False theta functions and their modularity properties

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Joint work with Caner Nazaroglu (University of Cologne)

May 13, 2019

Outline

1. Modular forms and mock modular forms

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Modular forms

"Modular forms are functions on the complex plane that are inordinately symmetric.

They satisfy so many internal symmetries that their mere existence seem like accidents.

But they do exist."
-Mazur

B. Mazur

Modularity

Definition:

$f : \mathbb{H} \to \mathbb{C}$ holomorphic is modular of weight k if for all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$

$$f\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k f(\tau)$$

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weight ↓

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plus growth condition

Fourier expansion ($q := e^{2\pi i \tau}$, $\tau \in \mathbb{H}$)

$$f(\tau) = \sum_{n \in \mathbb{Z}} c(n) q^n$$

Examples

- Dedekind η-function:

$$\eta(\tau) := q^{\frac{1}{24}} \prod_{n\geq 1} (1-q^n)$$

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Modularity:

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- Theta function:

$$\Theta(\tau) := \sum_{n\in\mathbb{Z}} q^{n^2}.$$

Identity of Gauss

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$$\Theta(\tau)^3 =: \sum_{n\geq 0} r(n)q^n.$$

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$$r(n) = 12\big(H(4n) - 2H(n)\big)$$

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with

$H(n) := \#\{$equivalence classes of integral binary quadratic forms of discriminant $n\}$.

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J. Bruinier

J. Funke

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$$\Delta_k(F) = 0$$

with $(\tau = \tau_1 + i\tau_2)$

$$\Delta_k := -\tau_2^2 \left(\frac{\partial^2}{\partial \tau_1^2} + \frac{\partial^2}{\partial \tau_2^2} \right) + ik\tau_2 \left(\frac{\partial}{\partial \tau_1} + i \frac{\partial}{\partial \tau_2} \right).$$

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Examples

- weight 2 Eisenstein series

$$\widehat{E}_2(\tau) := E_2(\tau) - \frac{3}{\pi\tau_2}$$

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- Class number generating function

shadow $\downarrow$

$$\widehat{\mathcal{H}}(\tau) := \sum_{\substack{n\geq 0 \\ n\equiv 0,3 \pmod 4}} H(n)q^n + \frac{i}{8\sqrt{2}\pi} \int_{-\overline{\tau}}^{i\infty} \frac{\Theta(w)}{(-i(\tau+w))^{\frac{3}{2}}} dw.$$

$\uparrow$ mock modular

Natural splitting

$\mathcal{F}$ harmonic Maass form

$$\mathcal{F} \quad = \quad \mathcal{F}^{+} \quad + \quad \mathcal{F}^{-}$$

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$$\mathcal{F} = \underset{\substack{\uparrow \\ \text{holomorphic} \\ \text{part}}}{\mathcal{F}^+} + \underset{\substack{\uparrow \\ \text{non-holomorphic} \\ \text{part}}}{\mathcal{F}^-}$$

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with

$$\mathcal{F}^+(\tau) := \sum_{n \gg -\infty} c^+(n) q^n,$$

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Alternative representation

The non-holomorphic part has the shape

$$\int_{-\bar{\tau}}^{i\infty} g(w)(\tau + w)^{2-k} dw.$$

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↑

modular, weight $2-k$

Ramanujan's last letter

"I am extremely sorry for not writing you a single letter up to now. I recently discovered very interesting functions which I call "Mock" ϑ-functions. Unlike the "False" ϑ-functions they enter into mathematics as beautifully as the theta functions. I am sending you with this letter some examples."

S. Ramanujan

Mock theta functions

These mock theta functions are 22 peculiar q-series.

