2-adic differential equations and isogenies between elliptic curves

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Transient Transcendence in Transylvania

May 14, 2019

Part 1

$$(x^2 - 4x)\, y' - (x - 2)\, y = g(x)$$

Homogeneous equation

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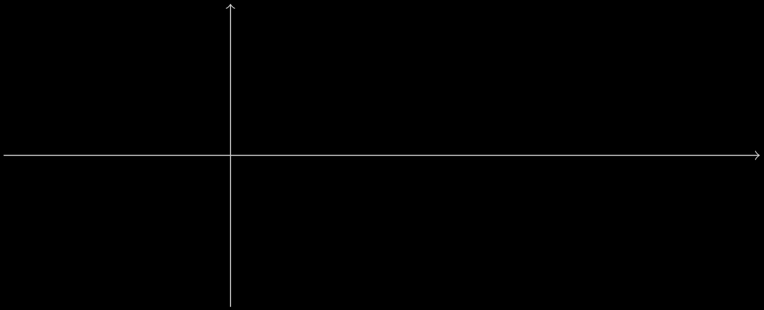

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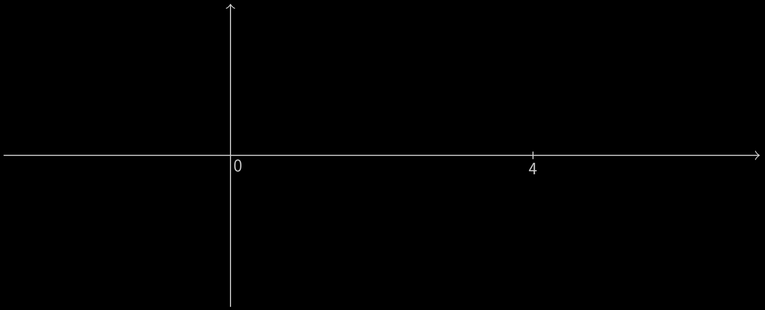

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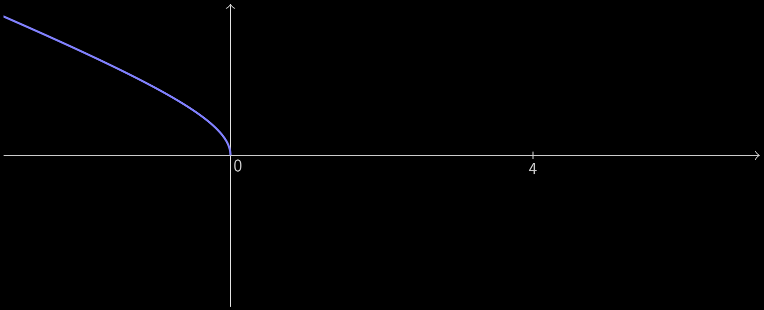

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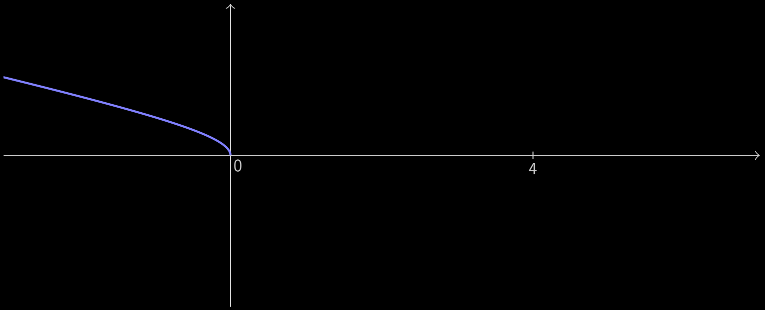

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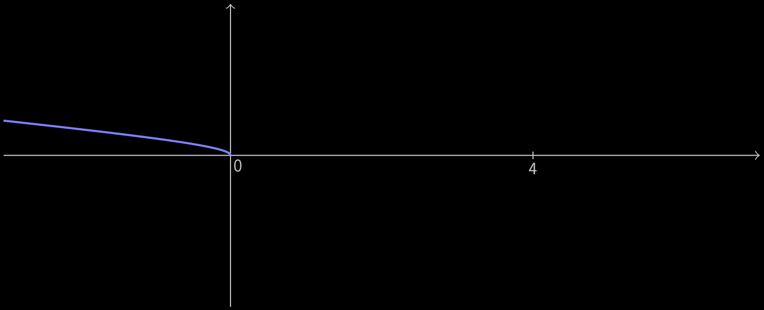

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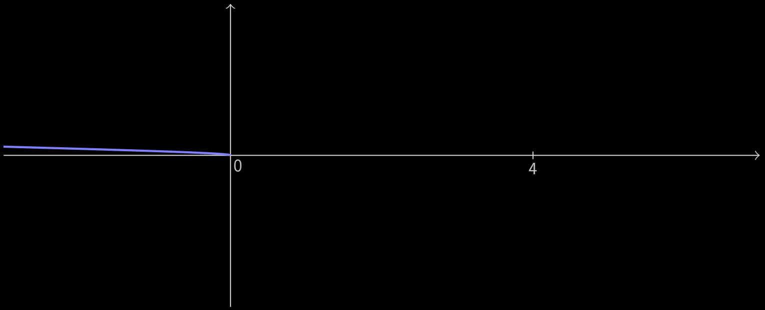

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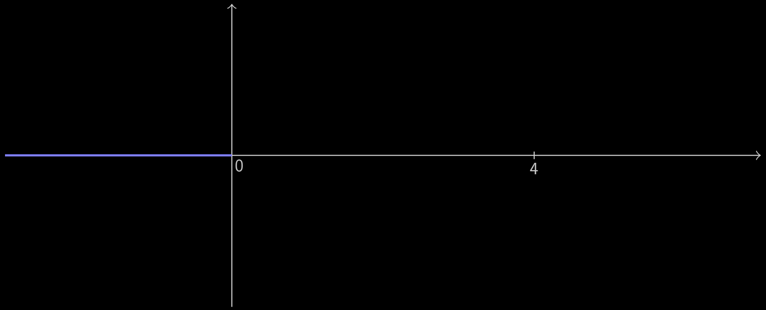

