Overview

Part 1: General presentation
Part 2: Guess’n’Prove
Part 3: Creative telescoping
Part 1: General presentation
**Enumerative Combinatorics:** the science of counting
Area of mathematics primarily concerned with counting discrete objects.

▷ Main outcome: theorems

**Computer Algebra:** effective mathematics
Area of computer science primarily concerned with the algorithmic manipulation of mathematical objects.

▷ Main outcome: algorithms

This lecture: **Computer Algebra** for **Enumerative Combinatorics**
→ **Algorithms** for proving **Theorems** on **Lattice Paths Combinatorics**.
An (innocent looking) combinatorial question

Let $\mathcal{I} = \{\uparrow, \leftarrow, \searrow\}$. An $\mathcal{I}$-walk is a path in $\mathbb{Z}^2$ using only steps from $\mathcal{I}$. Show that, for any integer $n$, the following quantities are equal:

(i) number $a_n$ of $\mathcal{I}$-walks of length $n$ confined to the upper half plane $\mathbb{Z} \times \mathbb{N}$ that start and end at the origin $(0,0)$ (*excursions*);

(ii) number $b_n$ of $\mathcal{I}$-walks of length $n$ confined to the quarter plane $\mathbb{N}^2$ that start at the origin $(0,0)$ and finish on the diagonal of $\mathbb{N}^2$ (*diagonal walks*).
An (innocent looking) combinatorial question

Let $\mathcal{S} = \{\uparrow, \leftarrow, \searrow\}$. An $\mathcal{S}$-walk is a path in $\mathbb{Z}^2$ using only steps from $\mathcal{S}$. Show that, for any integer $n$, the following quantities are equal:

(i) number $a_n$ of $\mathcal{S}$-walks of length $n$ confined to the upper half plane $\mathbb{Z} \times \mathbb{N}$ that start and end at the origin $(0, 0)$ (excursions);

(ii) number $b_n$ of $\mathcal{S}$-walks of length $n$ confined to the quarter plane $\mathbb{N}^2$ that start at the origin $(0, 0)$ and finish on the diagonal of $\mathbb{N}^2$ (diagonal walks).

For instance, for $n = 3$, this common value is $a_3 = b_3 = 3$:
Teaser 1: This problem can be solved using computer algebra!

Teaser 2: The answer has a beautiful formula!

\[ a_{3n} = b_{3n} = \frac{(3n)!}{n!^2 \cdot (n+1)!} \quad \text{and} \quad a_m = b_m = 0 \quad \text{if } m \text{ is not a multiple of } 3. \]

Teaser 3: A certain group attached to the step set \{↑, ←, ↘\} is finite!
Teasers

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Exercise 1

Teaser 3: A certain group attached to the step set \{↑, ←, ↓\} is finite!
Combinatorial context: lattice paths confined to cones

Let $\mathcal{S}$ be a subset of $\mathbb{Z}^d$ (step set, or model) and $p_0 \in \mathbb{Z}^d$ (starting point).

Example: $\mathcal{S} = \{(1,0), (-1,0), (1,-1), (-1,1)\}, p_0 = (0,0)$
Let $\mathcal{S}$ be a subset of $\mathbb{Z}^d$ (step set, or model) and $p_0 \in \mathbb{Z}^d$ (starting point).

A path (walk) of length $n$ starting at $p_0$ is a sequence $(p_0, p_1, \ldots, p_n)$ of elements in $\mathbb{Z}^d$ such that $p_{i+1} - p_i \in \mathcal{S}$ for all $i$.

**Example:** $\mathcal{S} = \{(1,0), (-1,0), (1,-1), (-1,1)\}$, $p_0 = (0,0)$
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Let $\mathcal{C}$ be a cone of $\mathbb{R}^d$ (if $x \in \mathcal{C}$ and $r \geq 0$ then $r \cdot x \in \mathcal{C}$).

Example: $\mathcal{S} = \{(1, 0), (-1, 0), (1, -1), (-1, 1)\}$, $p_0 = (0, 0)$ and $\mathcal{C} = \mathbb{R}_+^2$
Combinatorial context: lattice paths confined to cones

Let $\mathcal{S}$ be a subset of $\mathbb{Z}^d$ (step set, or model) and $p_0 \in \mathbb{Z}^d$ (starting point).

A path (walk) of length $n$ starting at $p_0$ is a sequence $(p_0, p_1, \ldots, p_n)$ of elements in $\mathbb{Z}^d$ such that $p_{i+1} - p_i \in \mathcal{S}$ for all $i$.

Let $\mathcal{C}$ be a cone of $\mathbb{R}^d$ (if $x \in \mathcal{C}$ and $r \geq 0$ then $r \cdot x \in \mathcal{C}$).

Example: $\mathcal{S} = \{(1,0), (-1,0), (1,-1), (-1,1)\}$, $p_0 = (0,0)$ and $\mathcal{C} = \mathbb{R}^2_+$

Questions

- What is the number $a_n$ of $n$-step walks contained in $\mathcal{C}$?
- For $i \in \mathcal{C}$, what is the number $a_{n;i}$ of such walks that end at $i$?
- What about their GF’s $A(t) = \sum_n a_n t^n$ and $A(t;x) = \sum_{n,i} a_{n;i} x^i t^n$?
Why should we care about counting walks?

Many objects from the real world can be encoded by walks:

- probability theory (voting, games of chance, branching processes, …)
- discrete mathematics (permutations, trees, words, urns, …)
- statistical physics (Ising model, …)
- operations research (queueing theory, …)
Why should we care about counting walks?

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- statistical physics (Ising model, . . .)
- operations research (queueing theory, . . .)
Counting walks is an old topic: the ballot problem [Bertrand, 1887]

Lattice path reformulation: find the number of paths with $a$ upsteps $\uparrow$ and $b$ downsteps $\downarrow$ that start at the origin and never touch the $x$-axis again.
Counting walks is an old topic: the ballot problem [Bertrand, 1887]

CALCUL DES PROBABILITÉS. — Solution d’un problème;
par M. J. BERTRAND.

« On suppose que deux candidats A et B soient soumis à un scrutin de
ballottage. Le nombre des votants est $\mu$. A obtient $m$ suffrages et est élu,
B en obtient $\mu - m$. On demande la probabilité pour que, pendant le dé-
pouillement du scrutin, le nombre des voix de A ne cesse pas une seule
fois de surpasser celles de son concurrent.

Lattice path reformulation: find the number of paths with $a$ upsteps $\rightarrow$ and
$b$ downsteps $\downarrow$ that start at the origin and never touch the $x$-axis again

Exercise 2: Prove that the coordinates of the endpoint are indeed $(a + b, a - b)$
An old topic: Pólya’s “promenade au hasard” / “Irrfahrt”

Motto: Drunkard: “Will I ever, ever get home again?”
Polya (1921): “You can’t miss; just keep going and stay out of 3D!”

(Adam and Delbrück, 1968)

[Pólya, 1921] Simple random walk \{±1\}^d on \mathbb{Z}^d is recurrent in dimensions \(d = 1, 2\) (“Alle Wege führen nach Rom”), and transient in dimension \(d \geq 3\)

Über eine Aufgabe der Wahrscheinlichkeitsrechnung betreffend die Irrfahrt im Straßennetz.
Lot of recent activity; many recent contributors:


etc.
...but it is still a very hot topic

Lot of recent activity; many recent contributors:


e tc.

Specific question

Ad hoc solution

Systematic approach
Chapter 10

Lattice Path Enumeration

Christian Krattenthaler
Universität Wien

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The simplest case: unconstrained walks

Geometric sequences

The number $a_n$ of $n$-step $\mathcal{S}$-walks in the whole plane $\mathbb{Z}^2$ is equal to

$$a_n = |\mathcal{S}|^n.$$ 

Proof: $a_n = |\mathcal{S}| \cdot a_{n-1} = |\mathcal{S}|^2 \cdot a_{n-2} = \ldots = |\mathcal{S}|^{n-1} \cdot a_1 = |\mathcal{S}|^n.$  □

▷ Remark: For $\mathcal{S} = \{\rightarrow, \uparrow\}$, the sequence $a_n$ is

$$1, 2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, \ldots$$

It satisfies the recurrence relation with constant coefficients $a_{n+1} - 2a_n = 0$. 

