ON A NICE INTEGRAL EQUALITY (II)

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Abstract. We give another proof of a nice equality of definite integrals.

The following infinite family of equalities between definite integrals was proven in https://arxiv.org/abs/1911.01423 by S. B. Ekhad, D. Zeilberger and W. Zudilin, using the Almkvist-Zeilberger "creative telescoping" algorithm:

$$\int_0^1 \frac{x^n\,(1-x)^n}{((x+a)\,(x+b))^{n+1}}\,\mathrm{d}x = \int_0^1 \frac{x^n\,(1-x)^n}{((a-b)\,x+(a+1)\,b)^{n+1}}\,\mathrm{d}x, \tag{1}$$

for any reals $a > b > 0$ and any non-negative integer n.

Here we offer a second proof based on "creative telescoping". The starting point is the observation that after multiplying both sides of (1) by t^n, and then summing on n from 0 to ∞, this family of identities is equivalent to the fact that the two integrals $I_1(t)$ and $I_2(t)$ between $x = 0$ and $x = 1$ of the rational functions $F_1(x,t)$ and $F_2(x,t)$ defined by

$$F_1 = \frac{1}{(x+a)\,(x+b)-tx\,(1-x)} \quad \text{and} \quad F_2 = \frac{1}{(a-b)\,x+(a+1)\,b-tx\,(1-x)}$$

are equal.

Creative telescoping (this time in the classical "differential-differential" setting) shows that F_1 and F_2 satisfy the equalities

$$(t-2\,ab-a-b)\,F_1 + \left(t^2-2\,t\,(2\,ab+a+b)+(a-b)^2\right)\frac{\partial F_1}{\partial t} + \frac{\partial}{\partial x}\,(F_1\cdot R_1) = 0$$

and

$$(t-2\,ab-a-b)\,F_2 + \left(t^2-2\,t\,(2\,ab+a+b)+(a-b)^2\right)\frac{\partial F_2}{\partial t} + \frac{\partial}{\partial x}\,(F_2\cdot R_2) = 0$$

where $R_1(t,x)$ and $R_2(t,x)$ are the rational functions

$$R_1 = ((a+b+t+2)\,x+2\,ab+a+b-t)\,x$$

and

$$R_2 = \frac{\left((2\,ab+a+b)\;t-(a-b)^2\right)x^2+b\,(a+1)\,(2\,ab+a+b-t)}{t+b-a}.$$

Hence, by integration between $x = 0$ and $x = 1$, one obtains that both I_1 and I_2 are solutions of the differential equation

$$(t-2\,ab-a-b)\cdot I(t) + \left(t^2-2\,t\,(2\,ab+a+b)+(a-b)^2\right)\cdot I'(t) = 2,$$

therefore $I_1 = I_2$ by Cauchy's theorem, since $I_1 - I_2$ is the solution of a differential equation of order 1 with leading term non-vanishing at $t = 0$, and its evaluation at $t = 0$ is zero, as $I_1(0) = I_2(0) = \frac{1}{a-b}\ln\left(\frac{a(b+1)}{(a+1)b}\right)$. □

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