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Example:

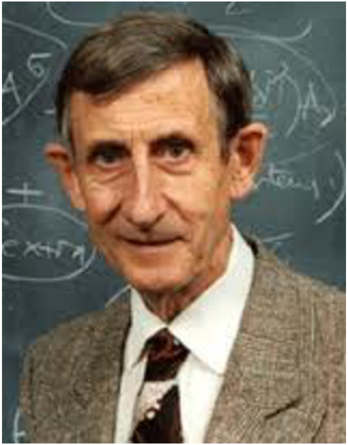
$$f(q) := \sum_{n\geq 0} \frac{q^{n^2}}{(-q;q)_n^2}$$

with

$$(a;q)_n := \prod_{m=0}^{n-1} (1 - aq^m).$$

Dyson's challenge for the future

"The mock theta-functions give us tantalizing hints of a grand synthesis still to be discovered. Somehow it should be possible to build them into a coherent group-theoretical structure, analogous to the structure of modular forms which Hecke built around the old theta functions of Jacobi. This remains a challenge for the future..."



F. Dyson

Mock modularity of $f(q)$

Theorem (Zwegers)

The function $f(q)$ is a mock modular form.

S. Zwegers

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Wrong sgn-factors prevent modularity.

Rogers false theta functions

Wrong sgn-factors prevent modularity.

Example

$$\sum_{n\in\mathbb{Z}}(-1)^n \operatorname{sgn}\left(n+\frac{1}{2}\right) q^{\left(n+\frac{1}{2}\right)^2},$$

where $\operatorname{sgn}(x) := \frac{x}{|x|}$ for $x \in \mathbb{R}\setminus\{0\}$, $\operatorname{sgn}(0) := 0$.

Quantum modular forms

Definition

$f : \mathcal{Q} \to \mathbb{H}$ ($\mathcal{Q} \subset \mathbb{Q}$) is a quantum modular form of weight k if for $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$

$$f(\tau) - (c\tau + d)^{-k} f\left(\frac{a\tau + b}{c\tau + d}\right)$$

is "nice".

D. Zagier

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Examples: false theta functions

D. Zagier

Strange function

Let

$$\widetilde{\eta}(\tau) := -\frac{1}{2}\sum_{n\in\mathbb{Z}} \left(\frac{12}{n}\right) |n| q^{\frac{n^2-1}{24}}$$

M. Kontsevich

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"Strange identity"

$$\widetilde{\eta}(\tau) \text{``}=\text{''}\, K(q) := 1 + \sum_{n\geq 1} (q;q)_n.$$

M. Kontsevich

Background

Vertex algebras

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- Singlet vertex algebras are non-rational, characters are 1-dimensional false theta functions.

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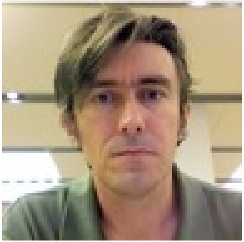
- Feigin and Frenkel investigated affine vertex algebras.
- Characters of rational vertex algebra are modular functions.
- Singlet vertex algebras are non-rational, characters are 1-dimensional false theta functions.
- Triplet algebras lead to higher-dimensional theta functions.

Vertex algebras

Class of functions ($0 \leq s < N \in \mathbb{N}$)

$$\frac{F_{N-s,N}(N\tau)}{\eta(\tau)}$$



A. Milas

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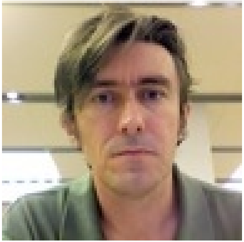
with

$$F_{j,N}(\tau) :\doteq \sum_{n \equiv j \pmod{2N}} \operatorname{sgn}(n) q^{\frac{n^2}{4N}}.$$



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False theta functions in combinatorics

Definition:

$\{a_j\}_{j=1}^{s}$ with

$$a_1 \leq a_2 \leq \ldots \leq a_k \geq a_k \geq \ldots \geq a_s$$

and $a_1 + \ldots + a_s = n$ is a unimodal sequence.