Alin Bostan  Computer Algebra for Combinatorics
Rational series [folklore]

If $\mathcal{S} \subset \mathbb{Z}^d$ is finite and $\mathbb{C} = \mathbb{R}^d$, then

$$a_n = |\mathcal{S}|^n, \text{ i.e. } A(t) = \sum_{n \geq 0} a_n t^n = \frac{1}{1 - |\mathcal{S}| t}.$$

More generally:

$$A(t; x) = \sum_{n, i} a_{n;i} x^i t^n = \frac{1}{1 - t \sum_{s \in \mathcal{S}} x^s}.$$
The next case: walks confined to a half-space

Recurrent sequences [Bousquet-Mélou, Petkovšek, 2000]

The sequence \((a_n)\) of \(n\)-step \(I\)-walks confined to a \textit{half-space} still satisfies a \textit{recurrence relation} (but not necessarily with constant coefficients).

\[ \Delta \]

Example: For \(\{\rightarrow, \uparrow\}\)-walks in the half-plane \(\mathbb{Z} \times \mathbb{N}\), the sequence \((a_n)\) is

\[ 1, 1, 2, 3, 6, 10, 20, 35, 70, 126, 252, 462, 924, 1716, \ldots \]

It is \textit{not geometric}, but satisfies \((n + 3)a_{n+2} - 2a_{n+1} - 4(n + 1)a_n = 0\) for all \(n\).
The next case: walks confined to a half-space

Algebraic series [Bousquet-Mélou, Petkovšek, 2000]

If \( \mathcal{S} \subset \mathbb{Z}^d \) is finite and \( \mathcal{C} \) is a rational half-space, then \( A(t; x) \) is algebraic, given by an explicit system of polynomial equations.

Example: For Dyck paths (ballot problem), \( A(t; 1) = \sum_{n \geq 0} C_n t^n = \frac{1 - \sqrt{1 - 4t}}{2t} \)
Back to the ballot problem [Bertrand, 1887]

Suppose that candidates A and B are running in an election. If \( a \) votes are cast for A and \( b \) votes are cast for B, where \( a > b \), then the probability that A stays ahead of B throughout the counting of the ballots is \( (a - b)/(a + b) \).

**Lattice path reformulation**: find the number of paths with \( a \) upsteps \( \uparrow \) and \( b \) downsteps \( \downarrow \) that start at the origin and never touch the \( x \)-axis again

**Reflection principle [Aebly, 1923]**: paths in \( \mathbb{N}^2 \) from \( (1, 1) \) to \( T(a + b, a - b) \) that do touch the \( x \)-axis are in bijection with paths in \( \mathbb{Z}^2 \) from \( (1, -1) \) to \( T \)

![Diagram of lattice paths](image)

**Answer**: \((paths in \mathbb{Z}^2 from (1, 1) to T) - (paths in \mathbb{Z}^2 from (1, -1) to T)\)

\[
\left( \binom{a + b - 1}{a - 1} \right) - \left( \binom{a + b - 1}{b - 1} \right) = \frac{a - b}{a + b} \binom{a + b}{a}
\]
Back to the ballot problem [Bertrand, 1887]

Suppose that candidates $A$ and $B$ are running in an election. If $a$ votes are cast for $A$ and $b$ votes are cast for $B$, where $a > b$, then the probability that $A$ stays ahead of $B$ throughout the counting of the ballots is $(a - b)/(a + b)$.

**Lattice path reformulation:** find the number of paths with $a$ upsteps $\uparrow$ and $b$ downsteps $\downarrow$ that start at the origin and never touch the $x$-axis again.

**Reflection principle [Aebly, 1923]:** paths in $\mathbb{N}^2$ from $(1, 1)$ to $T(a + b, a - b)$ that do touch the $x$-axis are in bijection with paths in $\mathbb{Z}^2$ from $(1, -1)$ to $T$.

**Answer:** when $a = n + 1$ and $b = n$, this is the **Catalan number**

$$C_n = \frac{1}{2n+1} \binom{2n+1}{n+1} = \frac{1}{n+1} \binom{2n}{n}$$
The next case: walks confined to a half-space

\[
f(i, j; n) = \begin{cases} 
0 & \text{if } i < 0 \text{ or } j < 0 \text{ or } n < 0, \\
\sum_{i', j' \in S} f(i - i', j - j'; n - 1) & \text{otherwise}. 
\end{cases}
\]
The next case: walks confined to a half-space
Entire books dedicated to walks in the quarter plane!
Our approach: Experimental Mathematics using Computer Algebra
Our approach: Experimental Mathematics using Computer Algebra

Algorithmes Efficaces en Calcul Formel

Alin Bostan
Frédéric Chyzak
Marc Giusti
Romain Lebreton
Grégoire Lecerf
Bruno Salvy
Éric Schost
Problem 6

A flea starts at \((0, 0)\) on the infinite two-dimensional integer lattice and executes a biased random walk: At each step it hops north or south with probability \(1/4\), east with probability \(1/4 + \epsilon\), and west with probability \(1/4 - \epsilon\). The probability that the flea returns to \((0, 0)\) sometime during its wanderings is 1/2. What is \(\epsilon\)?
Chapter 6
Biasing for a Fair Return
Folkmar Bornemann

It was often claimed that [direct and “exact” numerical solution of the equations of physics] would make the special functions redundant. ... The persistence of special functions is puzzling as well as surprising. What are they, other than just names for mathematical objects that are useful only in situations of contrived simplicity? Why are we so pleased when a complicated calculation “comes out” as a Bessel function, or a Laguerre polynomial? What determines which functions are “special”?

— Sir Michael Berry [Ber01]

People who like this sort of thing will find this the sort of thing they like.

— Barry Hughes, quoting Abraham Lincoln at the beginning of an appendix on “Special Functions for Random Walk Problems” [Hug95, p. 569]

Problem 6

A flea starts at (0, 0) on the infinite two-dimensional integer lattice and executes a biased random walk: At each step it hops north or south with probability 1/4, east with probability 1/4 + \(\epsilon\), and west with probability 1/4 − \(\epsilon\). The probability that the flea returns to (0, 0) sometime during its wanderings is 1/2. What is \(\epsilon\)?

Computer algebra conjectures and proves

\[
p(\epsilon) = 1 - \sqrt{\frac{A}{2}} \cdot 2F_1 \left( \begin{array}{c} \frac{1}{2}, \frac{1}{2} \\ \frac{1}{2} \end{array} \mid \frac{2\sqrt{1 - 16\epsilon^2}}{A} \right) \quad \text{with } A = 1 + 8\epsilon^2 + \sqrt{1 - 16\epsilon^2}.\]
Problem 6

A flea starts at \((0, 0)\) on the infinite two-dimensional integer lattice and executes a biased random walk: At each step it hops north or south with probability \(1/4\), east with probability \(1/4 + \epsilon\), and west with probability \(1/4 - \epsilon\). The probability that the flea returns to \((0, 0)\) sometime during its wanderings is \(1/2\). What is \(\epsilon\)?

▷ Computer algebra conjectures and proves

\[
\epsilon \approx 0.06191395447399094284817521647321217699963877499836207606146725885993101029759615845907105645752087861 \ldots
\]
From now on: we focus on nearest-neighbor walks in the quarter plane, i.e. walks in $\mathbb{N}^2$ starting at $(0, 0)$ and using steps in a fixed subset $\mathcal{I}$ of

$$\{\searrow, \leftarrow, \nwarrow, \uparrow, \nearrow, \rightarrow, \swarrow, \downarrow\}.$$ 

Example with $n = 45$, $i = 14$, $j = 2$ for:
Lattice walks with small steps in the quarter plane

\[ \text{From now on: we focus on nearest-neighbor walks in the quarter plane, i.e. walks in } \mathbb{N}^2 \text{ starting at } (0, 0) \text{ and using steps in a fixed subset } \mathcal{S} \text{ of } \{\searrow, \leftarrow, \nwarrow, \uparrow, \rightarrow, \swarrow, \downarrow\}. \]

\[ \text{Example with } n = 45, i = 14, j = 2 \text{ for:} \]

\[ \mathcal{S} = \]

\[ \text{Counting sequence: } f_{n;i,j} = \text{number of walks of length } n \text{ ending at } (i, j). \]
From now on: we focus on nearest-neighbor walks in the quarter plane, i.e. walks in \( \mathbb{N}^2 \) starting at \((0, 0)\) and using steps in a fixed subset \( \mathcal{S} \) of \(\{\searrow, \leftarrow, \nearrow, \uparrow, \nearrow, \rightarrow, \swarrow, \downarrow\}\).

Example with \( n = 45, i = 14, j = 2 \) for:

\[
\mathcal{S} = \begin{array}{c}
\searrow \\
\leftarrow \\
\nearrow \\
\uparrow \\
\nearrow \\
\rightarrow \\
\swarrow \\
\downarrow
\end{array}
\]

Counting sequence: \( f_{n;i,j} \) = number of walks of length \( n \) ending at \((i, j)\).

