Diagonals and Creative Telescoping
—From 1G to 4G algorithms—

Alin Bostan

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Creative Telescoping

\[
\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{2020 \cdot 2021} = ?
\]

[Knuth 1969] Ex. 1.2.6.63: Develop computer programs for simplifying sums that involve binomial coefficients.

Today: how computer algebra uses Bernoulli's 1682 idea –systematically and algorithmically–, to solve Knuth's 1969 exercise, and more
Creative Telescoping

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\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{2020 \cdot 2021} = ?
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▷ [J. Bernoulli 1682]: Use \( \frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1} \) to create a telescoping sum

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\left( \frac{1}{1} - \frac{1}{2} \right) + \left( \frac{1}{2} - \frac{1}{3} \right) + \cdots + \left( \frac{1}{2020} - \frac{1}{2021} \right) = 1 - \frac{1}{2021}.
\]
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[50] Develop computer programs for simplifying sums that involve binomial coefficients.

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Diagonals and Creative Telescoping
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[50] Develop computer programs for simplifying sums that involve binomial coefficients.

▷ Today: how computer algebra uses Bernoulli’s 1682 idea –systematically and algorithmically–, to solve Knuth’s 1969 exercise, and more
DIAGONALS
Definition

If $F$ is a multivariate power series

$$F = \sum_{i_1,\ldots,i_n \geq 0} a_{i_1,\ldots,i_n} x_1^{i_1} \cdots x_n^{i_n},$$

its diagonal is the univariate power series

$$\text{Diag}(F) \overset{\text{def}}{=} \sum_i a_{i,i} t^i.$$

This is false for more than 2 variables. E.g.

$$\text{Diag}(1 - x - y - z) = \sum_{n \geq 0} (3^n n^n, n^n, n^n) t^n = 2F_1$$

is transcendental
Definition

*If* $F$ *is a multivariate power series*

$$F = \sum_{i_1,\ldots,i_n \geq 0} a_{i_1,\ldots,i_n} x_1^{i_1} \cdots x_n^{i_n},$$

*its diagonal* is the univariate power series

$$\text{Diag}(F) \overset{\text{def}}{=} \sum_{i} a_{i,\ldots,i} t^i.$$

**Example:** if $n = 1$, then (trivially)

$$\text{Diag}(F) = F(t).$$
Definition

If $F$ is a multivariate power series

$$F = \sum_{i_1, \ldots, i_n \geq 0} a_{i_1, \ldots, i_n} x_1^{i_1} \cdots x_n^{i_n},$$

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$$\text{Diag}(F) \overset{\text{def}}{=} \sum_i a_{i, \ldots, i} t^i.$$

Example: if $n = 2$ and $F = \frac{1}{1 - x - y} = \sum_{i,j \geq 0} \binom{i+j}{i} x^i y^j$, then

$$\text{Diag} (F) = \sum_{n \geq 0} \binom{2n}{n} t^n = 1 + 2 t + 6 t^2 + 20 t^3 + 70 t^4 + \cdots.$$
Definition

If $F$ is a multivariate power series

$$F = \sum_{i_1, \ldots, i_n \geq 0} a_{i_1, \ldots, i_n} x_1^{i_1} \cdots x_n^{i_n},$$

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$$\text{Diag}(F) = \sum_{n \geq 0} \binom{2n}{n} t^n = 1 + 2 t + 6 t^2 + 20 t^3 + 70 t^4 + \cdots.$$

$\triangleright$ Diag($F$) is not a rational function, even though $F$ is rational.
Pólya’s theorem

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If $F$ is a multivariate power series

$$F = \sum_{i_1, \ldots, i_n \geq 0} a_{i_1, \ldots, i_n} x_1^{i_1} \cdots x_n^{i_n},$$

its **diagonal** is the univariate power series

$$\text{Diag}(F) \overset{\text{def}}{=} \sum_i a_{i, \ldots, i} t^i.$$

**Theorem (Pólya, 1922)**

Diagonals of *bivariate* rational functions are **algebraic**.
Pólya’s theorem

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$$\text{Diag}(F) \overset{\text{def}}{=} \sum_i a_{i_1,\ldots,i_t} t^i.$$

Theorem (Pólya, 1922)

Diagonals of bivariate rational functions are algebraic.

Proof: Since $F \left( x, \frac{t}{x} \right) = \sum_{i,j} a_{i,j} x^{i-j} t^j$ we have that $\text{Diag}(F) = [x^0]F \left( x, \frac{t}{x} \right)$.

Therefore, by Cauchy’s integral theorem,

$$\text{Diag}(F) = [x^{-1}] \frac{1}{x} F \left( x, \frac{t}{x} \right) = \frac{1}{2\pi i} \oint_{|x|=\epsilon} F \left( x, \frac{t}{x} \right) \frac{dx}{x}.$$ 

By the Residue Theorem: last integral is a sum of residues, all algebraic. □
Example: Dyck walks

Let $B_n$ be the number of Dyck bridges, i.e. $\{\text{NE, SE}\}$-walks of length $n$ in $\mathbb{Z}^2$ starting at $(0,0)$ and ending on the horizontal axis.

Equivalently, $B_n =$ number of $\{\text{N, E}\}$-walks in $\mathbb{Z}^2$ from $(0,0)$ to $(n,n)$

$$
\implies B(t) = \sum_{n \geq 0} B_n t^n = \text{Diag} \left( \frac{1}{1 - x - y} \right)
$$

Then:
$$
B(t) = \frac{1}{2\pi i} \oint_{|x|=\epsilon} \frac{dx}{x - x^2 - t} = \left. \frac{1}{1 - 2x} \right|_{x = \frac{1 - \sqrt{1 - 4t}}{2}} = \frac{1}{\sqrt{1 - 4t}}
$$
Example: Dyck walks

Let $B_n$ be the number of Dyck bridges, i.e. \{NE, SE\}-walks of length $n$ in $\mathbb{Z}^2$ starting at $(0,0)$ and ending on the horizontal axis.