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Generating function:

$$U(q) := \sum_{n \geq 0} u(n) q^n = \frac{1}{(q;q)_\infty^2} \sum_{n \geq 1} (-1)^{n+1} q^{\frac{n(n+1)}{2}}.$$

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Two-variable false theta function

Define

$$\psi(z;\tau) := i\sum_{n\in\mathbb{Z}} \operatorname{sgn}\left(n+\frac{1}{2}\right)(-1)^n q^{\frac{1}{2}\left(n+\frac{1}{2}\right)^2}\zeta^{n+\frac{1}{2}}.$$

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Remark

Removing the sgn yields a Jacobi form. The sgn breaks the inversion property.

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Definition:

Holomorphic Jacobi forms of weight $k \in \mathbb{Z}$ and index $m \in \mathbb{N}_0$ are holomorphic functions

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$$\phi\left(\frac{z}{c\tau+d}; \frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k e^{\frac{2\pi i c m z^2}{c\tau+d}} \phi(z;\tau),$$

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plus growth conditions.

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"Complete" ψ to obtain a function transforming like Jacobi forms.

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Completion: $w \in \mathbb{H}$, $z_2 := \mathrm{Im}(z)$

$$\widehat{\psi}(z;\tau,w) := i\sum_{n\in\mathbb{Z}} \mathrm{erf}\left(-i\sqrt{\pi i(w-\tau)}\left(n+\frac{1}{2}+\frac{z_2}{\tau_2}\right)\right) \times (-1)^n q^{\frac{1}{2}\left(n+\frac{1}{2}\right)^2}\zeta^{n+\frac{1}{2}}.$$

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Note that, for $-\frac{1}{2} < \frac{z_2}{\tau_2} < \frac{1}{2}$ and $\varepsilon > 0$,

$$\lim_{t\to\infty} \widehat{\psi}(z;\tau,\tau+it+\varepsilon) = \psi(z;\tau).$$

Theorem 1 (B.-Nazaroglu)

The function $\widehat{\psi}$ transforms like a Jacobi form.

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Sketch of proof:

Poisson summation.

Higher-dimensional-analogues

Notation:

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- Multiplier

$$\chi_{\tau,w} := \sqrt{\frac{i(w-\tau)}{\tau w}} \frac{\sqrt{\tau}\sqrt{w}}{\sqrt{i(w-\tau)}}.$$

Definition:

Define for $\boldsymbol{\mu} \in L^*$

$$\begin{aligned}\Psi_{Q,\boldsymbol{\mu}}(\boldsymbol{z};\tau) &= \Psi_{Q,\boldsymbol{\mu},\boldsymbol{\ell},\boldsymbol{c}}(\boldsymbol{z};\tau) \\ &:= \sum_{\boldsymbol{n}\in\boldsymbol{\mu}+\frac{\boldsymbol{\ell}}{2}+L} \operatorname{sgn}\left(B(\boldsymbol{c},\boldsymbol{n})\right) q^{Q(\boldsymbol{n})} e^{2\pi i B\left(\boldsymbol{n},\boldsymbol{z}+\frac{\boldsymbol{\ell}}{2}\right)},\end{aligned}$$

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Completion

$$\widehat{\Psi}_{Q,\boldsymbol{\mu}}(\boldsymbol{z};\tau,w) = \widehat{\Psi}_{Q,\boldsymbol{\mu},\boldsymbol{\ell},\boldsymbol{c}}(\boldsymbol{z};\tau,w) := \sum_{\boldsymbol{n}\in\boldsymbol{\mu}+\frac{\boldsymbol{\ell}}{2}+L} \operatorname{erf}\left(-i\sqrt{\pi i(w-\tau)}B\left(\boldsymbol{c},\boldsymbol{n}+\frac{\operatorname{Im}(\boldsymbol{z})}{\tau_2}\right)\right) q^{Q(\boldsymbol{n})} e^{2\pi i B\left(\boldsymbol{n},\boldsymbol{z}+\frac{\boldsymbol{\ell}}{2}\right)}.$$