Specializations:
- \( f_{n;0,0} \) = number of walks of length \( n \) returning to origin ("excursions");
- \( f_n = \sum_{i,j \geq 0} f_{n;i,j} \) = number of walks with prescribed length \( n \).
Generating functions and combinatorial problems

Complete generating function:

\[ F(t; x, y) = \sum_{n=0}^{\infty} \left( \sum_{i,j=0}^{\infty} f_{n;i,j} x^i y^j \right) t^n \in \mathbb{Q}[x, y][[t]]. \]
Generating functions and combinatorial problems

▷ Complete generating function:

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\]

▷ Specializations:
  - GF of excursions: \( F(t; 0, 0) \)
  - GF of walks: \( F(t; 1, 1) = \sum_{n \geq 0} f_n t^n \)
  - GF of horizontal returns: \( F(t; 1, 0) \)
  - GF of diagonal returns: \( \text{“} F(t; 0, \infty) \text{“} := [x^0] F(t; x, 1/x) \)

Our goal: Use computer algebra to give computational answers.
Generating functions and combinatorial problems

▷ Complete generating function:

\[ F(t; x, y) = \sum_{n=0}^{\infty} \left( \sum_{i,j=0}^{\infty} f_{n;i,j} x^i y^j \right) t^n \in \mathbb{Q}[x, y][[t]]. \]

▷ Specializations:

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Combinatorial questions:

Given \( \mathcal{S} \), what can be said about \( F(t; x, y) \), resp. \( f_{n;i,j} \), and their variants?

- Structure of \( F \): algebraic? transcendental? solution of ODE?
- Explicit form: of \( F \) of \( f_{n;i,j} \)?
- Asymptotics of \( f_{n;0,0} \) of \( f_n \)?
Generating functions and combinatorial problems

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\]

▷ Specializations:

- GF of excursions: \( F(t; 0, 0); \)
- GF of walks: \( F(t; 1, 1) = \sum_{n \geq 0} f_n t^n; \)
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Our goal: Use computer algebra to give computational answers.
Among the $2^8$ step sets $\mathcal{S} \subseteq \{-1, 0, 1\}^2 \setminus \{(0,0)\}$, some are:

- trivial
- simple
- intrinsic to the half plane
- symmetrical

One is left with 79 interesting distinct models.

Is any further classification possible?
Small-step models of interest

Among the $2^8$ step sets $\mathcal{S} \subseteq \{-1, 0, 1\}^2 \setminus \{(0, 0)\}$, some are:

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- simple,
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- symmetrical.

One is left with 79 interesting distinct models.

Is any further classification possible?
The 79 models

Non-singular

Singular
Task: classify their generating functions!
Two important models: **Kreweras** and **Gessel** walks

\[ \mathcal{I} = \{ \downarrow, \leftarrow, \uparrow \} \quad F_{\mathcal{I}}(t; x, y) \equiv K(t; x, y) \]

\[ \mathcal{I} = \{ \uparrow, \searrow, \leftarrow, \rightarrow \} \quad F_{\mathcal{I}}(t; x, y) \equiv G(t; x, y) \]

Example: A Kreweras excursion.
“Special” models of walks in the quarter plane

- Dyck:
- Motzkin:
- Pólya:
- Kreheras:
- Gessel:
- Gouyou-Beauchamps:
- King walks:
- Tandem walks:
Gessel’s walks

\[ I = \{\uparrow, \downarrow, \leftarrow, \rightarrow\} \]
Conjecture 1  The generating function of Gessel excursions is equal to
\[
G(t;0,0) = 3F_2\left(\begin{array}{ccc} 5/6 & 1/2 & 1 \\ 5/3 & 2 & \end{array} \left| 16t^2 \right. \right)
\]
\[
= \sum_{n=0}^{\infty} \frac{(5/6)_n(1/2)_n}{(5/3)_n(2)_n} (4t)^{2n} 
\]
\[
= 1 + 2t^2 + 11t^4 + 85t^6 + 782t^8 + \cdots
\]

Conjecture 2  The full generating function $G(t; x, y)$ is not D-finite.
Genesis of Gessel’s questions – the “simple walk” in different cones

The simple walk in the plane

[Pólya, 1921]:
▷ Formula $\binom{2n}{n}^2$ for $2n$-excursions
▷ Rational generating function

The simple walk in the half-plane and in the quarter-plane

▷ Formulas $\binom{2n+1}{n}C_n$, resp. $C_nC_{n+1}$, for $2n$-excursions [Arquès, 1986]
Genesis of Gessel’s questions – the “simple walk” in different cones

The simple walk in the cone with angle $45^\circ$

- Formula $c_n c_{n+2} - c_{n+1}^2$ for $2n$-excursions [Gouyou-Beauchamps, 1986]
- D-finite generating function [Gessel, Zeilberger, 1992]

What about the simple walk in the cone with angle $135^\circ$?
Algebraic reformulation: solving a functional equation

Generating function: \( G(t; x, y) = \sum_{n=0}^{\infty} \sum_{i=0}^{n} \sum_{j=0}^{n} g_{n;i,j} x^i y^j t^n \in Q[x, y][[t]] \)

"Kernel equation":

\[
G(t; x, y) = 1 + t \left( xy + x + \frac{1}{xy} + \frac{1}{x} \right) G(t; x, y) \\
- t \left( \frac{1}{x} + \frac{1}{xy} \right) G(t; 0, y) - t \frac{1}{xy} (G(t; x, 0) - G(t; 0, 0))
\]
Algebraic reformulation: solving a functional equation

Generating function: \( G(t; x, y) = \sum_{n=0}^{\infty} \sum_{i=0}^{n} \sum_{j=0}^{n} g_{n;i,j} x^i y^j t^n \in \mathbb{Q}[x, y][[t]] \)

"Kernel equation":

\[
G(t; x, y) = 1 + t \left( xy + x + \frac{1}{xy} + \frac{1}{x} \right) G(t; x, y) - t \left( \frac{1}{x} + \frac{1}{xy} \right) G(t; 0, y) - t \frac{1}{xy} (G(t; x, 0) - G(t; 0, 0))
\]

Task: Solve this functional equation!
Algebraic reformulation: solving a functional equation

Generating function: $G(t; x, y) = \sum_{n=0}^{\infty} \sum_{i=0}^{n} \sum_{j=0}^{n} g_{n;i,j} x^i y^j t^n \in \mathbb{Q}[x, y][[t]]$

"Kernel equation":

$$G(t; x, y) = 1 + t \left( xy + x + \frac{1}{xy} + \frac{1}{x} \right) G(t; x, y)$$
$$- t \left( \frac{1}{x} + \frac{1}{xy} \right) G(t; 0, y) - t \frac{1}{xy} (G(t; x, 0) - G(t; 0, 0))$$

Task: For the other models – solve 78 similar equations!
Important classes of univariate generating functions

- **algebraic**:\[ S(t) = \sum_{n=0}^{\infty} s_n t^n \in \mathbb{Q}[[t]] \]
delta algebraic if \( P(t, S(t)) = 0 \) for some \( P(x, y) \in \mathbb{Z}[x, y] \{0\} \);

- **D-finite**:\[ c_r(t) S(r)(t) + \cdots + c_0(t) S(t) = 0 \] for some \( c_i(t) \in \mathbb{Z}[t] \), not all zero;

- **hypergeometric**:\[ s_{n+1} / s_n \in \mathbb{Q}(n) \]
Important classes of univariate generating functions

- **algebraic**
  \[ S(t) = \sum_{n=0}^{\infty} s_n t^n \in \mathbb{Q}[[t]] \]
  is

  ▶ *algebraic* if \( P(t, S(t)) = 0 \) for some \( P(x, y) \in \mathbb{Z}[x, y] \setminus \{0\} \);
Important classes of univariate generating functions

$$S(t) = \sum_{n=0}^{\infty} s_n t^n \in \mathbb{Q}[[t]]$$ is

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- **D-finite** if $c_r(t) S^{(r)}(t) + \cdots + c_0(t) S(t) = 0$ for some $c_i \in \mathbb{Z}[t]$, not all zero;
Important classes of univariate generating functions

\[ S(t) = \sum_{n=0}^{\infty} s_n t^n \in \mathbb{Q}[[t]] \] is

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- **hypergeometric** if \( \frac{s_{n+1}}{s_n} \in \mathbb{Q}(n). \)
Important classes of univariate generating functions

\[ S(t) = \sum_{n=0}^{\infty} s_n t^n \in \mathbb{Q}[[t]] \text{ is} \]