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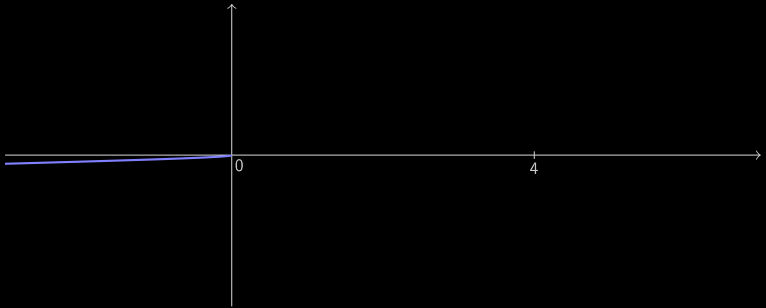

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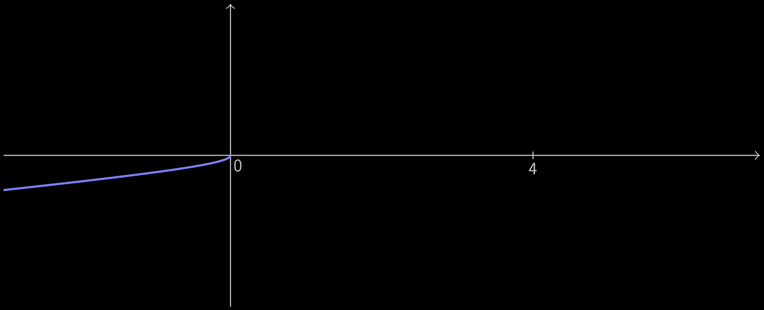

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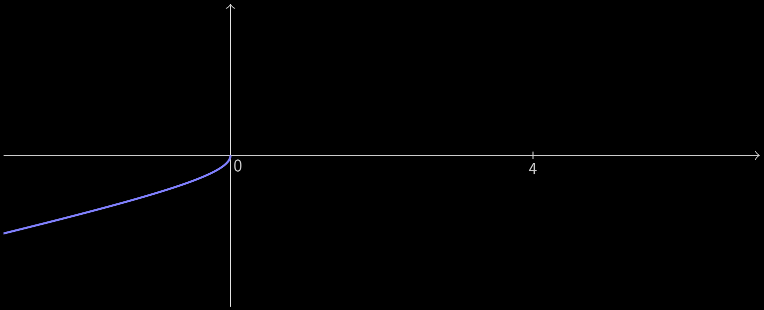

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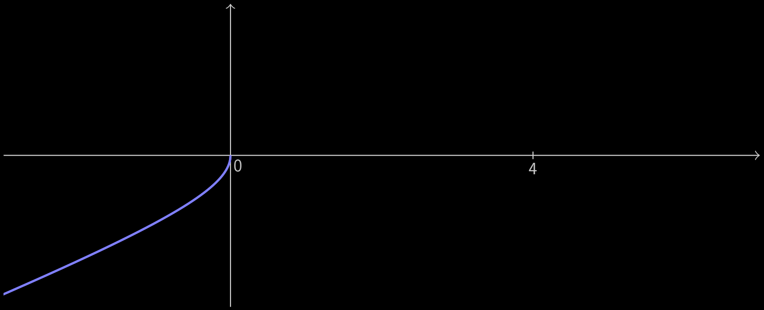

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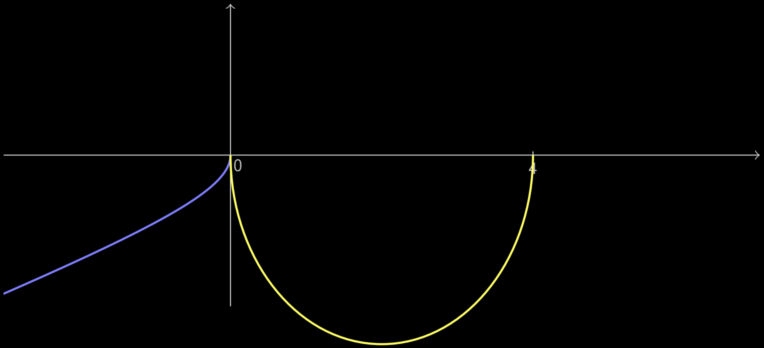

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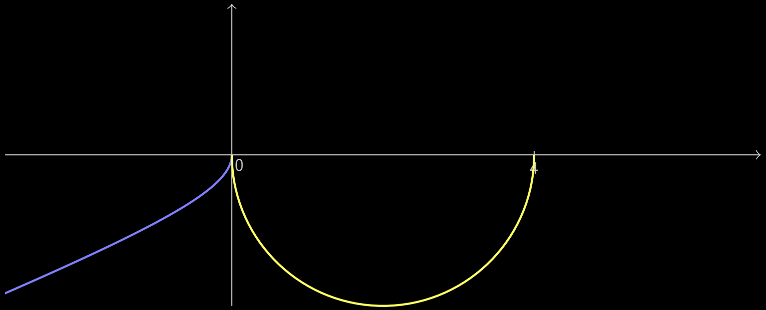

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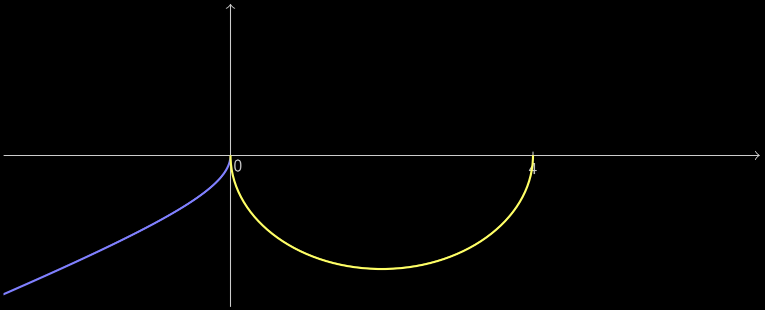

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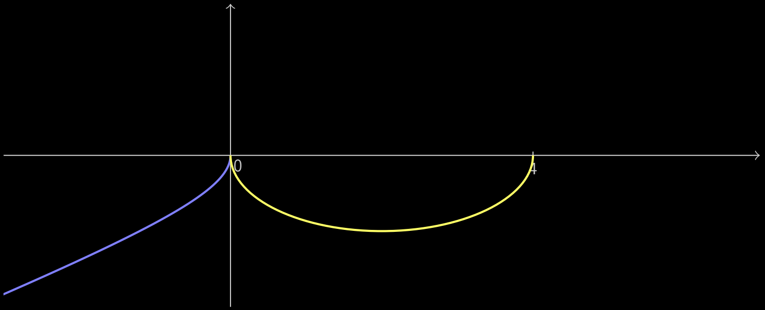