Equivalently, $B_n = \text{number of } \{N, E\}\text{-walks in } \mathbb{Z}^2 \text{ from } (0,0) \text{ to } (n,n) = \binom{2n}{n}$

$$\implies B(t) = \sum_{n \geq 0} B_n t^n = \text{Diag} \left( \frac{1}{1 - x - y} \right)$$

Then: $B(t) = \frac{1}{2\pi i} \oint_{|x| = \epsilon} \frac{dx}{x - x^2 - t} = \frac{1}{\sqrt{1 - 4t}} = \sum_{n \geq 0} \binom{2n}{n} t^n$
Rothstein-Trager resultant

Let $A, B \in \mathbb{K}[x]$ be such that $\deg(A) < \deg(B)$, with $B$ squarefree. In particular, the rational function $F = A/B$ has simple poles only.

Lemma. The residue $r_i$ of $F$ at the pole $p_i$ equals $r_i = \frac{A(p_i)}{B'(p_i)}$.

Proof. If $F = \sum_i \frac{r_i}{x - p_i}$, then $r_i = (F \cdot (x - p_i))|_{x=p_i} = \frac{A(x)}{\prod_{j \neq i}(x - p_j)}(p_i)$

Theorem. The residues $r_i$ of $F$ are roots of the resultant

$$R(t) = \text{Res}_x(B(x), A(x) - t \cdot B'(x)).$$

Proof. By Poisson's formula: $R(t) = \prod_i \left( A(p_i) - t \cdot B'(p_i) \right)$. □

Introduced by [Rothstein-Trager 1976] for the (indefinite) integration of rational functions.

Example: diagonal Rook paths

**Question:** A chess Rook can move any number of squares horizontally or vertically in one step. How many paths can a Rook take from the lower-left corner square to the upper-right corner square of an $N \times N$ chessboard? Assume that the Rook moves only right or up at each step.

1, 2, 14, 106, 838, 6802, 56190, 470010, …
Example: diagonal Rook paths

Generating function of the sequence

\[1, 2, 14, 106, 838, 6802, 56190, 470010, \ldots\]

is

\[
\text{Diag}(F) = [x^0] F(x, t/x) = \frac{1}{2\pi i} \oint F(x, t/x) \frac{dx}{x}, \quad \text{where} \quad F = \frac{1}{1 - \frac{x}{1-x} - \frac{y}{1-y}}.
\]

Residue theorem: Diag(\(F\)) is a sum of roots \(y\) of the Rothstein-Trager resultant

\[
> F:=1/(1-x/(1-x)-y/(1-y)):
> G:=normal(1/x*subs(y=t/x,F)):
> factor(resultant(denom(G),numer(G)-y*diff(denom(G),x),x));
\]

\[
t^2(1 - t)(2y - 1)(36ty^2 - 4y^2 + 1 - t)
\]

Answer: Generating series of diagonal Rook paths is

\[
\frac{1}{2} \left(1 + \sqrt{\frac{1 - t}{1 - 9t}}\right).
\]
Pólya’s theorem

**Definition**

If $F$ is a formal power series

$$F = \sum_{i_1, \ldots, i_n \geq 0} a_{i_1, \ldots, i_n} x_1^{i_1} \cdots x_n^{i_n},$$

its **diagonal** is

$$\text{Diag}(F) \overset{\text{def}}{=} \sum_i a_{i, \ldots, i} t^i.$$ 

**Theorem (Pólya, 1922)**

Diagonals of **bivariate** rational functions are **algebraic**. \(^a\)

\(^a\)The converse is also true [Furstenberg, 1967]
Definition

If $F$ is a formal power series

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Theorem (Pólya, 1922)

Diagonals of bivariate rational functions are algebraic.

This is false for more than 2 variables. E.g.

$$\text{Diag} \left( \frac{1}{1 - x - y - z} \right) = \sum_{n \geq 0} \frac{(3n)!}{n!^3} t^n = 2F_1 \left( \begin{array}{c} \frac{1}{3} \\ \frac{2}{3} \end{array} \bigg| 27t \right)$$

is transcendental.
Pólya’s theorem

Definition

If $F$ is a formal power series

$$F = \sum_{i_1, \ldots, i_n \geq 0} a_{i_1, \ldots, i_n} x_1^{i_1} \cdots x_n^{i_n},$$

its diagonal is

$$\text{Diag}(F) \overset{\text{def}}{=} \sum_{i} a_{i_i, \ldots, i_n} t^i.$$

Theorem (Pólya, 1922)

Diagonals of bivariate rational functions are algebraic and thus D-finite.

- Algebraic equation has exponential size [B., Dumont, Salvy, 2015]
- Differential equation has polynomial size [B., Chen, Chyzak, Li, 2010]
**Lipshitz’s theorem**

**Definition**

If $F$ is a formal power series

\[
F = \sum_{i_1,\ldots,i_n \geq 0} a_{i_1,\ldots,i_n} x_1^{i_1} \cdots x_n^{i_n},
\]

its **diagonal** is

\[
\text{Diag}(F) \overset{\text{def}}{=} \sum_i a_{i_1,\ldots,i_n} t^i.
\]

**Theorem (Lipshitz, 1988)**

Diagonals of **multivariate** rational functions are **D-finite**.
Question [Erickson 2010]
How many ways can a Rook move from (0, 0, 0) to (N, N, N), where each step is a positive integer multiple of (1, 0, 0), (0, 1, 0), or (0, 0, 1)?

1, 6, 222, 9918, 486924, 25267236, 1359631776, 75059524392, …

Answer [B., Chyzak, van Hoeij, Pech, 2011]: GF of 3D diagonal Rook paths is

\[
G(t) = 1 + 6 \cdot \int_{0}^{t} \frac{2F_1\left(\begin{array}{c} 1/3, 2/3 \\ 2 \\ \end{array} \right| \frac{27x(2-3x)}{(1-4x)^3}}{(1-4x)(1-64x)} \, dx
\]
Proof of Lipshitz’s theorem on the 3D Rooks example

**Problem:** Show that $\text{Diag}(F)$ is D-finite, where $F(x, y, z)$ is

\[
\left(1 - \sum_{n \geq 1} x^n - \sum_{n \geq 1} y^n - \sum_{n \geq 1} z^n\right)^{-1} = \frac{(1 - x)(1 - y)(1 - z)}{1 - 2(x + y + z) + 3(xy + yz + zx) - 4xyz}
\]
**Proof of Lipshitz’s theorem on the 3D Rooks example**

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$$

**Idea:** If one is able to find a nonzero differential operator of the form

$$
L(t, \partial_t, \partial_x, \partial_y) = P(t, \partial_t) + (\text{higher-order terms in } \partial_x \text{ and } \partial_y)
$$

that annihilates $G = \frac{1}{xy} \cdot F\left(x, \frac{y}{x}, \frac{t}{y}\right)$, then $P(t, \partial_t)$ annihilates $\text{Diag}(F)$.
Proof of Lipshitz’s theorem on the 3D Rooks example

**Problem:** Show that \( \text{Diag}(F) \) is D-finite, where \( F(x, y, z) \) is

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that annihilates \( G = \frac{1}{xy} \cdot F \left( x, \frac{y}{x}, \frac{t}{y} \right) \), then \( P(t, \partial_t) \) annihilates \( \text{Diag}(F) \).

