Exercises on the chapters "Padé approximants" and "Sparse matrices"

To prepare for 2020-11-02

Exercise 1. Prove algorithmically that there is no Padé approximant of type $(1,1)$ for $1+x^2$, i.e. no pair (R,V) of polynomials in $\mathbb{Q}[x]$ of degree at most 1 such that $V(0) \neq 0$ and $\dfrac{R}{V} = 1 + x^2 \bmod x^3$.

Exercise 2. Let $\mathbb{K} = \mathbb{Z}/5\mathbb{Z}$ be the finite field with 5 elements, let $M \in \mathcal{M}_3(\mathbb{K})$ and $b \in \mathbb{K}^3$ be

$$M = \begin{pmatrix} 1 & 4 & 4 \\ 4 & 0 & 3 \\ 1 & 2 & 4 \end{pmatrix}, \quad b = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}.$$

We want to find $y \in \mathbb{K}^3$ such that $My = b$, by using Wiedemann's algorithm.

1. For the choice $u = (1,0,0)^T$, show that the algorithm computes the sequence $(3,0,4,2,3,0,\ldots)$, then its minimal polynomial $x^2 + 2x + 2$, and that it eventually rejects this choice of u.
2. Apply the algorithm for the choice $u = (1,2,0)^T$, and deduce that the minimal polynomial of $(M^i b)_{i \geq 0}$ equals $x^3 + 3x + 1$.
3. Determine the solution y by using this minimal polynomial.