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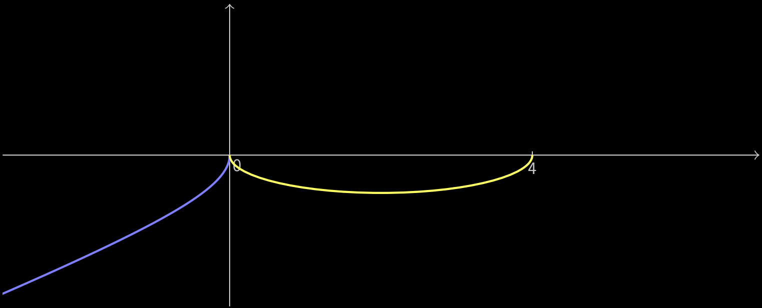

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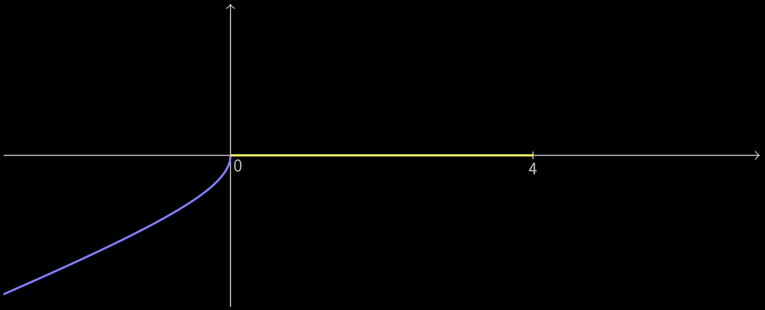

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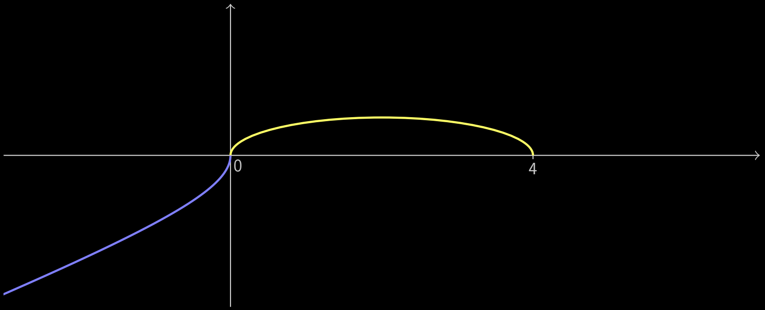

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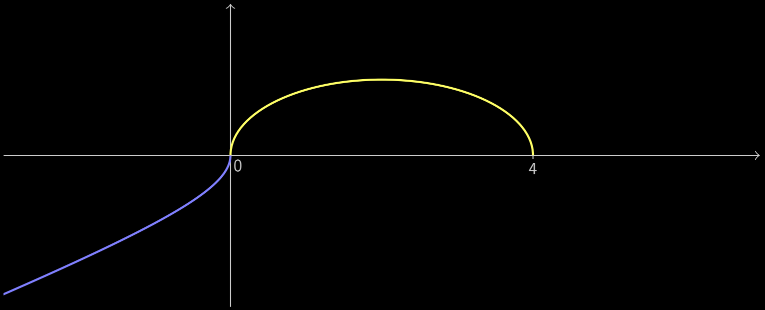

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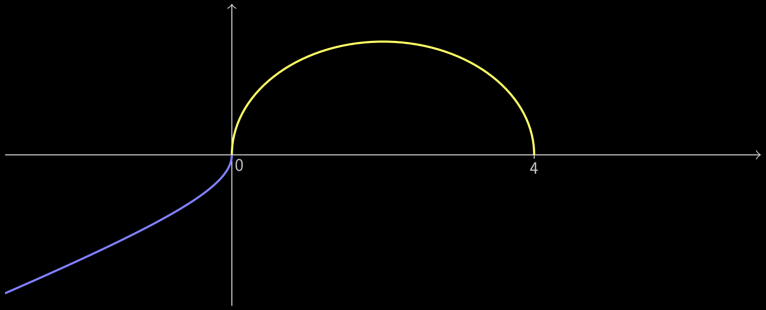

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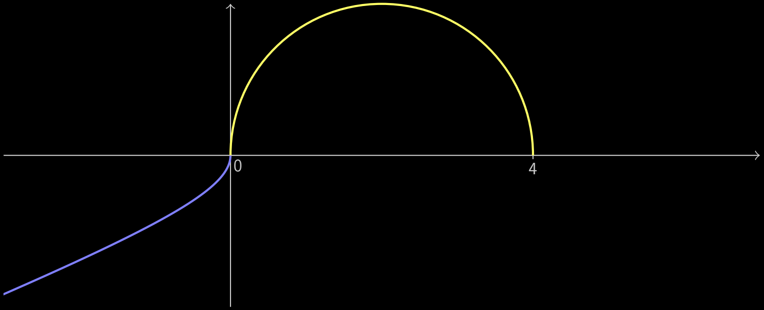

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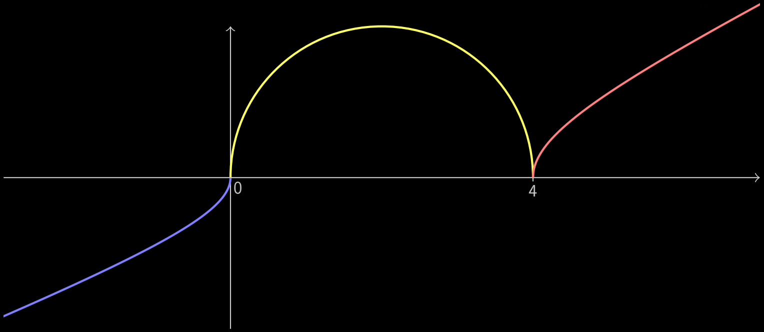