- algebraic if \( P(t, S(t)) = 0 \) for some \( P(x, y) \in \mathbb{Z}[x, y] \setminus \{0\} \);

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- hypergeometric if \( \frac{s_{n+1}}{s_n} \in \mathbb{Q}(n) \). E.g.,

\[ \ln(1-t); \quad \frac{\arcsin(\sqrt{t})}{\sqrt{t}}; \quad (1-t)^{\alpha}, \alpha \in \mathbb{Q} \]
Important classes of univariate generating functions

\[ S(t) = \sum_{n=0}^{\infty} s_n t^n \in \mathbb{Q}[[t]] \]

- **algebraic** if \( P(t, S(t)) = 0 \) for some \( P(x, y) \in \mathbb{Z}[x, y] \setminus \{0\} \);

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- **hypergeometric** if \( \frac{s_{n+1}}{s_n} \in \mathbb{Q}(n) \). E.g.,

\[
\binom{a}{b} \binom{c}{t} = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{t^n}{n!}, \quad (a)_n = a(a+1) \cdots (a+n-1).
\]
Important classes of univariate generating functions

\[ S(t) = \sum_{n=0}^{\infty} s_n t^n \in \mathbb{Q}[[t]] \] is

- **algebraic** if \( P(t, S(t)) = 0 \) for some \( P(x, y) \in \mathbb{Z}[x, y] \setminus \{0\} \);
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- **hypergeometric** if \( \frac{s_{n+1}}{s_n} \in \mathbb{Q}(n) \). E.g.,

\[
2 F_1\left(\begin{array}{c} \frac{1}{2} \frac{1}{2} \\ \frac{1}{1} \end{array} \bigg| t \right) = \frac{2}{\pi} \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-tx^2)}}.
\]
Important classes of univariate generating functions

- **algebraic** if $P(t, S(t)) = 0$ for some $P(x, y) \in \mathbb{Z}[x, y] \setminus \{0\};$

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\[
\begin{align*}
\binom{a}{b}^{(c)}(d e | t) &= \sum_{n=0}^{\infty} \left( \frac{(a)_n(b)_n(c)_n}{(d)_n(e)_n} \right) \frac{t^n}{n!}, \quad (a)_n = a(a+1) \cdots (a+n-1). \\
\end{align*}
\]
Important classes of univariate generating functions

- **algebraic** if \( P(t, S(t)) = 0 \) for some \( P(x, y) \in \mathbb{Z}[x, y] \setminus \{0\} \);
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**Theorem** [Schwarz, 1873; Beukers, Heckman, 1989]

Characterization of \( \{ \text{hypergeometric} \} \cap \{ \text{algebraic} \} \).
Important classes of multivariate generating functions

D-finite

algebraic

\( S \in \mathbb{Q}[x, y, t] \) is \textit{algebraic} if it is the root of a polynomial \( P \in \mathbb{Q}[x, y, t, T] \);
Important classes of multivariate generating functions

\[ \sum a_i(t, x, y) \frac{\partial^i S}{\partial x^i} = 0, \quad \sum b_i(t, x, y) \frac{\partial^i S}{\partial y^i} = 0, \quad \sum c_i(t, x, y) \frac{\partial^i S}{\partial t^i} = 0. \]

- \( S \in \mathbb{Q}[[x, y, t]] \) is \textit{algebraic} if it is the root of a polynomial \( P \in \mathbb{Q}[x, y, t, T] \);

- \( S \in \mathbb{Q}[[x, y, t]] \) is \textit{D-finite} if it satisfies a system of linear partial differential equations with polynomial coefficients.
Main results (I): algebraicity of Gessel walks

**Theorem** [Kreweras, 1965; 100 pages long combinatorial proof!]

\[ K(t; 0, 0) = \binom{1/3}{3/2} \binom{2/3}{2} \binom{1}{1} 27 t^3 = \sum_{n=0}^{\infty} \frac{4^n (3^n)}{(n+1)(2n+1)} t^{3n} \]

**Theorem** [Kauers, Koutschan, Zeilberger, 2009: former Gessel’s conj. 1]

\[ G(t; 0, 0) = \binom{5/6}{5/3} \binom{1/2}{2} \binom{1}{1} 16 t^2 = \sum_{n=0}^{\infty} \frac{(5/6)_n (1/2)_n}{(5/3)_n (2)_n} (4t)^{2n} \]

**Question:** What about the structure of \( K(t; x, y) \) and \( G(t; x, y) \)?
Main results (I): algebraicity of Gessel walks

Theorem [Kreweras, 1965; 100 pages long combinatorial proof!]

\[ K(t; 0, 0) = {}_3F_2 \left( \begin{array}{ccc} 1/3 & 2/3 & 1 \\ 3/2 & 2 \\ \end{array} \right | 27 t^3 \) = \sum_{n=0}^{\infty} \frac{4^n (3n)}{(n+1)(2n+1)} t^{3n}. \]

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Theorem [Gessel, 1986; Bousquet-Mélou, 2005] \( K(t; x, y) \) is algebraic.

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Main results (I): algebraicity of Gessel walks

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▷ Computer-driven discovery and proof.
▷ **Guess’n’Prove method, using Hermite-Padé approximants**

† Minimal polynomial \( P(x, y, t, G(t; x, y)) = 0 \) has \( > 10^{11} \) terms; \( \approx 30 \text{Gb} (!) \)
Main results (I): algebraicity of Gessel walks

**Theorem** [Kreweras, 1965; 100 pages long combinatorial proof!]

\[ K(t; 0, 0) = \binom{3}{0} F_2 \left( \begin{array}{ccc} 1/3 & 2/3 & 1 \\ 3/2 & 2 \\ \end{array} \bigg| 27 t^3 \right) = \sum_{n=0}^{\infty} \frac{4^n (3^n)}{(n + 1)(2n + 1)} t^{3n}. \]

**Theorem** [Kauers, Koutschan, Zeilberger, 2009: former Gessel’s conj. 1]

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▷ Computer-driven discovery and proof.
▷ **Guess’n’Prove** method, using Hermite-Padé approximants† —— Part 2
▷ Recent (human) proofs [B., Kurkova, Raschel, 2013; Bousquet-Mélou, 2015]

---

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Main results (II): Explicit form for $G(t; x, y)$

**Theorem [B., Kauers, van Hoeij, 2010]**

Let $V = 1 + 4t^2 + 36t^4 + 396t^6 + \cdots$ be a root of 
\[(V - 1)(1 + 3/V)^3 = (16t)^2,
\]

let $U = 1 + 2t^2 + 16t^4 + 2xt^5 + 2(x^2 + 83)t^6 + \cdots$ be a root of 
\[x(V - 1)(V + 1)U^3 - 2V(3x + 5xV - 8Vt)U^2 \]
\[\quad - xV(V^2 - 24V - 9)U + 2V^2(xV - 9x - 8Vt) = 0,
\]

let $W = t^2 + (y + 8)t^4 + 2(y^2 + 8y + 41)t^6 + \cdots$ be a root of 
\[y(1 - V)W^3 + y(V + 3)W^2 - (V + 3)W + V - 1 = 0.
\]

Then $G(t; x, y)$ is equal to 
\[
\frac{64(U(V+1)-2V)V^{3/2}}{x(U^2-V(U^2-8U+9-V))^2} - \frac{y(W-1)^4(1-Wy)V^{-3/2}}{t(y+1)(1-W)(W^2y+1)^2} - \frac{1}{tx(y + 1)}.
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▷ Computer-driven discovery and proof.
Main results (II): Explicit form for $G(t; x, y)$

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▷ Computer-driven discovery and proof.
▷ Proof uses guessed minimal polynomials for $G(t; x, 0)$ and $G(t; 0, y)$. 

