ON A NICE INTEGRAL EQUALITY (I)

ALIN BOSTAN

Abstract. We give an elementary proof of a nice equality of definite integrals.

The following infinite family of equalities between definite integrals was proven in https://arxiv.org/abs/1911.01423 by S. B. Ekhad, D. Zeilberger and W. Zudilin, using the Almkvist-Zeilberger "creative telescoping" algorithm:

$$\int_0^1 \frac{x^n (1-x)^n}{((x+a)(x+b))^{n+1}} \, \mathrm{d}x = \int_0^1 \frac{x^n (1-x)^n}{((a-b)x+(a+1)b)^{n+1}} \, \mathrm{d}x, \tag{1}$$

for any reals $a > b > 0$ and any non-negative integer n.

Here we offer an elementary alternative proof. The starting point is the observation that after multiplying both sides of (1) by t^n, and then summing on n from 0 to ∞, this family of identities is equivalent to the fact that the two integrals $I_1(t)$ and $I_2(t)$ between $x = 0$ and $x = 1$ of the rational functions $F_1(x,t)$ and $F_2(x,t)$ defined by

$$F_1 = \frac{1}{(x+a)(x+b) - tx(1-x)} \quad \text{and} \quad F_2 = \frac{1}{(a-b)x + (a+1)b - tx(1-x)}$$

are equal.

The second observation is that F_1 and F_2 admit the closed form antiderivatives:

$$G_1(x,t) = \frac{2}{B} \arctan\left(\frac{2\,(t+1)\,x + a + b - t}{B}\right), \; G_2(x,t) = \frac{2}{B} \arctan\left(\frac{2\,tx + a - b - t}{B}\right)$$

where

$$B = \sqrt{(4\,ab + 2\,a + 2\,b)\,t - (a-b)^2 - t^2}.$$

Using the identity

$$\arctan(x) - \arctan(y) = \arctan\left(\frac{x-y}{xy+1}\right),$$

we deduce that $I_1(t) = G_1(1,t) - G_1(0,t)$ and $I_2(t) = G_2(1,t) - G_2(0,t)$ admit the following closed forms:

$$I_1 = \frac{2}{B}\left(\arctan\left(\frac{t+a+b+2}{B}\right) - \arctan\left(\frac{a+b-t}{B}\right)\right)$$

$$= \frac{2}{B} \cdot \arctan\left(\frac{2\,(t+1)\,B}{B^2 + (a+b)^2 - t^2 + 2\,(a+b-t)}\right) = \frac{2}{B} \cdot \arctan\left(\frac{B}{2\,ab + a + b - t}\right)$$

and, similarly

$$I_2 = \frac{2}{B}\left(\arctan\left(\frac{t+a-b}{B}\right) - \arctan\left(\frac{a-b-t}{B}\right)\right)$$

$$= \frac{2}{B} \cdot \arctan\left(\frac{2\,tB}{B^2 + (a-b)^2 - t^2}\right) = \frac{2}{B} \cdot \arctan\left(\frac{B}{2\,ab + a + b - t}\right).$$

Therefore, $I_1 = I_2$. □

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