**Proof:**

1. \( \text{Diag}(F) = \left[ x^0 y^0 \right] F \left( x, \frac{y}{x}, \frac{t}{y} \right) \)
2. \( 0 = L(G) = P(G) + \partial_x(\cdot) + \partial_y(\cdot) \)
3. \( 0 = \left[ x^{-1} y^{-1} \right] L(G) = \left[ x^{-1} y^{-1} \right] P(G) = P(\left[ x^{-1} y^{-1} \right] G) = P(\text{Diag}(F)) \)
Proof of Lipshitz’s theorem on the 3D Rooks example

Problem: Show that $\text{Diag}(F)$ is D-finite, where $F(x, y, z)$ is

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\left( 1 - \sum_{n \geq 1} x^n - \sum_{n \geq 1} y^n - \sum_{n \geq 1} z^n \right)^{-1} = \frac{(1 - x)(1 - y)(1 - z)}{1-2(x+y+z)+3(xy+yz+zx)-4xyz}
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Idea: If one is able to find a nonzero differential operator of the form

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L(t, \partial_t, \partial_x, \partial_y) = P(t, \partial_t) + ( \text{higher-order terms in } \partial_x \text{ and } \partial_y )
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that annihilates $G = \frac{1}{xy} \cdot F \left( x, \frac{y}{x}, \frac{t}{y} \right)$, then $P(t, \partial_t)$ annihilates $\text{Diag}(F)$.

▷ Remaining task: Show that such an $L$ does exist.
Counting argument: By Leibniz’s rule, the \( \binom{N+4}{4} \) rational functions

\[
t^i \partial_i^j \partial_x^k \partial_y^\ell (G), \quad 0 \leq i + j + k + \ell \leq N
\]

are contained in the \( \mathbb{Q} \)-vector space of dimension \( \leq 18(N+1)^3 \) spanned by

\[
\frac{t^i x^j y^k}{\text{denom}(G)^{N+1}}, \quad 0 \leq i \leq 2N + 1, \ 0 \leq j \leq 3N + 2, \ 0 \leq k \leq 3N + 2.
\]
Proof of Lipshitz’s theorem on the 3D Rooks example

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\( N \) is such that \( \# \text{ unknowns} = \binom{N+4}{4} > 18(N + 1)^3 = \# \text{ equations} \), then there exists \( L(t, \partial_i, \partial_x, \partial_y) \) of total degree at most \( N \), such that \( LG = 0 \).
Counting argument: By Leibniz’s rule, the \( \binom{N+4}{4} \) rational functions

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\[ \frac{t^i x^j y^k}{\text{denom}(G)^{N+1}}, \quad 0 \leq i \leq 2N + 1, \; 0 \leq j \leq 3N + 2, \; 0 \leq k \leq 3N + 2. \]

\( \triangleright \) If \( N \) is such that \# unknowns = \( \binom{N+4}{4} \) > \( 18(N + 1)^3 \) = \# equations, then there exists \( P(t, \partial_t) \) of total degree at most \( N \), such that \( P(\text{Diag}(F)) = 0. \)
Proof of Lipshitz’s theorem on the 3D Rooks example

Counting argument: By Leibniz’s rule, the \( \binom{N+4}{4} \) rational functions

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▷ If \( N \) is such that \( \# \text{ unknowns} = \binom{N+4}{4} > 18(N + 1)^3 = \# \text{ equations} \), then there exists \( P(t, \partial_t) \) of total degree at most \( N \), such that \( P(\text{Diag}(F)) = 0 \).

▷ \( N = 425 \) is the smallest integer satisfying \( \binom{N+4}{4} > 18(N + 1)^3 \) (!)
Proof of Lipshitz’s theorem on the 3D Rooks example

Counting argument: By Leibniz’s rule, the \( \binom{N+4}{4} \) rational functions

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▷ \( N = 425 \) is the smallest integer satisfying \( \binom{N+4}{4} > 18(N + 1)^3 \)

▷ Finding the operator \( P \) by Lipshitz’ argument would require solving a linear system with 1,391,641,251 unknowns and 1,391,557,968 equations.
Counting argument: By Leibniz’s rule, the $\binom{N+4}{4}$ rational functions

$$t^i \partial^j_t \partial^k_x \partial^\ell_y (G), \quad 0 \leq i + j + k + \ell \leq N$$

are contained in the $\mathbb{Q}$-vector space of dimension $\leq 18(N + 1)^3$ spanned by

$$\frac{t^i x^j y^k}{\text{denom}(G)^{N+1}}, \quad 0 \leq i \leq 2N + 1, \ 0 \leq j \leq 3N + 2, \ 0 \leq k \leq 3N + 2.$$

▷ If $N$ is such that $\# \text{unknowns} = \binom{N+4}{4} > 18(N + 1)^3 = \# \text{equations}$, then there exists $P(t, \partial_t)$ of total degree at most $N$, such that $P(\text{Diag}(F)) = 0$.

▷ $N = 425$ is the smallest integer satisfying $\binom{N+4}{4} > 18(N + 1)^3$ (!)

▷ Finding the operator $P$ by Lipshitz’ argument would require solving a linear system with 1,391,641,251 unknowns and 1,391,557,968 equations (!)

