_{n\geq 0}: \quad 1,\ 2,\ 14,\ 106,\ 838,\ 6802,\ 56190,\ 470010,\ \ldots$$

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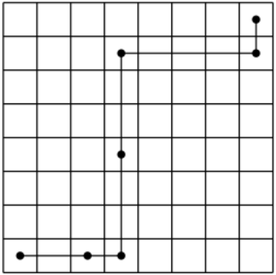


$$(r_n)_{n\geq 0}: \quad 1,\ 2,\ 14,\ 106,\ 838,\ 6802,\ 56190,\ 470010,\ \ldots$$

Answer: r_N = Nth coefficient in the Taylor expansion of $\frac{1}{2}\left(1+\sqrt{\frac{1-t}{1-9t}}\right)$.

Diagonal Rook paths via Creative Telescoping

Generating function of the sequence

$$1,\ 2,\ 14,\ 106,\ 838,\ 6802,\ 56190,\ 470010,\ \ldots$$

is

$$\mathsf{Diag}(F) = [x^0]\, F(x,t/x) = \frac{1}{2\pi i}\oint F(x,t/x)\,\frac{dx}{x}, \quad \text{where} \quad F = \frac{1}{1-\frac{x}{1-x}-\frac{y}{1-y}}.$$

Creative telescoping computes a differential equation satisfied by $\mathsf{Diag}(F)$:

```
> F:=1/(1-x/(1-x)-y/(1-y)):
> G:=normal(1/x*subs(y=t/x,F)):
> Zeilberger(G, t, x, Dt)[1];
```

$$(9t^2-10t+1)\partial_t^2 + (18t-14)\partial_t$$

Conclusion: Generating series of diagonal Rook paths is $\displaystyle \frac{1}{2}\left(1+\sqrt{\frac{1-t}{1-9t}}\right).$

Creative Telescoping for multiple rational integrals

Problem:

$$\left.\begin{aligned} \mathbf{x} = x_1, \dots, x_n &\text{ — integration variables} \\ t &\text{ — parameter} \\ H(t, \mathbf{x}) &\text{ — rational function} \\ \gamma &\text{ — } n\text{-cycle in } \mathbb{C}^n \end{aligned}\right\} \oint_\gamma H(t, \mathbf{x}) d\mathbf{x}$$

Principle of creative telescoping

$$\underbrace{\sum_{k=0}^{r} c_k(t) \frac{\partial^k H}{\partial t^k} = \overbrace{\sum_{i=1}^{n} \frac{\partial A_i}{\partial x_i}}^{\text{certificate}}}_{\text{telescopic relation}} \implies \overbrace{\left(\sum_{k=0}^{r} c_k(t) \partial_t^k \right)}^{\text{telescoper}} \cdot \oint_\gamma H d\mathbf{x} = 0$$

Task:

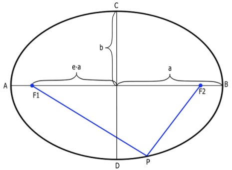
1. find the $c_k(t)$ which satisfy a telescopic relation,
2. ideally, without computing the certificate (A_i).

Example: Perimeter of an ellipse

Example [Euler, 1733]: Perimeter of an ellipse of eccentricity e, semi-major axis 1

Example [Euler, 1733]: Perimeter of an ellipse of eccentricity e, semi-major axis 1

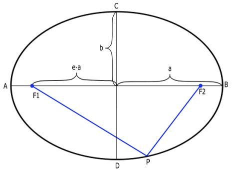
$$p(e) = 4\int_0^1 \sqrt{\frac{1-e^2u^2}{1-u^2}}\mathrm{d}u = 4\oiint \frac{\mathrm{d}u\mathrm{d}v}{1-\frac{1-e^2u^2}{(1-u^2)v^2}}$$



Principle: Find algorithmically

Example [Euler, 1733]: Perimeter of an ellipse of eccentricity e, semi-major axis 1

$$p(e) = 4\int_0^1 \sqrt{\frac{1-e^2u^2}{1-u^2}}\,\mathrm{d}u = 4\oint\!\!\oint \frac{\mathrm{d}u\mathrm{d}v}{1-\frac{1-e^2u^2}{(1-u^2)v^2}}$$



Principle: Find algorithmically

$$\left((e-e^3)\partial_e^2 + (1-e^2)\partial_e + e\right)\cdot\left(\frac{1}{1-\frac{1-e^2u^2}{(1-u^2)v^2}}\right) =$$

$$\partial_u\left(-\frac{e\left(-1-u+u^2+u^3\right)v^2\left(-3+2u+v^2+u^2\left(-2+3e^2-v^2\right)\right)}{\left(-1+v^2+u^2(e^2-v^2)\right)^2}\right)$$

$$+\,\partial_v\left(\frac{2e\left(-1+e^2\right)u\left(1+u^3\right)v^3}{\left(-1+v^2+u^2(e^2-v^2)\right)^2}\right)$$

▷ Conclusion: $p(e) = \frac{\pi}{2}\cdot {}_2F_1\left(\left.\begin{matrix}-\frac{1}{2} & \frac{1}{2}\\ & 1\end{matrix}\,\right|\, e^2\right) = 2\pi - \frac{\pi}{2}e^2 - \frac{3\pi}{32}e^4 - \cdots.$

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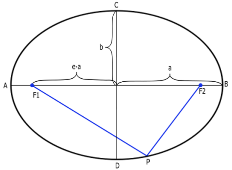

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▷ Drawback: Size(certificate) $\gg$ Size(telescoper).