Higher dimensional-analogues

Theorem 2 (B.-Nazaroglu)

1. For $\boldsymbol{m}, \boldsymbol{r} \in L$

$$\widehat{\Psi}_{Q,\boldsymbol{\mu}}(\boldsymbol{z} + \boldsymbol{m}\tau + \boldsymbol{r}; \tau, w) \\ = (-1)^{2Q(\boldsymbol{m}+\boldsymbol{r})} q^{-Q(\boldsymbol{m})} e^{-2\pi i B(\boldsymbol{m}, \boldsymbol{z})} \widehat{\Psi}_{Q,\boldsymbol{\mu}}(\boldsymbol{z}; \tau, w).$$

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2. We have

$$\widehat{\Psi}_{Q,\boldsymbol{\mu}}(\boldsymbol{z}; \tau + 1, w + 1) = e^{2\pi i Q\left(\boldsymbol{\mu} + \frac{\boldsymbol{\ell}}{2}\right)} \widehat{\Psi}_{Q,\boldsymbol{\mu}}(\boldsymbol{z}; \tau, w),$$

$$\widehat{\Psi}_{Q,\boldsymbol{\mu}}\left(\frac{\boldsymbol{z}}{\tau}; -\frac{1}{\tau}, -\frac{1}{w}\right) = \chi_{\tau,w} \frac{(-i\tau)^{\frac{N}{2}}}{\sqrt{|L^*/L|}} e^{2\pi i \frac{Q(\boldsymbol{z})}{\tau} - \pi i Q(\boldsymbol{\ell})} \times \sum_{\boldsymbol{\nu} \in L^*/L} e^{-2\pi i B(\boldsymbol{\mu}, \boldsymbol{\nu})} \widehat{\Psi}_{Q,\boldsymbol{\nu}}(\boldsymbol{z}; \tau, w).$$

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Applications

2 Applications:

- asymptotics for unimodal sequences

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- quantum modularity of $F_{j,N}$

Partitions

A partition of a nonnegative integer n is a nonincreasing sequence of positive integers whose sum is n.

L. Euler

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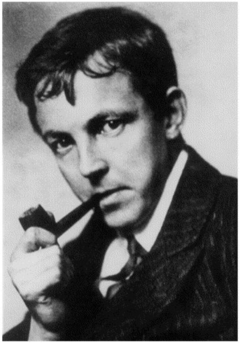
Generating function:

$$P(q) := \sum_{n\geq 0} p(n)q^n = \prod_{n\geq 1} \frac{1}{1-q^n}$$

L. Euler

Growth of $p(n)$

$$p(10) = 42$$

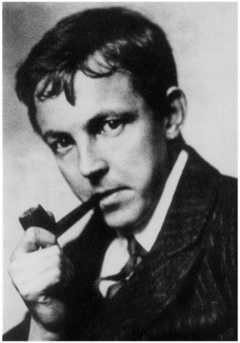


G. Hardy

Growth of $p(n)$

$$p(10) = 42$$

$$p(50) = 204226$$



G. Hardy

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$$p(10) = 42$$

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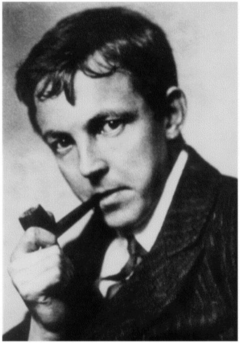
$$p(100) = 190569292$$



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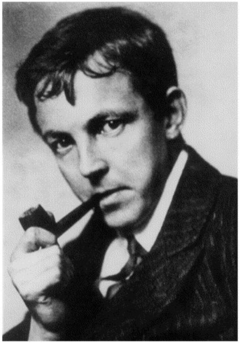

G. Hardy

Asymptotic behavior (Hardy–Ramanujan)

$$p(n) \sim \frac{1}{4\sqrt{3n}} e^{\pi\sqrt{\frac{2n}{3}}} \qquad (n \to \infty)$$

