Exercises on the chapters "Polynomial Matrices" and "Structured Matrices"

To prepare for 2020-11-09

Exercise 1. Let $M(x) \in \mathcal{M}_n(\mathbb{K}[x]_{\leq d})$ be an invertible polynomial matrix.

Assume one computes its inverse using Strassen's inversion algorithm for dense (scalar) matrices.

Estimate the complexity of this computation, counting operations in $\mathbb{K}$, in terms of the two parameters n and d, under the assumption that all matrices encountered during the inversion algorithm are invertible.

Exercise 2. Recall that a matrix $C \in \mathcal{M}_n(\mathbb{K})$ is called Cauchy if its (i, j) element is of the form $\frac{1}{x_i - y_j}$ for some $x_1, \dots, x_n, y_1, \dots, y_n \in \mathbb{K}$ with $x_i \neq y_j$ for all i, j.

Show that the product Cv of a Cauchy matrix $C \in \mathcal{M}_n(\mathbb{K})$ by a vector $v \in \mathbb{K}^n$ can be performed in $O(\mathsf{M}(n) \log n)$ operations in $\mathbb{K}$.

Exercise 3. Let $(a_n)_{n\geq 0}$ be a sequence with $a_0 = a_1 = 1$ satisfying the recurrence

$$(n + 3)a_{n+1} = (2n + 3)a_n + 3na_{n-1}.$$

Show that a_n is an integer for all n, by following the next steps:

(1) Compute the first 5 terms of the sequence, $a_0, \dots, a_4$;

(2) Show that $[1, x-1, x^2]$ is a Hermite-Padé approximant of type $(0, 1, 2)$ for $(1, f, f^2)$, where $f = \sum_n a_n x^n$;

(3) Deduce that $P(x, y) := 1 + (x - 1)y + x^2y^2$ satisfies $P(x, f(x)) = 0 \bmod x^5$;

(4) Show that the equation $P(x, y) = 0$ admits a root $y = g(x) \in \mathbb{Q}[[x]]$ whose coefficients satisfy the same linear recurrence as $(a_n)_{n\geq 0}$;

(5) Deduce that $a_{n+2} = a_{n+1} + \sum_{k=0}^{n} a_k \cdot a_{n-k}$ for all n, and conclude.