Example: 3D rook paths [B., Chyzak, van Hoeij, Pech, 2011]

Task: Given $G = \frac{1}{xy} \cdot \left(1 - \sum_{n\geq 1} x^n - \sum_{n\geq 1} (y/x)^n - \sum_{n\geq 1} (t/y)^n\right)^{-1}$
construct a linear differential operator $P(t, \partial_t)$, and two rational functions R and S in $\mathbb{Q}(t, x, y)$ such that

$$P(G) = \frac{\partial R}{\partial x} + \frac{\partial S}{\partial y}.$$

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Solution: Creative telescoping!

```
> G:=subs(y=y/x,z=t/y,1/(1-x/(1-x)-y/(1-y)-z/(1-z)))/y/x:
> P,R,S:=op(op(Mgfun:-creative_telescoping(G,t::diff,[x::diff,y::diff]))):
> P;
```

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$$\begin{aligned} P = \quad & t(t-1)(64t-1)(3t-2)(6t+1)\partial_t^3 \\ & +(4608t^4 - 6372t^3 + 813t^2 + 514t - 4)\partial_t^2 \\ & +4(576t^3 - 801t^2 - 108t + 74)\partial_t \end{aligned}$$

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- ▷ The whole computation takes <10 seconds on a personal laptop.
- ▷ Proves a recurrence conjectured by [Erickson 2010]

Several generations of Creative Telescoping algorithms

- 1G, *brutal elimination*: [Fasenmyer, 1947], [Lipshitz, 1988], [Zeilberger, 1990], [Takayama, 1990], [Wilf, Zeilberger, 1990], [Chyzak, Salvy, 2000]
- 2G, *linear diff/rec rational solving*: [Zeilberger, 1990], [Zeilberger, 1991], [Almkvist, Zeilberger, 1990], [Chyzak, 2000], [Koutschan, 2010]
- 3G, combines *1G + 2G + linear algebra*: [Apagodu, Zeilberger, 2005], [Koutschan 2010], [Chen, Kauers 2012], [Chen, Kauers, Koutschan 2014]

▷ Advantages:

- 1G–3G: very general algorithms;
- 2G/3G algorithms are able to solve non-trivial problems.

▷ Drawbacks:

- 1G: slow;
- 2G: bad or unknown complexity;
- 1G and 3G: non-minimality of telescopers;
- 1G–3G: all compute (big) certificates.

Several generations of Creative Telescoping algorithms

4G: roots in [Ostrogradsky, 1845], [Hermite, 1872] and [Picard, 1902].

- univariate:
  - rational $\int$: [B., Chen, Chyzak, Li, 2010];
  - hyperexponential $\int$: [B., Chen, Chyzak, Li, Xin, 2013]
  - hypergeometric $\sum$: [Chen, Huang, Kauers, Li, 2015], [Huang, 2016]
  - mixed $\int + \sum$: [B., Dumont, Salvy, 2016]
  - algebraic $\int$: [Chen, Kauers, Koutschan, 2016]
  - D-finite Fuchsian $\int$: [Chen, van Hoeij, Kauers, Koutschan, 2018]
  - D-finite $\int$: [B., Chyzak, Lairez, Salvy, 2018], [van der Hoeven, 2018]
- multiple:
  - rational bivariate $\oint\!\!\oint$: [Chen, Kauers, Singer, 2012]
  - rational: [B., Lairez, Salvy, 2013], [Lairez 2016]
  - binomial sums: [B., Lairez, Salvy, 2017]

▷ Advantages:

- complexity;
- minimality of telescopers;
- does not need to compute certificates;
- fast in practice.

▷ Drawback: not (yet) as general as 1G–3G algorithms.

Hermite's method for indefinite integration

Pb. Given coprime $f, g \in \mathbb{K}[x]$, "compute" $\displaystyle\int \frac{f}{g}$ in the following sense:

write $\displaystyle\frac{f}{g} = \partial_x \left(\frac{c}{d}\right) + \frac{a}{b}$ with $a, b, c, d \in \mathbb{K}[x]$, $\deg a < \deg b$ and b squarefree.

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Def. c/d is the *rational part* and $\int a/b$ is the *logarithmic part* of $\int f/g$.