Exact formula

Kloosterman sum:

$$A_k(n) := \sum_{h \pmod{k}^*} \omega_{h,k} e^{-\frac{2\pi i h n}{k}}$$

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Bessel function of order α:

$$I_\alpha(x) := \sum_{m \geq 0} \frac{1}{m!\Gamma(m+\alpha+1)} \left(\frac{x}{2}\right)^{2m+\alpha}$$

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Rademacher formula:

$$p(n) = \frac{2\pi}{(24n-1)^{\frac{3}{4}}} \sum_{k \geq 1} \frac{A_k(n)}{k} I_{\frac{3}{2}}\left(\frac{\pi\sqrt{24n-1}}{6k}\right)$$

Unimodal sequences

Asymptotics: (Auluck, Wright)

$$u(n) \sim \frac{1}{8 \cdot 3^{\frac{3}{4}} n^{\frac{5}{4}}} e^{2\pi\sqrt{\frac{n}{3}}} \qquad (n \to \infty).$$

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Notation: $K_k(n)$, $K_k(n, r)$ Kloosterman sums

Unimodal sequences

Theorem 3 (B.-Nazaroglu)

We have

$$u(n) = \frac{2\pi}{12n-1}\sum_{k\geq 1}\frac{K_k(n)}{k} I_2\left(\frac{\pi}{3k}\sqrt{12n-1}\right) - \frac{\pi}{2^{\frac{3}{4}}\sqrt{3}(24n+1)^{\frac{3}{4}}}$$

$$\times \sum_{k\geq 1}\sum_{r \pmod{2k}} \frac{K_k(n,r)}{k^2}\int_{-1}^{1}\left(1-x^2\right)^{\frac{3}{4}} \cot\left(\frac{\pi}{2k}\left(\frac{x}{\sqrt{6}}-r-\frac{1}{2}\right)\right)$$

$$\times I_{\frac{3}{2}}\left(\frac{\pi}{3\sqrt{2}k}\sqrt{\left(1-x^2\right)(24n+1)}\right) dx.$$

Theorem 3 (B.-Nazaroglu)

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$$\times I_{\frac{3}{2}}\left(\frac{\pi}{3\sqrt{2}k}\sqrt{\left(1-x^2\right)\left(24n+1\right)}\right) dx.$$

Idea of Proof: Use Circle Method and modular completion.

Quantum modular forms

Recall

$$F_{j,N}(\tau) \doteq \sum_{n \equiv j \pmod{2N}} \operatorname{sgn}(n) q^{\frac{n^2}{4N}}.$$

Quantum modular forms

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Asymptotics

$$F_{j,N}\left(it + \frac{h}{k}\right) \sim \sum_{m \geq 0} a_{h,k}(m) t^m \quad \left(t \to 0^+\right),$$

$$F^*_{j,N}\left(it - \frac{h}{k}\right) \sim \sum_{m \geq 0} a_{h,k}(m) (-t)^m \quad \left(t \to 0^+\right),$$

Quantum modular forms

where

$$F^*_{j,N}(\tau) :\doteq \sum_{\substack{n\in\mathbb{Z} \\ n\equiv j \pmod{2N}}} \operatorname{sgn}(n)\Gamma\left(\frac{1}{2}; \frac{\pi n^2 \tau_2}{p}\right) q^{-\frac{n^2}{4N}}.$$

Quantum modular forms

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Note that

$$F_{j,N}^{*}(\tau) \doteq \int_{-\overline{\tau}}^{i\infty} \frac{f_{j,N}(w)}{(-i(w+\tau))^{\frac{1}{2}}} dw,$$

where

$$f_{j,N}(\tau) := \frac{1}{2N} \sum_{n\equiv j \pmod{2N}} nq^{\frac{n^2}{4N}}.$$

False transformation properties

Theorem 4 (B.-Nazaroglu)