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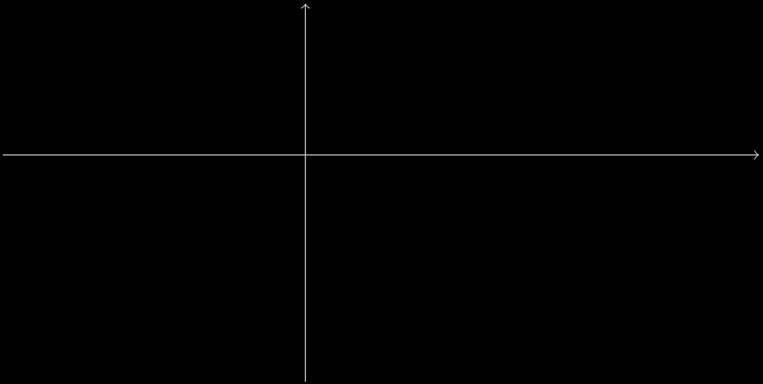

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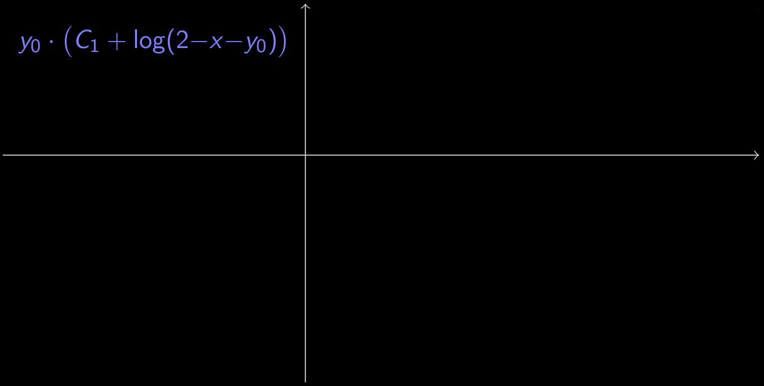

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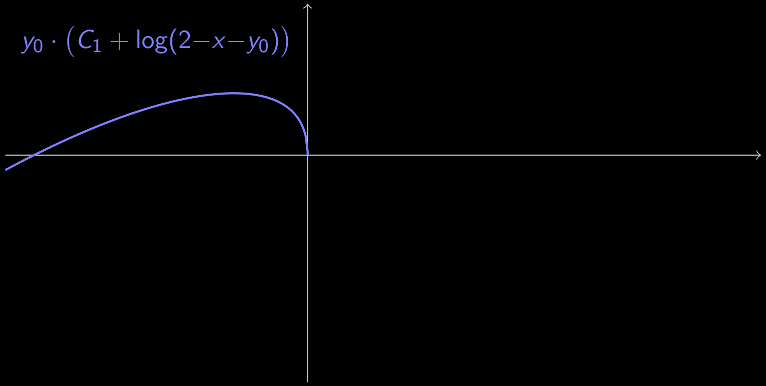

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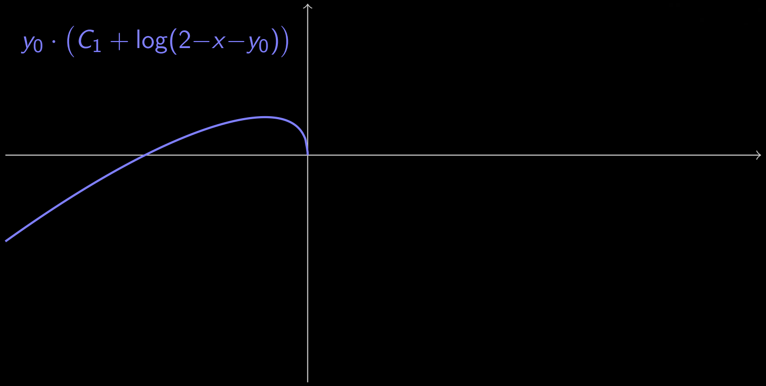

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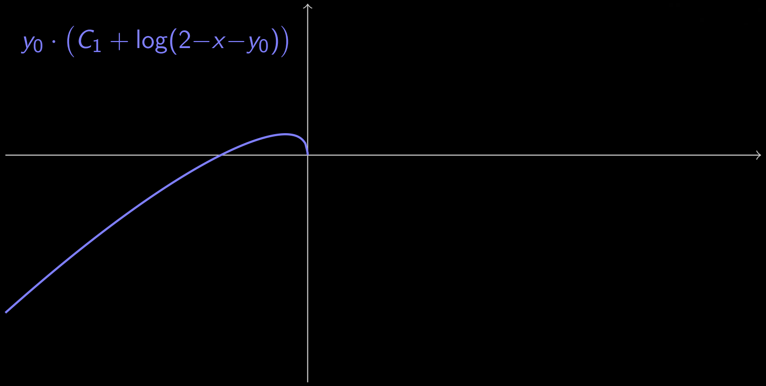

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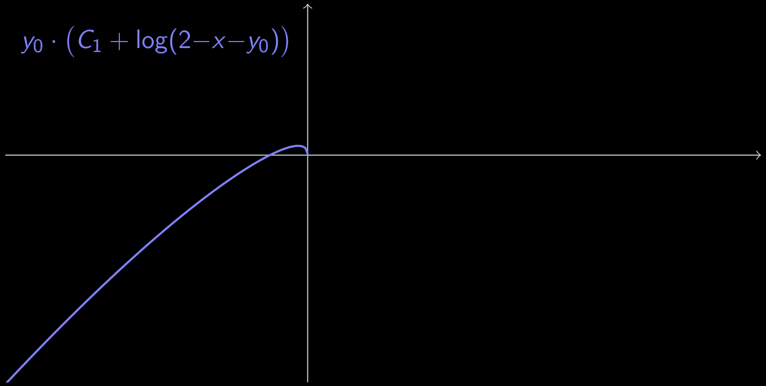

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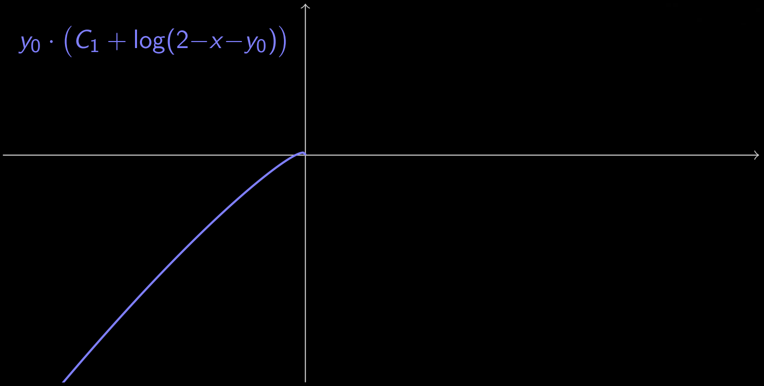