▷ A better solution is provided by creative telescoping.
CREATIVE TELESCOPING
Creative Telescoping

General framework in computer algebra –initiated by Zeilberger in the ‘90s– for proving identities on multiple integrals and sums with parameters.
Examples I: hypergeometric summation

\[
\sum_{k \in \mathbb{Z}} (-1)^k \left( \frac{a+b}{a+k} \right) \left( \frac{b+c}{b+k} \right) \left( \frac{c+a}{c+k} \right) = \frac{(a+b+c)!}{a!b!c!} \quad \text{[Dixon 1903]}
\]

\[
A_n = \sum_{k=0}^{n} \left( \binom{n}{k} \right)^2 \left( \binom{n+k}{k} \right)^2 \text{ satisfies the recurrence [Apéry 1978]:}
\]
\[
(n+1)^3 A_{n+1} = (34n^3 + 51n^2 + 27n + 5) A_n - n^3 A_{n-1}.
\]

(\textit{Neither Cohen nor I had been able to prove this in the intervening two months [Van der Poorten 1979]})

\[
\sum_{k=0}^{n} \left( \binom{n}{k} \right)^2 \left( \binom{n+k}{k} \right)^2 = \sum_{k=0}^{n} \left( \binom{n}{k} \right) \left( \binom{n+k}{k} \right) \sum_{j=0}^{k} \left( \binom{k}{j} \right)^3 \quad \text{[Strehl 1992]}
\]
Examples I: hypergeometric summation

\[ \sum_{k \in \mathbb{Z}} (-1)^k \binom{a + b}{a + k} \binom{b + c}{b + k} \binom{c + a}{c + k} = \frac{(a + b + c)!}{a!b!c!} \quad [\text{Dixon 1903}] \]

\[ A_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n + k}{k}^2 \quad \text{satisfies the recurrence [Apéry 1978]}: \]

\[ (n + 1)^3 A_{n+1} = (34n^3 + 51n^2 + 27n + 5) A_n - n^3 A_{n-1}. \]

(The specific problem was mentioned to Don Zagier, who solved it with irritating speed [Van der Poorten 1979])

\[ \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n + k}{k}^2 = \sum_{k=0}^{n} \binom{n}{k} \binom{n + k}{k} \sum_{j=0}^{k} \binom{k}{j}^3 \quad [\text{Strehl 1992}] \]
Examples II: Integrals and Diagonals

\[ \int_0^1 \frac{\cos(zu)}{\sqrt{1-u^2}} \, du = \int_1^{+\infty} \frac{\sin(zu)}{\sqrt{u^2-1}} \, du = \frac{\pi}{2} J_0(z) \]

\[ \frac{1}{2\pi i} \oint \frac{(1+2xy+4y^2) \exp\left(\frac{4x^2y^2}{1+4y^2}\right)}{y^{n+1}(1+4y^2)^{\frac{3}{2}}} \, dy = \frac{H_n(x)}{[n/2]!} \quad \text{[Doetsch 1930]} \]

\[ \int_0^{+\infty} x J_1(ax) I_1(ax) Y_0(x) K_0(x) \, dx = -\frac{\ln(1-a^4)}{2\pi a^2} \quad \text{[Glasser-Montaldi’94]} \]

\[ \text{Diag} \, \frac{1}{(1-x-y)(1-z-u) - xyzu} = \sum_{n \geq 0} A_n t^n \quad \text{[Straub 2014].} \]
Summation by Creative Telescoping

\[ I_n := \sum_{k=0}^{n} \binom{n}{k} = 2^n. \]

Principle: **IF** one knows Pascal’s triangle:

\[ \binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1} = 2\binom{n}{k} + \binom{n}{k-1} - \binom{n}{k}, \]

then summing over \( k \) telescopes and yields

\[ I_{n+1} = 2I_n. \]

The initial condition \( I_0 = 1 \) concludes the proof.
\[ F_n = \sum_{k} u_{n,k} = ? \]

**IF** one knows \( P(n, S_n) \) (telescoper) and \( R(n, k, S_n, S_k) \) (certificate) such that

\[
(P(n, S_n) + \Delta_k R(n, k, S_n, S_k)) \cdot u_{n,k} = 0
\]

(where \( \Delta_k \) is the difference operator, \( \Delta_k \cdot u_{n,k} = u_{n,k+1} - u_{n,k} \)),

then the sum “telescopes”, leading to

\[
P(n, S_n) \cdot F_n = 0.
\]
Zeilberger’s Algorithm [1990]

Input: a hypergeometric term $u_{n,k}$, i.e., $\frac{u_{n+1,k}}{u_{n,k}}$ and $\frac{u_{n,k+1}}{u_{n,k}}$ are in $\mathbb{Q}(n,k)$

Output:
- a linear recurrence, called telescoper, $(P)$ satisfied by $F_n = \sum_k u_{n,k}$
- a certificate $(Q)$, for checking the result: $P(n, S_n) \cdot u_{n,k} = \Delta_k Q \cdot u_{n,k}$. 

\begin{verbatim}
> T := binomial(n,k);
> Zpair:=SumTools[Hypergeometric][Zeilberger](T,n,k,Sn):
> tel:=Zpair[1];

S_n - 2 \Delta
\end{verbatim}
Zeilberger’s Algorithm [1990]

**Input:** a hypergeometric term $u_{n,k}$, i.e., $\frac{u_{n+1,k}}{u_{n,k}}$ and $\frac{u_{n,k+1}}{u_{n,k}}$ are in $\mathbb{Q}(n,k)$

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```

\[ S_n - 2 \]
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\[
S_n - 2
\]

▷ This is a proof that \( I_n := \sum_{k=0}^n \binom{n}{k} \) satisfies \( I_{n+1} = 2 \cdot I_n \).
Zeilberger's Algorithm [1990]

Input: a hypergeometric term $u_{n,k}$, i.e., $\frac{u_{n+1,k}}{u_{n,k}}$ and $\frac{u_{n,k+1}}{u_{n,k}}$ are in $\mathbb{Q}(n,k)$

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```plaintext
> T := binomial(n,k);
> Zpair:=SumTools[Hypergeometric][Zeilberger](T,n,k,Sn):
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S_n - 2

▶ This is a proof that $I_n := \sum_{k=0}^{n} (\binom{n}{k})$ satisfies $I_{n+1} = 2 \cdot I_n$.
▶ Can check using the certificate:

```plaintext
> cert:=Zpair[2];
> iszero:=(subs(n=n+1,T) - 2*T) - (subs(k=k+1,cert) - cert);
> simplify(convert(%,GAMMA));

0
```
Example: from the SIAM challenges

\[ U_{n,k} := \binom{2n}{2k} \binom{2k}{k} \left( \frac{2n - 2k}{n - k} \right) \left( \frac{1}{4} + c \right)^k \left( \frac{1}{4} - c \right)^k \frac{1}{4^{2n-2k}}, \]

\[ p_n = \sum_{k=0}^{n} U_{n,k} = \text{probability of return to (0,0) at step } 2n. \]