For $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, we have

$$F_{j,N}(\tau) - \operatorname{sgn}(c\tau_1 + d)(c\tau + d)^{-\frac{1}{2}} \sum_{r=1}^{N-1} \psi_{j,r}\left(M^{-1}\right) F_{r,N}\left(\frac{a\tau+b}{c\tau+d}\right) = -i\sqrt{2N} \int_{-\frac{d}{c}}^{i\infty} \frac{f_{j,N}(\mathfrak{z})}{\sqrt{-i(\mathfrak{z}-\tau)}} d\mathfrak{z}.$$

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Corollary 5

The functions $F_{j,N}$ are vector-valued quantum modular forms with quantum set $\mathbb{Q}$.

Outline

1. Modular forms and mock modular forms
2. False theta functions
3. Modularity properties of false theta functions
4. Applications
5. Higher dimensional false theta functions

Higher-dimensional theta functions

- Arise from triplet algebras

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- A typical example:

$$F(q) := \sum_{\boldsymbol{\alpha}\in\mathscr{S}} \varepsilon(\boldsymbol{\alpha}) \sum_{\boldsymbol{n}\in\boldsymbol{\alpha}+\mathbb{N}_0^2} q^{Q(\boldsymbol{n})} + \frac{1}{2}\sum_{n\in\mathbb{Z}} \operatorname{sgn}\left(n+\frac{1}{N}\right) q^{\left(n+\frac{1}{N}\right)^2},$$

Higher-dimensional theta functions

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where $\boldsymbol{n} = (n_1, n_2)$, $\quad Q(\boldsymbol{n}) := 3n_1^2 + 3n_1n_2 + n_2^2$,

$$\mathscr{S} := \left\{\left(1-\tfrac{1}{N}, \tfrac{2}{N}\right), \left(\tfrac{1}{N}, 1-\tfrac{2}{N}\right), \left(1, \tfrac{1}{N}\right), \left(0, 1-\tfrac{1}{N}\right), \left(\tfrac{1}{N}, 1-\tfrac{1}{N}\right), \left(1-\tfrac{1}{N}, \tfrac{1}{N}\right)\right\},$$

Higher-dimensional theta functions

$$\varepsilon(\boldsymbol{\alpha}) := \begin{cases} -2 & \text{if } \boldsymbol{\alpha} \in \left\{ \left(1-\frac{1}{N}, \frac{2}{N}\right), \left(\frac{1}{N}, 1-\frac{2}{N}\right) \right\}, \\ 1 & \text{otherwise.} \end{cases}$$

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Note that

$$F(q) = \frac{1}{2} \sum_{\boldsymbol{\alpha} \in \mathcal{S}^*} \varepsilon(\boldsymbol{\alpha}) \sum_{\boldsymbol{n} \in \boldsymbol{\alpha} + \mathbb{Z}^2} \operatorname{sgn}(n_2) \left(\operatorname{sgn}(n_1) + \operatorname{sgn}(n_2)\right) q^{Q(\boldsymbol{n})},$$

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where

$$\mathcal{S}^* := \left\{ \left(\frac{1}{N}, 1-\frac{2}{N}\right), \left(0, 1-\frac{1}{N}\right), \left(\frac{1}{N}, 1-\frac{1}{N}\right) \right\}.$$

Jacobi forms

Classical fact: Fourier coefficients of holomorphic Jacobi forms are modular forms.

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Classical fact: Fourier coefficients of holomorphic Jacobi forms are modular forms.