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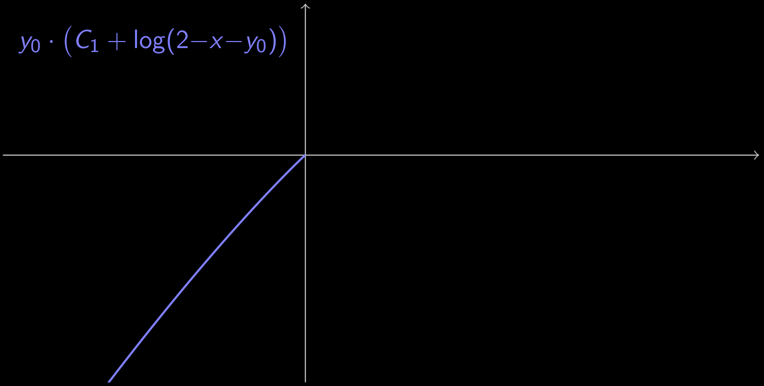

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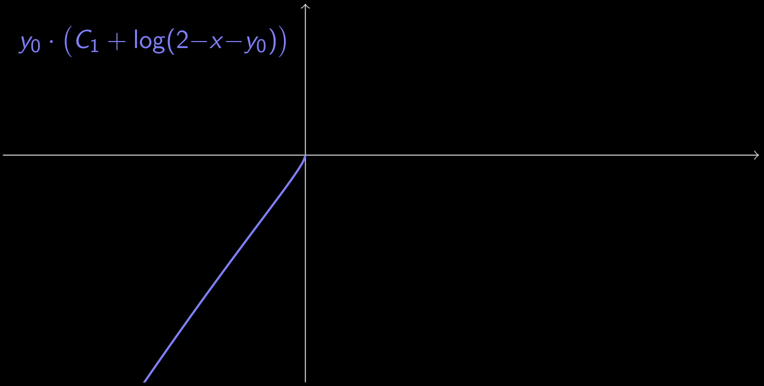

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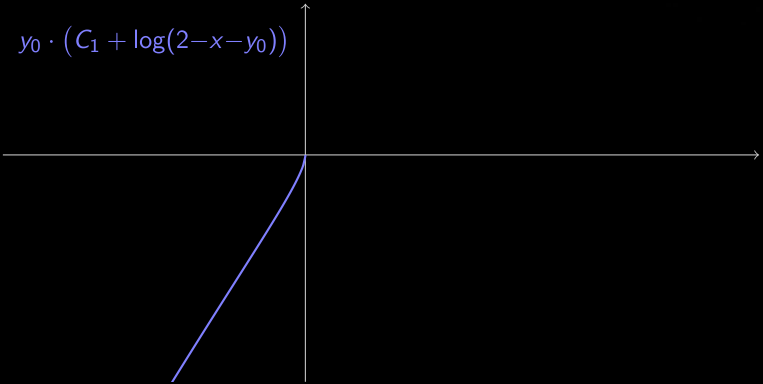

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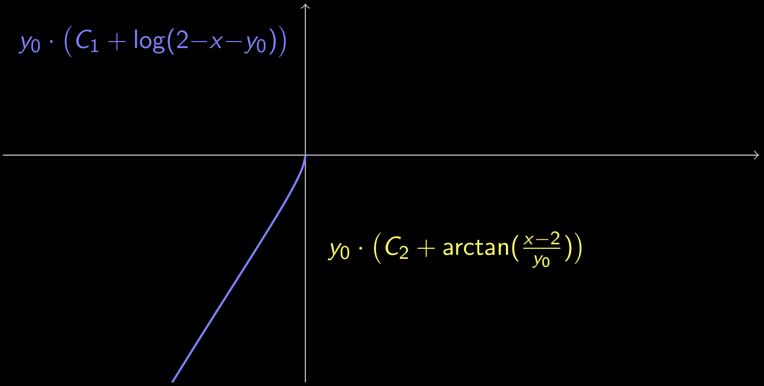

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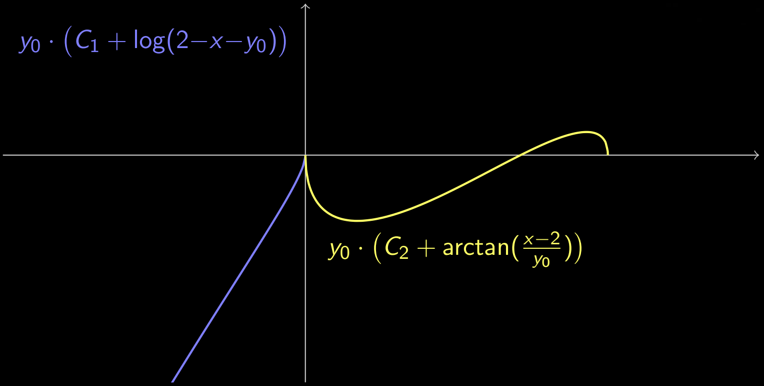

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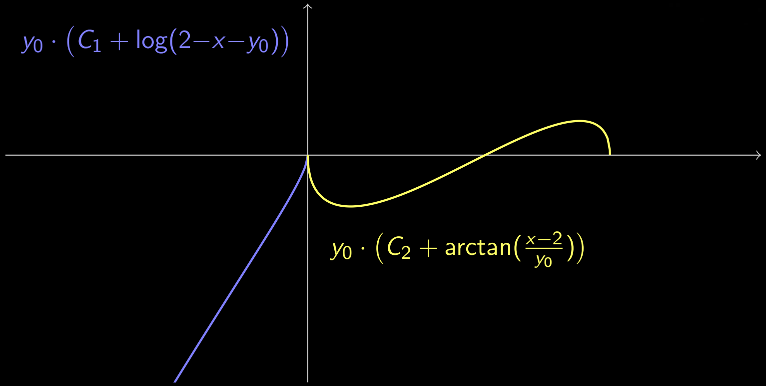