Computer Algebra for Combinatorics
Main results (II): Explicit form for $G(t; x, y)$

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▷ Computer-driven discovery and proof.
▷ Proof uses guessed minimal polynomials for $G(t; x, 0)$ and $G(t; 0, y)$.
▷ Recent (human) proofs [B., Kurkova, Raschel, 2013; Bousquet-Mélou, 2015]
Main results (III): Models with D-Finite $F(t; 1, 1)$

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Equation sizes = (order, degree)

- Computerized discovery: enumeration + guessing [B., Kauers, 2009]
- 23: Confirmed by a human proof in [B., Kurkova, Raschel, 2013]

Alin Bostan
Computer Algebra for Combinatorics
Main results (III): Models with D-Finite $F(t; 1, 1)$

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$A = 1 + \sqrt{2}, \ B = 1 + \sqrt{3}, \ C = 1 + \sqrt{6}, \ \lambda = 7 + 3\sqrt{6}, \ \mu = \sqrt{\frac{4\sqrt{6} - 1}{19}}$

▷ Computerized discovery: conv. acc. + LLL/PSLQ [B., Kauers, 2009]
▷ Confirmed by human proofs using ACSV in [Metczer, Wilson, 2015]
The characteristic polynomial

\[ \chi_S := x + \frac{1}{x} + y + \frac{1}{y} \]
The characteristic polynomial $\chi_\mathcal{S} := x + \frac{1}{x} + y + \frac{1}{y}$ is left invariant under

$$\psi(x, y) = \left(x, \frac{1}{y}\right), \quad \phi(x, y) = \left(\frac{1}{x}, y\right),$$
The group of a model: the simple walk case

The characteristic polynomial $\chi_S := x + \frac{1}{x} + y + \frac{1}{y}$ is left invariant under

$$\psi(x, y) = \left( x, \frac{1}{y} \right), \quad \phi(x, y) = \left( \frac{1}{x}, y \right),$$

and thus under any element of the group

$$\langle \psi, \phi \rangle = \left\{ (x, y), \left( x, \frac{1}{y} \right), \left( \frac{1}{x}, \frac{1}{y} \right), \left( \frac{1}{x}, y \right) \right\}.$$
The generating polynomial $\chi_S := \sum_{(i,j) \in \mathcal{S}} x^i y^j = \sum_{i=-1}^{1} B_i(y) x^i = \sum_{j=-1}^{1} A_j(x) y^j$
The generating polynomial $\chi_S := \sum_{(i,j) \in \mathcal{L}} x^i y^j = \sum_{i=-1}^{1} B_i(y) x^i = \sum_{j=-1}^{1} A_j(x) y^j$

is left invariant under the birational involutions

$$\psi(x, y) = \left( x, \frac{A_{-1}(x)}{A_1(x)} \frac{1}{y} \right), \quad \phi(x, y) = \left( \frac{B_{-1}(y)}{B_1(y)} \frac{1}{x}, y \right),$$
The generating polynomial $\chi_S := \sum_{(i,j) \in S} x^i y^j = \sum_{i=-1}^{1} B_i(y)x^i = \sum_{j=-1}^{1} A_j(x)y^j$

is left invariant under the birational involutions

$$\psi(x, y) = \left(x, \frac{A_{-1}(x)}{A_{+1}(x)} y\right), \quad \phi(x, y) = \left(\frac{B_{-1}(y)}{B_{+1}(y)} \frac{1}{x}, y\right),$$

and thus under any element of the (dihedral) group

$$G_S := \langle \psi, \phi \rangle.$$
Examples of groups

Order 4,
Examples of groups

Order 4, order 6,
Examples of groups

Order 4,

order 6,

order 8,
Examples of groups

Order 4, order 6, order 8, order $\infty$. 
Examples of groups

Order 4, order 6, order 8, order $\infty$.

\[
\Phi \quad \left(\frac{y}{x}, y\right) \quad \Psi \quad \left(\frac{y}{x}, \frac{1}{x}\right) \quad \Phi \\
\Psi \quad \left(x, \frac{x}{y}\right) \quad \Phi \quad \left(\frac{1}{y}, \frac{x}{y}\right) \quad \Psi
\]
Another important concept: the orbit sum (OS)

When $G_S$ is finite, the orbit sum of $S$ is the polynomial in $\mathbb{Q}[x, x^{-1}, y, y^{-1}]$:

$$OS_S := \sum_{\theta \in G_S} (-1)^\theta \theta(xy)$$

▷ E.g., for the simple walk, with $G_S = \{(x,y),(x,\frac{1}{y}), (\frac{1}{x},\frac{1}{y}), (\frac{1}{x},y)\}$:

$$OS_S = x \cdot y - \frac{1}{x} \cdot y + \frac{1}{x} \cdot \frac{1}{y} - x \cdot \frac{1}{y}$$

▷ For 4 models, the orbit sum is zero:

E.g., for the Kreweras model:

$$OS_S = x \cdot y - \frac{1}{xy} \cdot y + \frac{1}{xy} \cdot x - y \cdot x + y \cdot \frac{1}{xy} - x \cdot \frac{1}{xy} = 0$$
The 79 models: finite and infinite groups

79 models
The 79 models: finite and infinite groups

23 admit a finite group
[Mishna’07]

56 have an infinite group
[Bousquet-Mélou, Mishna’10]
The 79 models: finite and infinite groups

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all $F(t; x, y)$ D-finite

19 transcendental
(OS $\neq 0$)
[Gessel, Zeilberger’92]
[Bousquet-Mélou’02]

4 algebraic
(OS $= 0$)
(3 Kreweras-type + Gessel)
[BMM’10] + [B., Kauers’10]
The 79 models: finite and infinite groups

79 models

23 admit a finite group
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56 have an infinite group
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→ all \( F(t; x, y) \) D-finite

19 transcendental
(OS \( \neq 0 \))
[Gessel, Zeilberger’92]
[Bousquet-Mélou’02]

4 algebraic (OS \( = 0 \))
(3 Kreweras-type + Gessel)
[BMM’10] + [B., Kauers’10]

→ all non-D-finite

- [Mishna, Rechnitzer’07] and [Melczer, Mishna’13] for 5 singular models
- [Kurkova, Raschel’13] and [B., Raschel, Salvy’13] for all others
The kernel $J = 1 - t \cdot \sum_{(i,j) \in \mathcal{S}} x^i y^j = 1 - t \left( x + \frac{1}{x} + y + \frac{1}{y} \right)$ is invariant under the change of $(x, y)$ into, respectively:

$\left( \frac{1}{x}, y \right), \left( \frac{1}{x}, \frac{1}{y} \right), (x, \frac{1}{y})$. 
The kernel \( J = 1 - t \cdot \sum_{(i,j) \in \mathcal{S}} x^i y^j = 1 - t \left( x + \frac{1}{x} + y + \frac{1}{y} \right) \) is invariant under the change of \((x, y)\) into, respectively:

\[
\left( \frac{1}{x}, y \right), \left( \frac{1}{x}, \frac{1}{y} \right), (x, \frac{1}{y}).
\]

Kernel equation:

\[
J(t; x, y)xyF(t; x, y) = xy - txF(t; x, 0) - tyF(t; 0, y)
\]
The kernel $J = 1 - t \cdot \sum_{(i,j) \in \mathcal{S}} x^i y^j = 1 - t \left( x + \frac{1}{x} + y + \frac{1}{y} \right)$ is invariant under the change of $(x, y)$ into, respectively:

$$\left( \frac{1}{x}, y \right), \left( \frac{1}{x}, \frac{1}{y} \right), (x, \frac{1}{y}).$$

Kernel equation:

$$J(t; x, y) xy F(t; x, y) = xy - tx F(t; x, 0) - ty F(t; 0, y)$$

$$- J(t; x, y) \frac{1}{x} y F(t; \frac{1}{x}, y) = - \frac{1}{x} y + t \frac{1}{x} F(t; \frac{1}{x}, 0) + ty F(t; 0, y)$$
The kernel $J = 1 - t \cdot \sum_{(i,j) \in S} x^i y^j = 1 - t \left( x + \frac{1}{x} + y + \frac{1}{y} \right)$ is invariant under the change of $(x, y)$ into, respectively:

$$\left( \frac{1}{x}, y \right), \left( \frac{1}{x}, \frac{1}{y} \right), \left( x, \frac{1}{y} \right).$$

Kernel equation:

$$J(t; x, y)xyF(t; x, y) = xy - txF(t; x, 0) - tyF(t; 0, y)$$

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$$J(t; x, y) \frac{1}{x} \frac{1}{y} F(t; \frac{1}{x}, \frac{1}{y}) = \frac{1}{x} \frac{1}{y} - t \frac{1}{x} F(t; \frac{1}{x}, 0) - \frac{1}{y} F(t; 0, \frac{1}{y})$$
The kernel $J = 1 - t \cdot \sum_{(i,j) \in \mathcal{S}} x^i y^j = 1 - t \left( x + \frac{1}{x} + y + \frac{1}{y} \right)$ is invariant under the change of $(x, y)$ into, respectively:

$$\left( \frac{1}{x}, y \right), \left( \frac{1}{x}, \frac{1}{y} \right), (x, \frac{1}{y}).$$