Fourier coefficients: $(z_0 \in \mathbb{C})$

$$h_\ell(\tau) = h_{\ell,z_0}(\tau) := q^{-\frac{\ell^2}{4m}} \int_{z_0}^{z_0+1} \phi(z;\tau) e^{-2\pi i \ell z} dz.$$

Theorem 5 (B.-Rolen-Zwegers)

Let $m \in -\frac{1}{2}\mathbb{N}$, $\phi: \mathbb{C} \to \mathbb{C}$ meromorphic function satisfying for $\lambda, \mu \in \mathbb{Z}$

$$\phi(z + \lambda\tau + \mu) = (-1)^{2m\mu+\lambda\varepsilon} e^{-2\pi i m\left(\lambda^2\tau+2\lambda z\right)} \phi(z)$$

with $\varepsilon \in \{0, 1\}$.

L. Rolen

Fourier coefficients of Jacobi forms

We have for $\ell \in m + \mathbb{Z}$

$$h_{\ell,z_0}(\tau) = \sum_{w \in S_{z_0,\tau}} \sum_{n \in \mathbb{N}} \frac{D_{-n,w}(\tau)}{(n-1)!} \left[\left(\frac{1}{2\pi i} \frac{\partial}{\partial z} \right)^{n-1} \vartheta^+_{\ell,\varepsilon,-m}(z;\tau) \right]_{z=w},$$

with the partial theta function

$$\vartheta^+_{\ell,\varepsilon,M}(z;\tau) := \sum_{n \geq 0} (-1)^{n\varepsilon} q^{\frac{2Mn-\ell^2}{4M}} \zeta^{2Mn-\ell}.$$

Fourier coefficients of Jacobi forms

We have for $\ell \in m + \mathbb{Z}$

Laurent cofficients of ϕ

$$h_{\ell,z_0}(\tau) = \sum_{w \in S_{z_0,\tau}} \sum_{n \in \mathbb{N}} \frac{D_{-n,w}(\tau)}{(n-1)!} \left[\left(\frac{1}{2\pi i} \frac{\partial}{\partial z} \right)^{n-1} \vartheta^{+}_{\ell,\varepsilon,-m}(z;\tau) \right]_{z=w},$$

set of representative of poles of ϕ

with the partial theta function

$$\vartheta^{+}_{\ell,\varepsilon,M}(z;\tau) := \sum_{n \geq 0} (-1)^{n\varepsilon} q^{\frac{2Mn-\ell^2}{4M}} \zeta^{2Mn-\ell}.$$

Theorem 6 (B.-Kaszian-Milas-Zwegers)

We have

$$F(q) = \frac{\eta(\tau)^5}{\eta(2\tau)} \underset{\uparrow}{\mathrm{CT}}_{[\zeta_1,\,\zeta_2]} \frac{\vartheta(z_1;2\tau)\vartheta(z_2;2\tau)\vartheta(z_1+z_2;2\tau)}{\vartheta(z_1;\tau)\vartheta(z_2;\tau)\vartheta(z_1+z_2;\tau)},$$

constant term

J. Kaszian

Jacobi forms

J. Kaszian

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$$F(q) = \frac{\eta(\tau)^5}{\eta(2\tau)} \underset{\substack{\uparrow \\ \text{constant term}}}{\mathrm{CT}}_{[\zeta_1,\, \zeta_2]} \frac{\vartheta(z_1;2\tau)\vartheta(z_2;2\tau)\vartheta(z_1+z_2;2\tau)}{\vartheta(z_1;\tau)\vartheta(z_2;\tau)\vartheta(z_1+z_2;\tau)},$$

where $\zeta_j := e^{2\pi i z_j}$ satisfy $|q| < |\zeta_j| < 1$, $|q| < |\zeta_1\zeta_2| < 1$ and

$$\vartheta(z;\tau) := -iq^{\frac{1}{8}}\zeta^{-\frac{1}{2}} \prod_{n\geq 1} (1-q^n)\left(1-\zeta q^{n-1}\right)\left(1-\zeta^{-1}q^n\right).$$

Future research

- Find modular completions of the higher-dimensional false theta functions.
- Build a theory of completed false theta functions.
- Find different representations, e.g. as Poincaré series or theta lifts.
- Build a theory of Fourier coefficients of multi-variable Jacobi forms.