> p:=SumTools[Hypergeometric][Zeilberger](U,n,k,Sn);

\[ \left( 4n^2 + 16n + 16 \right) Sn^2 + \left( -4n^2 + 32c^2n^2 + 96c^2n - 12n + 72c^2 - 9 \right) Sn \]
\[ + 128c^4n + 64c^4n^2 + 48c^4, \ldots \text{(BIG certificate)} \ldots \]
Creative Telescoping for Integrals

$I(t) = \int_\gamma H(t, x) \, dx = \, ?$

**IF** one knows $P(t, \partial_t)$ (telescooper) and $Q(t, x, \partial_t, \partial_x)$ (certificate) such that

$$(P(t, \partial_t) + \partial_x Q(t, x, \partial_t, \partial_x)) \cdot H(t, x) = 0,$$

then the integral “telescopes”, leading to

$$P(t, \partial_t) \cdot I(t) = 0.$$
The Almkvist-Zeilberger Algorithm [1990]

**Input:** a hyperexponential function $H(t, x)$, i.e., $\frac{\partial H}{\partial t}$ and $\frac{\partial H}{\partial x}$ are in $Q(t, x)$

**Output:**

- a linear differential operator $P(t, \partial_t)$ satisfied by $I(t) = \int_{\gamma} H(t, x) \, dx$
- a $G(t, x) \in Q(t, x)$ such that $P(t, \partial_t) \cdot H(t, x) = \frac{\partial}{\partial x} \left( G(t, x) \cdot H(t, x) \right)$.
The Almkvist-Zeilberger Algorithm [1990]

Input: a hyperexponential function $H(t, x)$, i.e., $\frac{\partial H}{\partial t}$ and $\frac{\partial H}{\partial x}$ are in $Q(t, x)$

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- a linear differential operator $P(t, \partial_t)$ satisfied by $I(t) = \oint_{\gamma} H(t, x) \, dx$
- a $G(t, x) \in Q(t, x)$ such that $P(t, \partial_t) \cdot H(t, x) = \frac{\partial}{\partial x} \left( G(t, x) \cdot H(t, x) \right)$.

Algorithm: Write $\mathbb{L} = Q(t)$. For $r = 0, 1, 2, \ldots$ do

1. compute $a(x) := \frac{\partial H}{\partial x} \in \mathbb{L}(x)$ and $b_k(x) := \frac{\partial^k H}{\partial t^k} \in \mathbb{L}(x)$ for $k = 0, \ldots, r$

2. decide whether the inhomogeneous parametrized LDE

$$\frac{\partial G}{\partial x} + a(x) \cdot G = \sum_{k=0}^{r} c_k \cdot b_k(x)$$

admits a rational solution $G \in \mathbb{L}(x)$, for some $c_0, \ldots, c_r \in \mathbb{L}$ not all zero

3. if so, then return $P := \sum_{k=0}^{r} c_k \partial_t^k$ and $G$; else increase $r$ by 1 and repeat
**Example: Diagonal Rook paths**

**Question:** A chess Rook can move any number of squares horizontally or vertically in one step. How many paths can a Rook take from the lower-left corner square to the upper-right corner square of an $N \times N$ chessboard? Assume that the Rook moves only right or up at each step.

\[
(r_n)_{n \geq 0} : \quad 1, 2, 14, 106, 838, 6802, 56190, 470010, \ldots
\]
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\[ (r_n)_{n \geq 0} : \quad 1, 2, 14, 106, 838, 6802, 56190, 470010, \ldots \]

**Answer:** \( r_N = N \)th coefficient in the Taylor expansion of \( \frac{1}{2} \left( 1 + \sqrt{\frac{1 - t}{1 - 9t}} \right) \).
Diagonal Rook paths via Creative Telescoping

Generating function of the sequence

$$1, 2, 14, 106, 838, 6802, 56190, 470010, \ldots$$

is

$$\text{Diag}(F) = [x^0] F(x, t/x) = \frac{1}{2\pi i} \int F(x, t/x) \frac{dx}{x}, \quad \text{where} \quad F = \frac{1}{1 - \frac{x}{1-x} - \frac{y}{1-y}}.$$

Creative telescoping computes a differential equation satisfied by $\text{Diag}(F)$:

$$\partial_t^2 + (18t - 14) \partial_t$$

Conclusion: Generating series of diagonal Rook paths is

$$\frac{1}{2} \left( 1 + \sqrt{\frac{1-t}{1-9t}} \right).$$
Creative Telescoping for multiple rational integrals

**Problem:**
\[ x = x_1, \ldots, x_n \quad \text{— integration variables} \]
\[ t \quad \text{— parameter} \]
\[ H(t, x) \quad \text{— rational function} \]
\[ \gamma \quad \text{— } n\text{-cycle in } \mathbb{C}^n \]
\[ \int_{\gamma} H(t, x) \, dx \]

**Principle of creative telescoping**

\[ \sum_{k=0}^{r} c_k(t) \frac{\partial^k H}{\partial t^k} = \sum_{i=1}^{n} \frac{\partial A_i}{\partial x_i} \]
\[ \Rightarrow \left( \sum_{k=0}^{r} c_k(t) \frac{\partial^k}{\partial t^k} \right) \cdot \int_{\gamma} H \, dx = 0 \]

**Task:**

1. find the \( c_k(t) \) which satisfy a telescopic relation,
2. ideally, without computing the certificate \((A_i)\).
Example: Perimeter of an ellipse

Example [Euler, 1733]: Perimeter of an ellipse of eccentricity \( e \), semi-major axis 1
Example: Perimeter of an ellipse

Example [Euler, 1733]: Perimeter of an ellipse of eccentricity \( e \), semi-major axis 1

\[
p(e) = 4 \int_0^1 \sqrt{\frac{1 - e^2 u^2}{1 - u^2}} \, du = 4 \iint \frac{dudv}{1 - \frac{1-e^2u^2}{(1-u^2)v^2}}
\]