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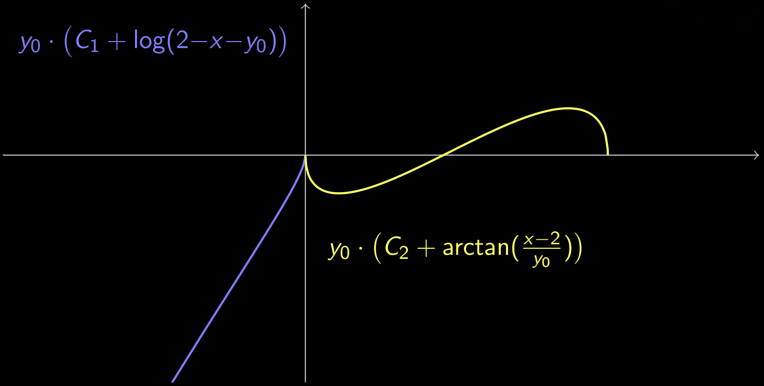

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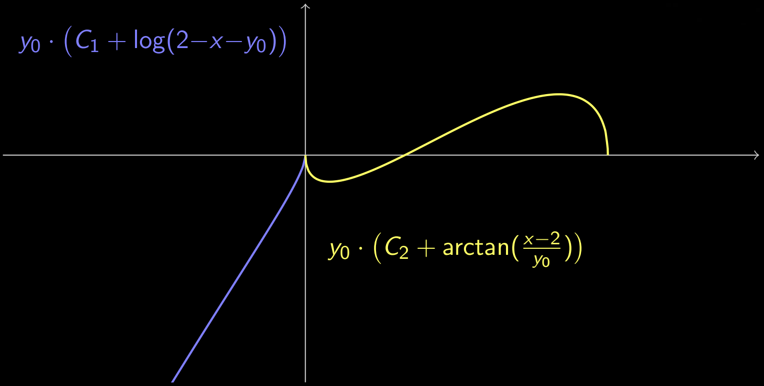

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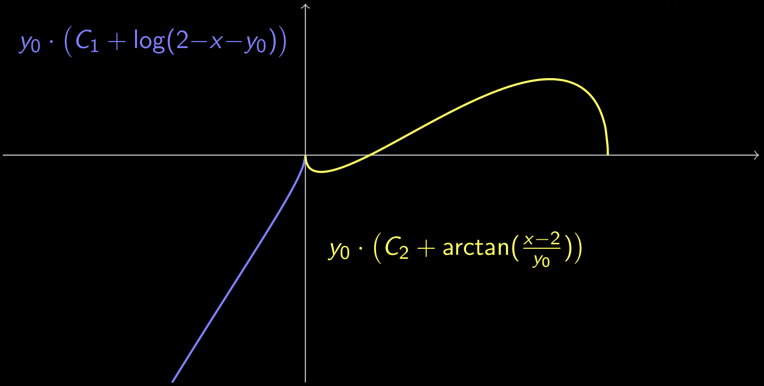

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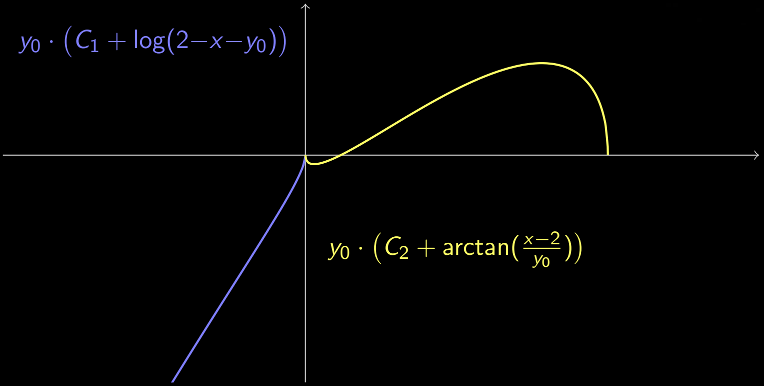

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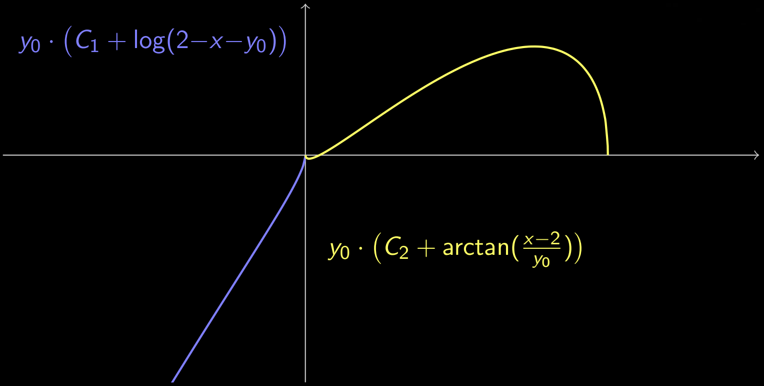

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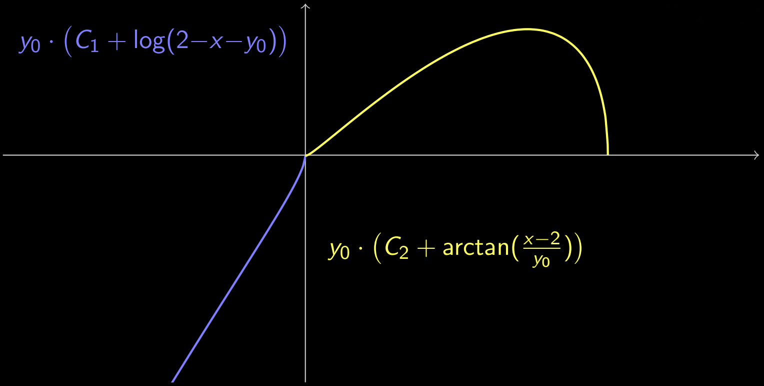

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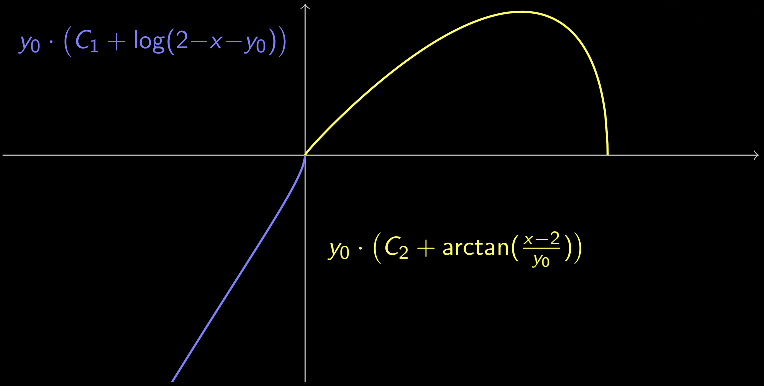