Kernel equation:

$$J(t; x, y) xy F(t; x, y) = xy - tx F(t; x, 0) - ty F(t; 0, y)$$

$$- J(t; x, y) \frac{1}{x} y F(t; \frac{1}{x}, y) = - \frac{1}{x} y + t \frac{1}{x} F(t; \frac{1}{x}, 0) + ty F(t; 0, y)$$

$$J(t; x, y) \frac{1}{x} \frac{1}{y} F(t; \frac{1}{x}, \frac{1}{y}) = \frac{1}{x} \frac{1}{y} t \frac{1}{x} F(t; \frac{1}{x}, 0) - t \frac{1}{y} F(t; 0, \frac{1}{y})$$

$$- J(t; x, y) x \frac{1}{y} F(t; x, \frac{1}{y}) = - x \frac{1}{y} + tx F(t; x, 0) + t \frac{1}{y} F(t; 0, \frac{1}{y})$$
The kernel \( J = 1 - t \cdot \sum_{(i,j) \in \mathcal{S}} x^i y^j = 1 - t \left( x + \frac{1}{x} + y + \frac{1}{y} \right) \) is invariant under the change of \((x, y)\) into, respectively:

\[
\left( \frac{1}{x}, y \right), \left( \frac{1}{x}, \frac{1}{y} \right), (x, \frac{1}{y}).
\]

Kernel equation:

\[
J(t; x, y)xyF(t; x, y) = xy - txF(t; x, 0) - tyF(t; 0, y)
\]

\[
- J(t; x, y) \frac{1}{x} y F(t; \frac{1}{x}, y) = - \frac{1}{x} y + t \frac{1}{x} F(t; \frac{1}{x}, 0) + tyF(t; 0, y)
\]

\[
J(t; x, y) \frac{1}{x} \frac{1}{y} F(t; \frac{1}{x}, \frac{1}{y}) = \frac{1}{x} \frac{1}{y} - t \frac{1}{x} F(t; \frac{1}{x}, 0) - t \frac{1}{y} F(t; 0, \frac{1}{y})
\]

\[
- J(t; x, y) x \frac{1}{y} F(t; x, \frac{1}{y}) = - x \frac{1}{y} + txF(t; x, 0) + t \frac{1}{y} F(t; 0, \frac{1}{y})
\]

Summing up yields the orbit equation:

\[
\sum_{\theta \in G} (-1)^{\theta} \theta(xy F(t; x, y)) = \frac{xy - \frac{1}{x} y + \frac{1}{x} \frac{1}{y} - x \frac{1}{y}}{J(t; x, y)}
\]
The kernel $J = 1 - t \cdot \sum_{(i,j) \in S} x^i y^j = 1 - t \left( x + \frac{1}{x} + y + \frac{1}{y} \right)$ is invariant under the change of $(x, y)$ into, respectively:

$$
\left( \frac{1}{x}, y \right), \left( \frac{1}{x}, \frac{1}{y} \right), (x, \frac{1}{y}).
$$

Kernel equation:

$$
J(t; x, y) xy F(t; x, y) = xy - tx F(t; x, 0) - ty F(t; 0, y)
$$

$$
- J(t; x, y) \frac{1}{x} y F(t; \frac{1}{x}, y) = - \frac{1}{x} y + t \frac{1}{x} F(t; \frac{1}{x}, 0) + ty F(t; 0, y)
$$

$$
J(t; x, y) \frac{1}{x} \frac{1}{y} F(t; \frac{1}{x}, \frac{1}{y}) = \frac{1}{x} \frac{1}{y} - t \frac{1}{x} F(t; \frac{1}{x}, 0) - t \frac{1}{y} F(t; 0, \frac{1}{y})
$$

$$
- J(t; x, y) x \frac{1}{y} F(t; x, \frac{1}{y}) = - x \frac{1}{y} + tx F(t; x, 0) + t \frac{1}{y} F(t; 0, \frac{1}{y})
$$

Taking positive parts yields:

$$
[x^+ y^+] \sum_{\theta \in G} (-1)^{\theta} \theta(\theta F(t; x, y)) = [x^+ y^+] \frac{xy - \frac{1}{x} y + \frac{1}{x} \frac{1}{y} - x \frac{1}{y}}{J(t; x, y)}
$$
The kernel \( J = 1 - t \cdot \sum_{(i,j) \in \mathcal{S}} x^i y^j = 1 - t \left(x + \frac{1}{x} + y + \frac{1}{y}\right) \) is invariant under the change of \((x, y)\) into, respectively:

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Kernel equation:

\[
J(t; x, y)xyF(t; x, y) = xy - txF(t; x, 0) - tyF(t; 0, y)
- J(t; x, y)\frac{1}{x}yF(t; \frac{1}{x}, y) = -\frac{1}{x}y + t\frac{1}{x}F(t; \frac{1}{x}, 0) + tyF(t; 0, y)
- J(t; x, y)\frac{1}{x}yF(t; \frac{1}{x}, \frac{1}{y}) = \frac{1}{x}y - t\frac{1}{x}F(t; \frac{1}{x}, 0) - t\frac{1}{y}F(t; 0, \frac{1}{y})
- J(t; x, y)x\frac{1}{y}F(t; x, \frac{1}{y}) = -x\frac{1}{y} +txF(t; x, 0) + t\frac{1}{y}F(t; 0, \frac{1}{y})
\]

Summing up and taking positive parts yields:

\[
xy F(t; x, y) = [x>y] \frac{xy - \frac{1}{x}y + \frac{1}{x} \frac{1}{y} - x\frac{1}{y}}{J(t; x, y)}
\]
D-Finiteness via the finite group [Bousquet-Mélou, Mishna, 2010]

The kernel $J = 1 - t \cdot \sum_{(i,j) \in \mathcal{S}} x^i y^j = 1 - t \left( x + \frac{1}{x} + y + \frac{1}{y} \right)$ is invariant under the change of $(x, y)$ into, respectively:

$\left( \frac{1}{x}, y \right), \left( \frac{1}{x}, \frac{1}{y} \right), (x, \frac{1}{y})$.

Kernel equation:

\[
J(t; x, y) xy F(t; x, y) = xy - txF(t; x, 0) - tyF(t; 0, y) \\
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J(t; x, y) \frac{1}{x} \frac{1}{y} F(t; \frac{1}{x}, \frac{1}{y}) = \frac{1}{x y} - t \frac{1}{x} F(t; \frac{1}{x}, 0) - t \frac{1}{y} F(t; 0, \frac{1}{y}) \\
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\]

$GF = \text{PosPart} \left( \frac{\text{OS kernel}}{\text{kernel}} \right)$
The kernel $J = 1 - t \cdot \sum_{(i,j) \in \mathcal{S}} x^i y^j = 1 - t \left( x + \frac{1}{x} + y + \frac{1}{y} \right)$ is invariant under the change of $(x, y)$ into, respectively:

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$J(t; x, y) \frac{1}{x} y F(t; \frac{1}{x}, \frac{1}{y}) = \frac{1}{x} \frac{1}{y} - t \frac{1}{x} F(t; \frac{1}{x}, 0) - t \frac{1}{y} F(t; 0, \frac{1}{y})$

$- J(t; x, y) x \frac{1}{y} F(t; x, \frac{1}{y}) = - x \frac{1}{y} + tx F(t; x, 0) + t \frac{1}{y} F(t; 0, \frac{1}{y})$

$GF = \text{PosPart} \left( \frac{\text{OS}}{\text{ker}} \right) \text{ is D-finite [Lipshitz, 1988]}$
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J(t; x, y) \frac{1}{x} \frac{1}{y} F(t; \frac{1}{x}, \frac{1}{y}) = \frac{1}{x} \frac{1}{y} - t \frac{1}{x} F(t; \frac{1}{x}, 0) - t \frac{1}{y} F(t; 0, \frac{1}{y}) \\
- J(t; x, y) x \frac{1}{y} F(t; x, \frac{1}{y}) = - x \frac{1}{y} + tx F(t; x, 0) + t \frac{1}{y} F(t; 0, \frac{1}{y})
\]

\[
\text{GF} = \text{PosPart} \left( \frac{\text{OS}}{\text{ker}} \right) \text{ is D-finite [Lipshitz, 1988]}
\]

▷ Argument works if \( \text{OS} \neq 0 \): algebraic version of the reflection principle
The kernel $J = 1 - t \cdot \sum_{(i,j) \in \mathcal{S}} x^i y^j = 1 - t \left( x + \frac{1}{x} + y + \frac{1}{y} \right)$ is invariant under the change of $(x, y)$ into, respectively:

$$
\left( \frac{1}{x}, y \right), \left( \frac{1}{x}, \frac{1}{y} \right), \left( x, \frac{1}{y} \right).
$$

Kernel equation:

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J(t; x, y) x y F(t; x, y) = x y - t x F(t; x, 0) - t y F(t; 0, y)
$$

$$
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$$

$$
J(t; x, y) \frac{1}{x} \frac{1}{y} F(t; \frac{1}{x}, \frac{1}{y}) = \frac{1}{x y} - t \frac{1}{x} F(t; \frac{1}{x}, 0) - t \frac{1}{y} F(t; 0, \frac{1}{y})
$$

$$
- J(t; x, y) x \frac{1}{y} F(t; x, \frac{1}{y}) = - x \frac{1}{y} + t x F(t; x, 0) + t \frac{1}{y} F(t; 0, \frac{1}{y})
$$

$$
GF = \text{PosPart} \left( \frac{\text{OS}}{\text{ker}} \right) \text{ is D-finite [Lipshitz, 1988]}
$$

Creative Telescoping finds a differential equation for $\text{PosPart}(\text{OS/ker})$
Main results (IV): explicit expressions for models 1–19

**Theorem [B., Chyzak, van Hoeij, Kauers, Pech, 2017]**

Let $\mathcal{S}$ be one of the 19 models with finite group $\mathcal{G}_\mathcal{S}$, and orbit sum $\text{OS} \neq 0$. Then

- $F_{\mathcal{S}}$ is expressible using iterated integrals of $\binom{2}{1}$ expressions.
- Among the $19 \times 4$ specializations of $F_{\mathcal{S}}(t; x, y)$ at $(x, y) \in \{0, 1\}^2$, only 4 are algebraic: for $\mathcal{S} = \includegraphics[width=0.05\textwidth]{up} \includegraphics[width=0.05\textwidth]{down}$ at $(1, 1)$, and $\mathcal{S} = \includegraphics[width=0.05\textwidth]{left} \includegraphics[width=0.05\textwidth]{right}$ at $(1, 0), (0, 1), (1, 1)$
Main results (IV): explicit expressions for models 1–19

**Theorem [B., Chyzak, van Hoeij, Kauers, Pech, 2017]**

Let $\mathcal{S}$ be one of the 19 models with finite group $G, \mathcal{S}$, and orbit sum $OS \neq 0$. Then

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- Among the $19 \times 4$ specializations of $F_{\mathcal{S}}(t; x, y)$ at $(x, y) \in \{0, 1\}^2$, only 4 are algebraic: for $\mathcal{S} = \leftarrow \rightarrow$ at $(1, 1)$, and $\mathcal{S} = \downarrow \uparrow$ at $(1, 0), (0, 1), (1, 1)$

**Example (King walks in the quarter plane, A025595)**

$$F_{\leftarrow \rightarrow}(t; 1, 1) = \frac{1}{t} \int_0^t \frac{1}{(1 + 4x)^3} \cdot 2F_1 \left( \begin{array}{c} \frac{3}{2} \ 2 \\ \ \end{array} ; \frac{16x(1 + x)}{(1 + 4x)^2} \right) dx$$

$$= 1 + 3t + 18t^2 + 105t^3 + 684t^4 + 4550t^5 + 31340t^6 + 219555t^7 + \cdots$$
Main results (IV): explicit expressions for models 1–19

Theorem [B., Chyzak, van Hoeij, Kauers, Pech, 2017]
Let $\mathcal{S}$ be one of the 19 models with finite group $\mathcal{G}_\mathcal{S}$, and orbit sum $OS \neq 0$. Then

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Example (King walks in the quarter plane, A025595)

$$F_{\begin{array}{c} \hline \end{array}}(t; 1, 1) = \frac{1}{t} \int_{0}^{t} \frac{1}{(1 + 4x)^3} \cdot 2F_1 \left( \begin{array}{c} \frac{3}{2}, \frac{3}{2} \\ \frac{3}{2}, \frac{3}{2} \end{array} \right) \frac{16x(1 + x)}{(1 + 4x)^2} \, dx$$

$$= 1 + 3t + 18t^2 + 105t^3 + 684t^4 + 4550t^5 + 31340t^6 + 219555t^7 + \cdots$$

▷ Computer-driven discovery and proof; no human proof yet.
▷ Proof uses creative telescoping, ODE factorization, ODE solving.
#### Bonus: hypergeometric functions occurring in $F(t; x, y)$

<table>
<thead>
<tr>
<th>$\mathcal{I}$</th>
<th>occurring $2F_1$</th>
<th>$w$</th>
<th>$\mathcal{I}$</th>
<th>occurring $2F_1$</th>
<th>$w$</th>
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</thead>
<tbody>
<tr>
<td>1</td>
<td>$2F_1\left(\frac{1}{2}, \frac{1}{2} \bigg</td>
<td>\frac{1}{1}\right) w$</td>
<td>$16t^2$</td>
<td>11</td>
<td>$2F_1\left(\frac{1}{2}, \frac{1}{2} \bigg</td>
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<td>2</td>
<td>$2F_1\left(\frac{1}{2}, \frac{1}{2} \bigg</td>
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<td>$16t^2$</td>
<td>12</td>
<td>$2F_1\left(\frac{1}{4}, \frac{3}{4} \bigg</td>
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<tr>
<td>3</td>
<td>$2F_1\left(\frac{1}{4}, \frac{3}{4} \bigg</td>
<td>\frac{1}{1}\right) w$</td>
<td>$\frac{64t^2}{(12t^2+1)^2}$</td>
<td>13</td>
<td>$2F_1\left(\frac{1}{4}, \frac{3}{4} \bigg</td>
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<tr>
<td>4</td>
<td>$2F_1\left(\frac{1}{2}, \frac{1}{2} \bigg</td>
<td>\frac{1}{1}\right) w$</td>
<td>$16t(t+1)$</td>
<td>14</td>
<td>$2F_1\left(\frac{1}{4}, \frac{3}{4} \bigg</td>
</tr>
<tr>
<td>5</td>
<td>$2F_1\left(\frac{1}{4}, \frac{3}{4} \bigg</td>
<td>\frac{1}{1}\right) w$</td>
<td>$64t^4$</td>
<td>15</td>
<td>$2F_1\left(\frac{1}{4}, \frac{3}{4} \bigg</td>
</tr>
<tr>
<td>6</td>
<td>$2F_1\left(\frac{1}{4}, \frac{3}{4} \bigg</td>
<td>\frac{1}{1}\right) w$</td>
<td>$\frac{64t^3(t+1)}{(1-4t^2)^2}$</td>
<td>16</td>
<td>$2F_1\left(\frac{1}{4}, \frac{3}{4} \bigg</td>
</tr>
<tr>
<td>7</td>
<td>$2F_1\left(\frac{1}{2}, \frac{1}{2} \bigg</td>
<td>\frac{1}{1}\right) w$</td>
<td>$\frac{16t^2}{4t^2+1}$</td>
<td>17</td>
<td>$2F_1\left(\frac{1}{3}, \frac{2}{3} \bigg</td>
</tr>
<tr>
<td>8</td>
<td>$2F_1\left(\frac{1}{4}, \frac{3}{4} \bigg</td>
<td>\frac{1}{1}\right) w$</td>
<td>$\frac{64t^3(2t+1)}{(8t^2-1)^2}$</td>
<td>18</td>
<td>$2F_1\left(\frac{1}{3}, \frac{2}{3} \bigg</td>
</tr>
<tr>
<td>9</td>
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<td>19</td>
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</tr>
<tr>
<td>10</td>
<td>$2F_1\left(\frac{1}{4}, \frac{3}{4} \bigg</td>
<td>\frac{1}{1}\right) w$</td>
<td>$\frac{64t^2(t^2+t+1)}{(12t^2+1)^2}$</td>
<td>20</td>
<td>$2F_1\left(\frac{1}{2}, \frac{1}{2} \bigg</td>
</tr>
</tbody>
</table>

All related to the complete elliptic integrals $\int_0^{\pi/2} (1 - k^2 \sin^2 \theta)^{1/2} d\theta$
Theorem [B., Raschel, Salvy, 2014]

Let $\mathcal{I}$ be one of the 51 non-singular models with infinite group $G_\mathcal{I}$. Then $F_\mathcal{I}(t;0,0)$, and in particular $F_\mathcal{I}(t;x,y)$, are non-D-finite.
Main results (V): non-D-finiteness in models with an infinite group

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**Theorem [B., Raschel, Salvy, 2014]**

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▷ [Bernardi, Bousquet-Mélou, Raschel, 2016] For 9 of these 51 models, $F_\mathcal{S}(t;x,y)$ is nevertheless D-algebraic!
▷ [Dreyfus, Hardouin, Roques, Singer, 2017]: hypertranscendence of the remaining 42 models.
The 56 models with infinite group

In blue, non-singular models, solved by [B., Raschel, Salvy, 2014]

In red, singular models, solved by [Melmzer, Mishna, 2013]
Example with infinite group: the scarecrows

[B., Raschel, Salvy, 2014]: $F_{\mathcal{G}}(t;0,0)$ is not D-finite for the models

For the 1st and the 3rd, the excursions sequence $[t^n] F_{\mathcal{G}}(t;0,0)$

$$1, 0, 0, 2, 4, 8, 28, 108, 372, \ldots$$

is $\sim K \cdot 5^n \cdot n^{-\alpha}$, with $\alpha = 1 + \pi / \arccos(1/4) = 3.383396 \ldots$

[Denisov, Wachtel, 2015]

The irrationality of $\alpha$ prevents $F_{\mathcal{G}}(t;0,0)$ from being D-finite.