Principle: Find algorithmically
Example: Perimeter of an ellipse

Example [Euler, 1733]: Perimeter of an ellipse of eccentricity $e$, semi-major axis 1

$$p(e) = 4 \int_0^1 \sqrt{\frac{1 - e^2 u^2}{1 - u^2}} \, du = 4 \iint \frac{dudv}{1 - \frac{1-e^2 u^2}{(1-u^2)v^2}}$$

**Principle:** Find algorithmically

\[
\left((e - e^3)\partial_e^2 + (1 - e^2)\partial_e + e\right) \cdot \left(\frac{1}{1 - \frac{1-e^2 u^2}{(1-u^2)v^2}}\right) = \\
\partial_u \left(\frac{-e(-1-u+u^2+u^3)v^2(-3+2u+v^2+u^2(-2+3e^2-v^2))}{(-1+v^2+u^2(e^2-v^2))^2}\right) + \\
\partial_v \left(\frac{2e(-1+e^2)u(1+u^3)v^3}{(-1+v^2+u^2(e^2-v^2))^2}\right)
\]

**Conclusion:**

$$p(e) = \frac{\pi}{2} \cdot 2 F_1 \left(\begin{array}{c}
-\frac{1}{2} \\
\frac{1}{2}
\end{array} \Bigg| e^2 \right) = 2\pi - \frac{\pi}{2} e^2 - \frac{3\pi}{32} e^4 - \cdots.$$
Example: Perimeter of an ellipse

**Example [Euler, 1733]: Perimeter of an ellipse** of eccentricity $e$, semi-major axis 1

$$p(e) = 4 \int_0^1 \sqrt{1 - e^2 u^2} \, du = 4 \iint \frac{dudv}{1 - \frac{1-e^2 u^2}{(1-u^2)v^2}}$$

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$$\left((e - e^3) \partial_e^2 + (1 - e^2) \partial_e + e\right) \cdot \left(\frac{1}{1 - \frac{1-e^2 u^2}{(1-u^2)v^2}}\right) =$$

$$\partial_u \left(\frac{e(-1-u+u^2+u^3)v^2(-3+2u+v^2+u^2(-2+3e^2-v^2))}{(-1+v^2+u^2(e^2-v^2))^2}\right) + \partial_v \left(\frac{2e(-1+e^2)u(1+u^3)v^3}{(-1+v^2+u^2(e^2-v^2))^2}\right)$$

▷ **Drawback:** Size(certificate) $\gg$ Size(telescoper).
Example: 3D rook paths [B., Chyzak, van Hoeij, Pech, 2011]

**Task:** Given $G = \frac{1}{xy} \cdot (1 - \sum_{n \geq 1} x^n - \sum_{n \geq 1} (y/x)^n - \sum_{n \geq 1} (t/y)^n)^{-1}$ construct a linear differential operator $P(t, \partial_t)$, and two rational functions $R$ and $S$ in $\mathbb{Q}(t, x, y)$ such that

$$P(G) = \frac{\partial R}{\partial x} + \frac{\partial S}{\partial y}.$$
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$$P(G) = \frac{\partial R}{\partial x} + \frac{\partial S}{\partial y}.$$ 

**Solution:** Creative telescoping!

```plaintext
> G:=subs(y=y/x,z=t/y,1/(1-x/(1-x)-y/(1-y)-z/(1-z))/y/x:
> P,R,S:=op(op(Mgfun:-creative_telescoping(G,t::diff,[x::diff,y::diff]))):
> P;
```
Example: 3D rook paths [B., Chyzak, van Hoeij, Pech, 2011]

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> P;
```

\[
P = t(t - 1)(64t - 1)(3t - 2)(6t + 1)\partial_t^3 + (4608t^4 - 6372t^3 + 813t^2 + 514t - 4)\partial_t^2 + 4(576t^3 - 801t^2 - 108t + 74)\partial_t
\]
Example: 3D rook paths [B., Chyzak, van Hoeij, Pech, 2011]

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P(G) = \frac{\partial R}{\partial x} + \frac{\partial S}{\partial y}.
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**Solution:** Creative telescoping!