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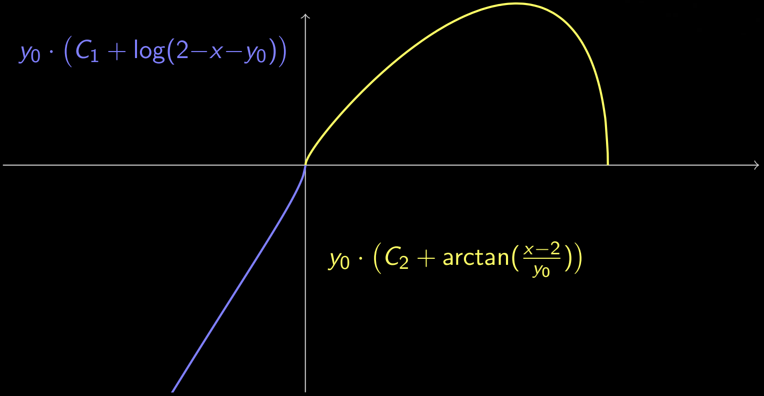

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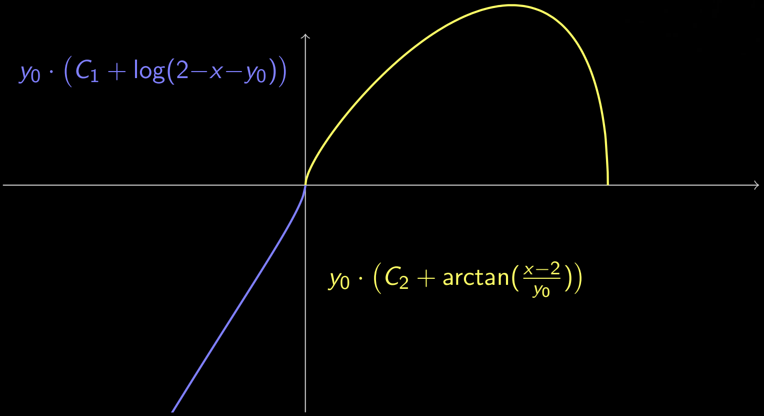

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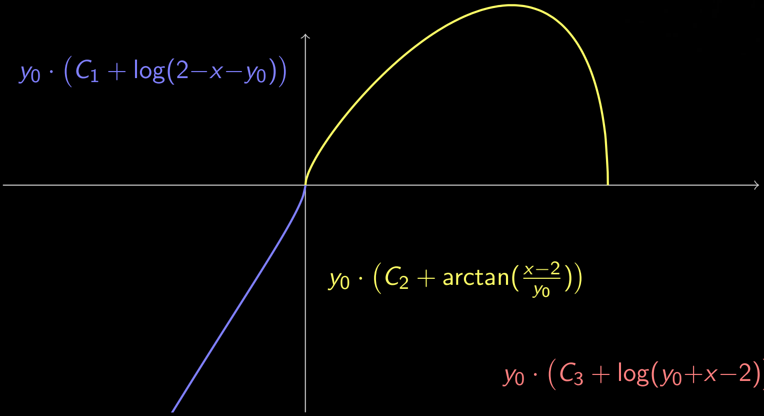

$$y_0 \cdot \big(C_1 + \log(2-x-y_0)\big)$$

$$y_0 \cdot \big(C_2 + \arctan(\tfrac{x-2}{y_0})\big)$$

$$y_0 \cdot \big(C_3 + \log(y_0+x-2)\big)$$

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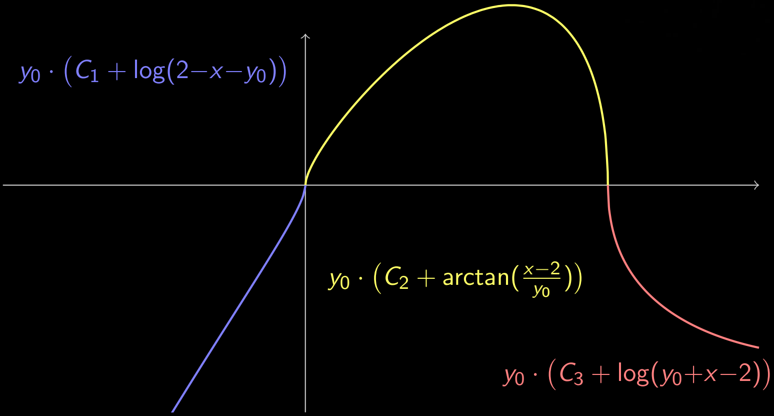

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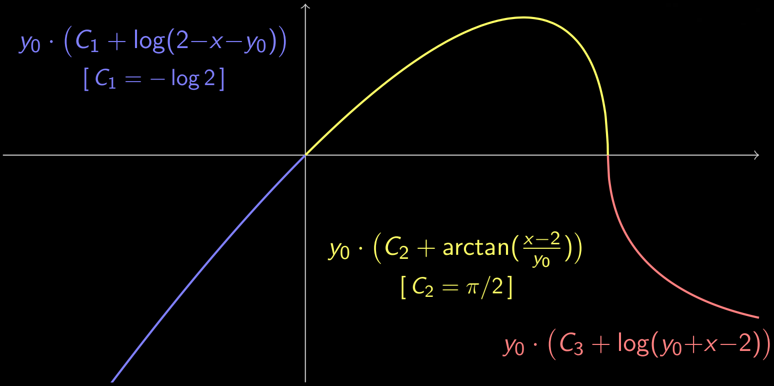

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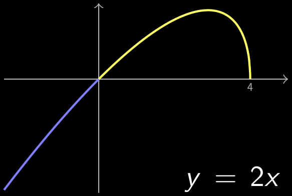
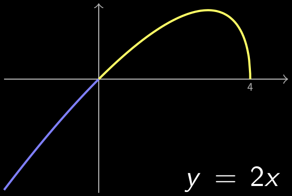


$$y = 2x - \frac{x^2}{6} - \frac{x^3}{60} - \frac{x^4}{420} - \frac{x^5}{2520} - \cdots$$