[Katz, 1970; Chudnovsky, 1985; André, 1989]
Theorem

Let $\mathcal{S}$ be one of the 74 non-singular models of small-step walks in $\mathbb{N}^2$. The following assertions are equivalent:

1. the full generating function $F_{\mathcal{S}}(t; x, y)$ is D-finite
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4. the group $G_{\mathcal{S}}$ is finite (and $|G_{\mathcal{S}}| = 2 \cdot \min\{\ell \in \mathbb{N}^* | \frac{\ell}{\alpha+1} \in \mathbb{Z}\}$)
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6. the step set $\mathcal{S}$ has either an axial symmetry, or zero drift and $|\mathcal{S}| \neq 5$. 

Moreover, under 1–6:

- $F_{\mathcal{S}}(t; x, y)$ is algebraic $\iff$ $O_{\mathcal{S}} = 0$
- $\exists U \in \mathbb{Q}(x, t), V \in \mathbb{Q}(y, t)$ s.t. $U(x) + V(y) = xy$ on the curve $\chi_{\mathcal{S}}(x, y) = \frac{1}{t}$.

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Proof uses various tools: algebra, complex analysis, probability theory, computer algebra, etc.
Summary: walks with small steps in $\mathbb{N}^2$

quadrant models $\mathcal{I}$: 79

$|\mathcal{G}_\mathcal{I}| < \infty$: 23

orbit sum $\neq 0$: 19
kernel method + CT
D-finite

orbit sum $= 0$: 4
Guess-and-Prove
algebraic

$|\mathcal{G}_\mathcal{I}| = \infty$: 56
asymptotics + GB
non-D-finite

Theorem: differential finiteness $\iff$ finiteness of the group!
Extensions: Walks in $\mathbb{N}^2$ with small repeated steps

2D quadrant models: 527

$|G_{\mathcal{A}}| < \infty$: 118

- orbit sum $\neq 0$: 95
  - kernel method: 94
    - D-finite

- CA: D-finite

$|G_{\mathcal{A}}| = \infty$: 409

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  - 22: reducible to Kreweras/Gessel
    - CA: D-finite

- non-D-finite?

D-finite

[Alin Bostan, Computer Algebra for Combinatorics]

[B., Bousquet-Mélou, Kauers, Melczer, 2016] + [Du, Hou, Wang, 2017]

Question: differential finiteness $\iff$ finiteness of the group?
Answer: probably yes
Extensions: Walks in $\mathbb{N}^2$ with small repeated steps

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    - CA: algebraic

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Question: differential finiteness $\iff$ finiteness of the group?

Answer: probably yes
Extensions: walks with small steps in $\mathbb{N}^3$

$2^{3^3-1} \approx 67$ million models, of which $\approx 11$ million inherently 3D

3D octant models $\mathcal{O}$ with $\leq 6$ steps: 20804

- $|\mathcal{G}_\mathcal{O}| < \infty$: 170
- $|\mathcal{G}_\mathcal{O}| = \infty$: 20634

- Orbit sum $\neq 0$: 108
- Orbit sum $= 0$: 62

- Kernel method: 2D-reducible: 43
- Not 2D-reducible: 19

- D-finite
- D-finite
- Non-D-finite?

[B., Bousquet-Mélou, Kauers, Melczer, 2016] + [Du, Hou, Wang, 2017]; completed by [Bacher, Kauers, Yatchak, 2016]
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Question: differential finiteness $\iff$ finiteness of the group?

Answer: probably no
19 mysterious 3D-models
Two different computations suggest:

\[ k_{4n} \approx C \cdot \frac{256^n}{n^{3.3257570041744\ldots}} , \]

so excursions are very probably transcendental (and even non-D-finite)
Extensions: Walks in $\mathbb{N}^2$ with large steps

quadrant models with steps in $\{-2, -1, 0, 1\}^2$: 13 110

<table>
<thead>
<tr>
<th>orbit</th>
<th>&lt; $\infty$: 240</th>
<th>orbit</th>
<th>= $\infty$: 12 870</th>
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D-finite | D-finite? | non-D-finite? | non-D-finite

[B., Bousquet-Mélou, Melczer, 2018]

- Example: For the model

$$xyF(t; x, y) = [x > 0 \, y > 0] \frac{(x - 2x^{-2})(y - (x - x^{-2})y^{-1})}{1 - t(xy^{-1} + y + x^{-2}y^{-1})}$$
Extensions: Walks in $\mathbb{N}^2$ with large steps

quadrant models with steps in $\{-2, -1, 0, 1\}^2$: 13 110

\[
\begin{align*}
|\text{orbit}| < \infty & : 240 \\
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\end{align*}
\]

\[
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D-finite \quad D-finite? \quad \alpha \text{ rational: 16} \quad \alpha \text{ irrational: 12 854}

\begin{align*}
\text{non-D-finite?} & \\
\text{non-D-finite}
\end{align*}

[B., Bousquet-Mélou, Melczer, 2018]

Question: \textbf{differential finiteness} $\iff$ finiteness of the orbit?

Answer: ?
Two challenging models with large steps

**Conjecture 1 [B., Bousquet-Mélou, Melczer, 2018]**

For the model the excursions generating function $F(t^{1/2}; 0, 0)$ equals

$$
\frac{1}{3t} - \frac{1}{6t} \cdot \left( \frac{1 - 12t}{(1 + 36t)^{1/3}} \right) \cdot 2F_1\left( \frac{1}{6} \left| \frac{108t(1 + 4t)^2}{(1 + 36t)^2} \right. \right) + \\
\sqrt{1 - 12t} \cdot 2F_1\left( \frac{1}{6} \left| \frac{108t(1 + 4t)^2}{(1 - 12t)^2} \right. \right).
$$

**Conjecture 2 [B., Bousquet-Mélou, Melczer, 2018]**

For the model the excursions generating function $F(t; 0, 0)$ equals

$$
\frac{(1 - 24U + 120U^2 - 144U^3) (1 - 4U)}{(1 - 3U) (1 - 2U)^{3/2} (1 - 6U)^{9/2}},
$$

where $U = t^4 + 53t^8 + 4363t^{12} + \cdots$ is the unique series in $\mathbb{Q}[[t]]$ satisfying

$$
U (1 - 2U)^3 (1 - 3U)^3 (1 - 6U)^9 = t^4 (1 - 4U)^4.
$$
Conclusion

- Computer algebra may solve difficult combinatorial problems
- Classification of $F(t; x, y)$ fully completed for 2D small step walks
- Robust algorithmic methods, based on efficient algorithms:
  - Guess’n’Prove
  - Creative Telescoping
- Brute-force and/or use of naive algorithms = hopeless.
  E.g. size of algebraic equations for $G(t; x, y) \approx 30Gb.$
Computer algebra may solve difficult combinatorial problems

Classification of $F(t; x, y)$ fully completed for 2D small step walks

Robust algorithmic methods, based on efficient algorithms:
- Guess’n’Prove
- Creative Telescoping

Brute-force and/or use of naive algorithms = hopeless.
E.g. size of algebraic equations for $G(t; x, y) \approx 30$Gb.

Lack of “purely human” proofs for some results.

Open: is $F(t; 1, 1)$ non-D-finite for all 56 models with infinite group?

Many beautiful open questions for 2D models with repeated or large steps, and in dimension $> 2$. 
• The complete generating function for Gessel walks is algebraic, with M. Kauers. Proceedings of the American Mathematical Society, 2010.
• On 3-dimensional lattice walks confined to the positive octant, with M. Bousquet-Mélou, M. Kauers and S. Melczer. Annals of Comb., 2016.
• Hypergeometric expressions for generating functions of walks with small steps in the quarter plane, with F. Chyzak, M. van Hoeij, M. Kauers and L. Pech, European Journal of Combinatorics, 2017.
• Counting walks with large steps in an orthant, with M. Bousquet-Mélou and S. Melczer, preprint, 2018.