```plaintext
> G:=subs(y=y/x,z=t/y,1/(1-x/(1-x)-y/(1-y)-z/(1-z)))/y/x:
> P,R,S:=op(op(Mgfun:-creative_telescoping(G,t::diff,[x::diff,y::diff]))):
> P;
```

\[
P = t(t - 1)(64t - 1)(3t - 2)(6t + 1)\partial_t^3
\]
\[
+ (4608t^4 - 6372t^3 + 813t^2 + 514t - 4)\partial_t^2
\]
\[
+ 4(576t^3 - 801t^2 - 108t + 74)\partial_t
\]

▷ The whole computation takes < 10 seconds on a personal laptop.
▷ Proves a recurrence conjectured by [Erickson 2010]
Several generations of Creative Telescoping algorithms

- **3G, combines 1G + 2G + linear algebra:** [Apagodu, Zeilberger, 2005], [Koutschan 2010], [Chen, Kauers 2012], [Chen, Kauers, Koutschan 2014]

▶ Advantages:
- 1G–3G: very general algorithms;
- 2G/3G algorithms are able to solve non-trivial problems.

▶ Drawbacks:
- 1G: slow;
- 2G: bad or unknown complexity;
- 1G and 3G: non-minimality of telescopers;
- 1G–3G: all compute (big) certificates.
Several generations of Creative Telescoping algorithms

4G: roots in [Ostrogradsky, 1845], [Hermite, 1872] and [Picard, 1902].

- **univariate:**
  - rational $\int$: [B., Chen, Chyzak, Li, 2010];
  - hyperexponential $\int$: [B., Chen, Chyzak, Li, Xin, 2013]
  - hypergeometric $\sum$: [Chen, Huang, Kauers, Li, 2015], [Huang, 2016]
  - mixed $\int + \sum$: [B., Dumont, Salvy, 2016]
  - algebraic $\int$: [Chen, Kauers, Koutschan, 2016]
  - D-finite Fuchsian $\int$: [Chen, van Hoeij, Kauers, Koutschan, 2018]
  - D-finite $\int$: [B., Chyzak, Lairez, Salvy, 2018], [van der Hoeven, 2018]

- **multiple:**
  - rational bivariate $\frac{\partial}{\partial x}$: [Chen, Kauers, Singer, 2012]
  - rational: [B., Lairez, Salvy, 2013], [Lairez 2016]
  - binomial sums: [B., Lairez, Salvy, 2017]

▷ Advantages:
  - complexity;
  - minimality of telescopers;
  - does not need to compute certificates;
  - fast in practice.

▷ Drawback: not (yet) as general as 1G–3G algorithms.
Hermite’s method for indefinite integration

Pb. Given coprime \( f, g \in \mathbb{K}[x] \), “compute” \( \int \frac{f}{g} \) in the following sense:

write \( \frac{f}{g} = \partial_x \left( \frac{c}{d} \right) + \frac{a}{b} \) with \( a, b, c, d \in \mathbb{K}[x] \), \( \deg a < \deg b \) and \( b \) squarefree.
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Def. $c/d$ is the rational part and $\int a/b$ is the logarithmic part of $\int f/g$. 
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Def. $c/d$ is the rational part and $\int a/b$ is the logarithmic part of $\int f/g$.

Idea [Ostrogradsky 1833–1845, Hermite 1872]
1. Compute the (rational) partial fraction decomposition of $f/g$, that is

$$\frac{f}{g} = P + \sum_{i=1}^{m} \sum_{j=1}^{i} \frac{f_{i,j}}{g_i^j}$$

where $P, f_{i,j}, g_i \in \mathbb{K}[x]$ with $\deg f_{i,j} < \deg g_i$ and $g = g_1 g_2^2 \cdots g_m^m$ a squarefree factorization: $g_i$’s are squarefree and mutually coprime, $g_m \neq 1$. 
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where \( P, f_{i,j}, g_i \in \mathbb{K}[x] \) with \( \deg f_{i,j} < \deg g_i \) and \( g = g_1 g_2^2 \cdots g_m^m \) a squarefree factorization: \( g_i \)'s are squarefree and mutually coprime, \( g_m \neq 1 \).

2. Then put into \( c/d \) all that comes from \( P \) and \( j > 1 \), and into \( a/b \) all that comes from \( j = 1 \). To do this, use integration by parts and extended gcds.
Hermite’s method for indefinite integration

Task: Given coprime $f, g \in \mathbb{K}[x]$, compute $a, b, c, d \in \mathbb{K}[x]$, with $\deg a < \deg b$ and $b$ squarefree, such that $\frac{f}{g} = \partial_x \left( \frac{c}{d} \right) + \frac{a}{b}$.
Hermite’s method for indefinite integration

Task: Given coprime \( f, g \in \mathbb{K}[x] \), compute \( a, b, c, d \in \mathbb{K}[x] \), with \( \deg a < \deg b \) and \( b \) squarefree, such that

\[
\frac{f}{g} = \partial_x \left( \frac{c}{d} \right) + \frac{a}{b}.
\]

Algorithm:

1. Compute the partial fraction decomposition \( \frac{f}{g} = P + \sum_{i=1}^{m} \sum_{j=1}^{i} \frac{f_{i,j}}{g_i} \).
Hermite’s method for indefinite integration

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Algorithm:

1. Compute the partial fraction decomposition $\frac{f}{g} = P + \sum_{i=1}^{m} \sum_{j=1}^{i} \frac{f_{i,j}}{g_{i}}$.

2. Perform Hermite reduction:
   Let $i \in \{1, 2, \ldots, m\}$. As $\gcd(g_{i}, g_{i}') = 1$, there exist $u, v \in \mathbb{K}[x]$ such that $ug_{i} + vg_{i}' = f_{i,j}$.
Hermite’s method for indefinite integration

Task: Given coprime $f, g \in \mathbb{K}[x]$, compute $a, b, c, d \in \mathbb{K}[x]$, with $\deg a < \deg b$ and $b$ squarefree, such that $\frac{f}{g} = \partial_x \left( \frac{c}{d} \right) + \frac{a}{b}$.

Algorithm:

1. Compute the **partial fraction decomposition** $\frac{f}{g} = P + \sum_{i=1}^{m} \sum_{j=1}^{i} \frac{f_{i,j}}{g_i}$

2. Perform **Hermite reduction**:
   Let $i \in \{1, 2, \ldots, m\}$. As $\gcd(g_i, g_i') = 1$, there exist $u, v \in \mathbb{K}[x]$ such that $ug_i + vg_i' = f_{i,j}$.

Exercise: Show that one can assume $\deg u < \deg g_i$ and $\deg v < \deg g_i$. 
Hermite’s method for indefinite integration

**Task:** Given coprime \( f, g \in \mathbb{K}[x] \), compute \( a, b, c, d \in \mathbb{K}[x] \), with
\[
\deg a < \deg b \text{ and } b \text{ squarefree},
\]
such that
\[
\frac{f}{g} = \partial_x \left( \frac{c}{d} \right) + \frac{a}{b}.
\]

**Algorithm:**

1. Compute the partial fraction decomposition
\[
\frac{f}{g} = P + \sum_{i=1}^{m} \sum_{j=1}^{i} \frac{f_{i,j}}{g_i^j}
\]
2. Perform Hermite reduction:
   Let \( i \in \{1, 2, \ldots, m\} \). As \( \gcd(g_i, g_i') = 1 \), there exist \( u, v \in \mathbb{K}[x] \) such that
   \[
u g_i + v g_i' = f_{i,j}.
\]

**Exercise:** Show that one can assume \( \deg u < \deg g_i \) and \( \deg v < \deg g_i \). Then, if \( j > 1 \), integration by parts gives
\[
\int \frac{f_{i,j}}{g_i^j} = \int \frac{u}{g_i^{j-1}} + \int \frac{v g_i'}{g_i^j} = \frac{-v}{(j-1) g_i^{j-1}} + \int \frac{u + v'}{g_i^{j-1}}.
\]
Hermite’s method for indefinite integration

Task: Given coprime \( f, g \in \mathbb{K}[x] \), compute \( a, b, c, d \in \mathbb{K}[x] \), with \( \deg a < \deg b \) and \( b \) squarefree, such that \( \frac{f}{g} = \partial_x \left( \frac{c}{d} \right) + \frac{a}{b} \).

Algorithm:

1. Compute the partial fraction decomposition
   \[
   \frac{f}{g} = P + \sum_{i=1}^{m} \sum_{j=1}^{i} \frac{f_{i,j}}{g_i^j}
   \]

2. Perform Hermite reduction:
   Let \( i \in \{1, 2, \ldots, m\} \). As \( \gcd(g_i, g'_i) = 1 \), there exist \( u, v \in \mathbb{K}[x] \) such that
   \[
   u g_i + v g'_i = f_{i,j}.
   \]
   
   Exercise: Show that one can assume \( \deg u < \deg g_i \) and \( \deg v < \deg g_i \).

   Then, if \( j > 1 \), integration by parts gives
   \[
   \int \frac{f_{i,j}}{g_i^j} \, dx = \int \frac{u}{g_i^{j-1}} \, dx + \int \frac{v g'_i}{g_i^j} \, dx = \frac{-v}{(j-1)g_i^{j-1}} + \int \frac{u + \frac{v'}{j-1}}{g_i^{j-1}} \, dx.
   \]
   Repeating the process at most \( j \) times results in
   \[
   \int \frac{f_{i,j}}{g_i^j} \, dx = \frac{k_{i,j}}{g_i^{j-1}} + \int \frac{l_{i,j}}{g_i} \, dx, \quad \text{with } \deg k_{i,j} < \deg g_i \text{ and } \deg l_{i,j} < \deg g_i.
   \]
Problem: Given $H = P/Q \in \mathbb{K}(t, x)$ compute $\int_{\gamma} H(t, x) dx$
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Hermite reduction: $H$ can be written in reduced form

$$H = \partial_x(g) + \frac{a}{Q^*},$$

where $Q^*$ is the squarefree part of $Q$ and $\deg_x(a) < d^* := \deg_x(Q^*)$. 
Problem: Given $H = P/Q \in \mathbb{K}(t,x)$ compute $\int_{\gamma} H(t,x) \, dx$

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Algorithm [B., Chen, Chyzak, Li, 2010]

1. For $i = 0, 1, \ldots, d^*$ compute Hermite reduction of $\partial_t^i(H)$:

$$\partial_t^i(H) = \partial_x(g_i) + \frac{a_i}{Q^*}, \quad \deg_x(a_i) < d^*$$
Problem: Given $H = P/Q \in \mathbb{K}(t, x)$ compute $\int_H H(t, x) dx$

**Hermite reduction:** $H$ can be written in reduced form

$$H = \partial_x(g) + \frac{a}{Q^*},$$

where $Q^*$ is the squarefree part of $Q$ and $\deg_x(a) < d^* := \deg_x(Q^*)$.

**Algorithm [B., Chen, Chyzak, Li, 2010]**

1. For $i = 0, 1, \ldots, d^*$ compute Hermite reduction of $\partial^i_t(H)$:

   $$\partial^i_t(H) = \partial_x(g_i) + \frac{a_i}{Q^*}, \quad \deg_x(a_i) < d^*$$

2. Find the first linear relation over $\mathbb{K}(t)$ of the form $\sum_{k=0}^{r} c_k a_k = 0$. 
4G Creative Telescoping: univariate case

Problem: Given \( H = P/Q \in \mathbb{K}(t, x) \) compute \( \int_{\gamma} H(t, x) \, dx \)

Hermite reduction: \( H \) can be written in reduced form

\[
H = \partial_x(g) + \frac{a}{Q^*},
\]

where \( Q^* \) is the squarefree part of \( Q \) and \( \deg_x(a) < d^* := \deg_x(Q^*) \).

Algorithm [B., Chen, Chyzak, Li, 2010]
(1) For \( i = 0, 1, \ldots, d^* \) compute Hermite reduction of \( \partial_t^i(H) \):

\[
\partial_t^i(H) = \partial_x(g_i) + \frac{a_i}{Q^*}, \quad \deg_x(a_i) < d^*
\]

(2) Find the first linear relation over \( \mathbb{K}(t) \) of the form \( \sum_{k=0}^r c_k a_k = 0 \).

\( L = \sum_{k=0}^r c_k \partial_t^k \) is a telescopers (and \( \sum_{k=0}^r c_k g_k \) the corresponding certificate).
Algorithm for the integration of rational functions [B., Lairez, Salvy, 2013]

- **Input:** \( R(e, x) \) a rational function in \( e \) and \( x = x_1, \ldots, x_n \).
- **Output:** A linear ODE \( T(e, \partial_e)y = 0 \) satisfied by \( y(e) = \int R(e, x)dx \).
- **Complexity:** \( O(D^{8n+2}) \), where \( D = \deg R \).
- **Output size (tight!):** \( T \) has order \( \leq D^n \) in \( \partial_e \) and degree \( \leq D^{3n+2} \) in \( e \).

- Avoids the (costly) computation of certificates, of size \( \Omega(Dn^2/2) \).
- Previous algorithms: complexity (at least) doubly exponential in \( n \).
- Very efficient in practice.
Main ingredients of the integration algorithm

Griffiths–Dwork method for the generic case
Linear reduction classical in algebraic geometry;
Generalization of Hermite’s reduction.

Fast linear algebra on polynomial matrices
Macaulay matrices encoding Gröbner bases computations;
Sophisticated algorithms due to Villard, Storjohann, Zhou, etc.

Deformation technique for the general case
Input perturbation using a new free variable.

▷ Highly non-trivial extension by [Lairez, 2016]: tremendously improves the efficiency of the algorithm in [B., Lairez, Salvy, 2013]
1. Compute a telescopener for the diagonal of the rational power series

\[
\frac{1}{1 - x - y} = \sum_{i,j \geq 0} \binom{i+j}{i} x^i y^j
\]

in two different ways:

1. using the 2G (Almkvist-Zeilberger) creative telescoping algorithm;
2. using the 4G (Hermite reduction-based) creative telescoping algorithm.

2. Let \( f, g \in \mathbb{Q}[x] \) be two coprime polynomials. Let \( h \in \mathbb{Q}[x] \) be another polynomial such that \( \deg h < \deg f + \deg g \).

1. Show that the equation

\[
sf + tg = h
\]

admits an unique solution \((s, t) \in \mathbb{Q}[x]^2\) s.t. \(\deg s < \deg g\), \(\deg t < \deg f\).

2. Design an algorithm for computing the solution \((s, t)\) starting from \((f, g, h)\) in quasi-optimal complexity.