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Idea [Ostrogradsky 1833–1845, Hermite 1872]
1. Compute the (rational) partial fraction decomposition of f/g, that is

$$\frac{f}{g} = P + \sum_{i=1}^{m} \sum_{j=1}^{i} \frac{f_{i,j}}{g_i^j}$$

where $P, f_{i,j}, g_i \in \mathbb{K}[x]$ with $\deg f_{i,j} < \deg g_i$ and $g = g_1 g_2^2 \cdots g_m^m$ a squarefree factorization: g_i's are squarefree and mutually coprime, $g_m \neq 1$.

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Idea [Ostrogradsky 1833–1845, Hermite 1872]
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$$\frac{f}{g} = P + \sum_{i=1}^{m} \sum_{j=1}^{i} \frac{f_{i,j}}{g_i^j}$$

where $P, f_{i,j}, g_i \in \mathbb{K}[x]$ with $\deg f_{i,j} < \deg g_i$ and $g = g_1 g_2^2 \cdots g_m^m$ a squarefree factorization: g_i's are squarefree and mutually coprime, $g_m \neq 1$.
2. Then put into c/d all that comes from P and $j > 1$, and into a/b all that comes from $j = 1$. To do this, use integration by parts and extended gcds.

Hermite's method for indefinite integration

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$\deg a < \deg b$ and b squarefree, such that $\dfrac{f}{g} = \partial_x \left(\dfrac{c}{d}\right) + \dfrac{a}{b}$.

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   Let $i \in \{1, 2, \ldots, m\}$. As $\gcd(g_i, g_i') = 1$, there exist $u, v \in \mathbb{K}[x]$ such that

$$u g_i + v g_i' = f_{i,j}.$$

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Then, if $j > 1$, integration by parts gives

$$\int \frac{f_{i,j}}{g_i^j} = \int \frac{u}{g_i^{j-1}} + \int \frac{v g_i'}{g_i^j} = \frac{-v}{(j-1) g_i^{j-1}} + \int \frac{u + \frac{v'}{j-1}}{g_i^{j-1}}.$$

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   Repeating the process at most j times results in
   $$\int \frac{f_{i,j}}{g_i^j} = \frac{k_{i,j}}{g_i^{j-1}} + \int \frac{\ell_{i,j}}{g_i}, \quad \text{with } \deg k_{i,j} < \deg g_i \text{ and } \deg \ell_{i,j} < \deg g_i.$$

4G Creative Telescoping: univariate case

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where $Q^\star$ is the squarefree part of Q and $\deg_x(a) < d^\star := \deg_x(Q^\star)$.

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▷ $L = \sum_{k=0}^r c_k \partial_t^k$ is a telescoper (and $\sum_{k=0}^r c_k g_k$ the corresponding certificate).

4G Creative Telescoping: multiple case

Algorithm for the integration of rational functions [B., Lairez, Salvy, 2013]

- **Input:** $R(e, \mathbf{x})$ a rational function in e and $\mathbf{x} = x_1, \ldots, x_n$.
- **Output:** A linear ODE $T(e, \partial_e)y = 0$ satisfied by $y(e) = \oint\!\!\oint R(e, \mathbf{x})d\mathbf{x}$.
- **Complexity:** $\mathcal{O}(D^{8n+2})$, where $D = \deg R$.
- **Output size (tight!):** T has order $\leq D^n$ in ∂_e and degree $\leq D^{3n+2}$ in e

▷ Avoids the (costly) computation of certificates, of size $\Omega(D^{n^2/2})$.

▷ Previous algorithms: complexity (at least) doubly exponential in n.

▷ Very efficient in practice.

Griffiths–Dwork method for the generic case

Linear reduction classical in algebraic geometry;
Generalization of Hermite's reduction.

Fast linear algebra on polynomial matrices

Macaulay matrices encoding Gröbner bases computations;
Sophisticated algorithms due to Villard, Storjohann, Zhou, etc.

Deformation technique for the general case

Input perturbation using a new free variable.

▷ Highly non-trivial extension by [Lairez, 2016]: tremendously improves the efficiency of the algorithm in [B., Lairez, Salvy, 2013]

Two exercises for next time (8/2/2021)

1. Compute a telescoper for the diagonal of the rational power series

$$\frac{1}{1-x-y} = \sum_{i,j\geq 0} \binom{i+j}{i} x^i y^j$$

   in two different ways:

   1. using the 2G (Almkvist-Zeilberger) creative telescoping algorithm;
   2. using the 4G (Hermite reduction-based) creative telescoping algorithm.

2. Let $f, g \in \mathbb{Q}[x]$ be two coprime polynomials. Let $h \in \mathbb{Q}[x]$ be another polynomial such that $\deg h < \deg f + \deg g$.

   1. Show that the equation

$$sf + tg = h$$

      admits an unique solution $(s, t) \in \mathbb{Q}[x]^2$ s.t. $\deg s < \deg g, \quad \deg t < \deg f$.
   2. Design an algorithm for computing the solution (s, t) starting from (f, g, h) in quasi-optimal complexity.