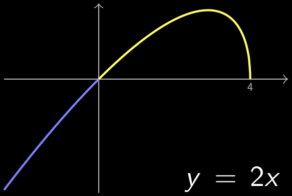
Analyticity



$$y = 2x - \frac{x^2}{6} - \frac{x^3}{60} - \frac{x^4}{420} - \frac{x^5}{2520} - \cdots$$

$$= 2x - \sum_{n=0}^{\infty} \frac{x^{n+2}}{2 \cdot \binom{2n}{n} \cdot (2n+1) \cdot (2n+3)}$$

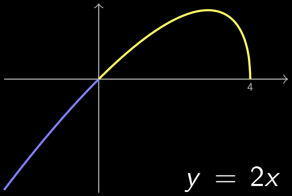
Analyticity



$$y = 2x - \frac{x^2}{6} - \frac{x^3}{60} - \frac{x^4}{420} - \frac{x^5}{2520} - \cdots$$

$$= 2x - \sum_{n=0}^{\infty} \frac{x^{n+2}}{2 \cdot \binom{2n}{n} \cdot (2n+1) \cdot (2n+3)}$$

Analyticity



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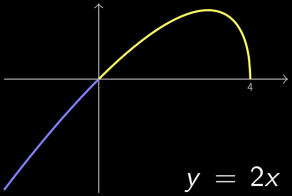
$$|\quad| \sim \frac{8}{\sqrt{\pi}} \cdot n^{3/2} \cdot 4^n$$

Analyticity



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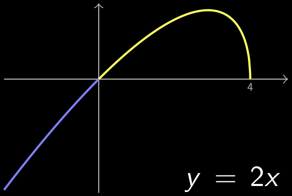
$$|\quad|_p$$

Analyticity



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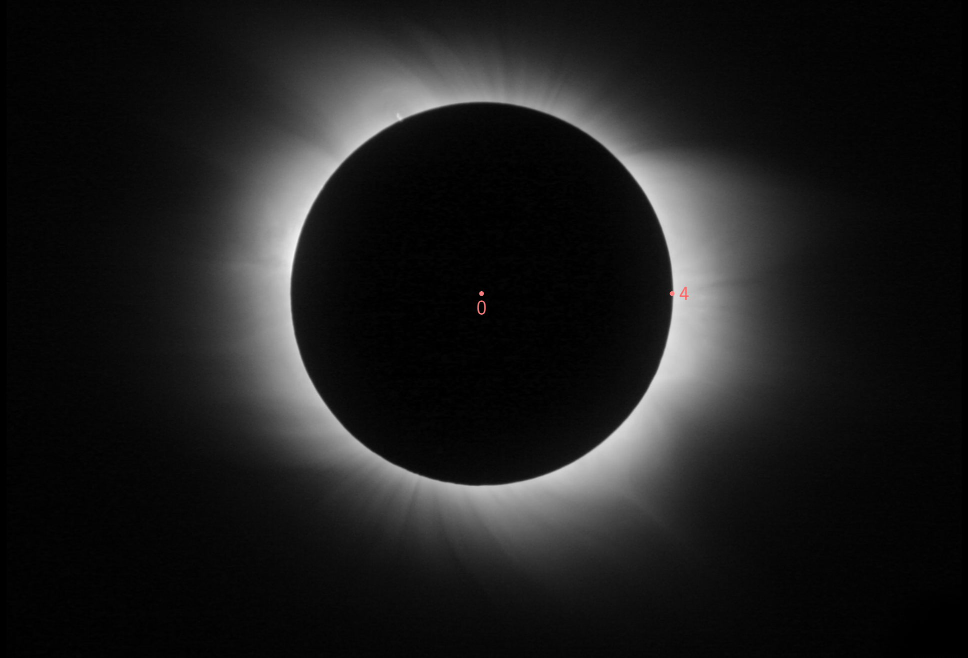
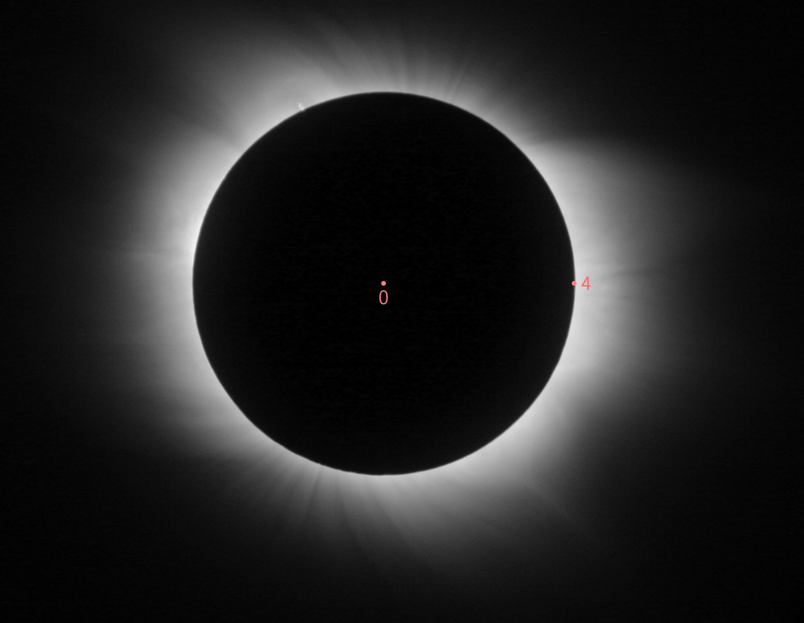
$$|\quad|_p = O(n)$$

Domain of analyticity

Domain of analyticity over ℂ



Domain of analyticity over $\mathbb{Q}_p$



Domain of analyticity over $\mathbb{Q}_2$

Domain of analyticity over $\mathbb{Q}_2$

Part 2

$$y^2 + xy = x^3 + ax + b$$

Elliptic curve

$y^2 + xy = x^3 + ax + b$

Elliptic curve

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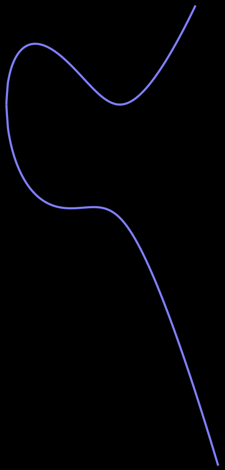

Elliptic curve

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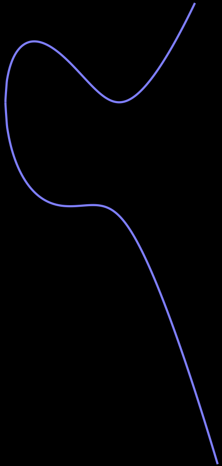
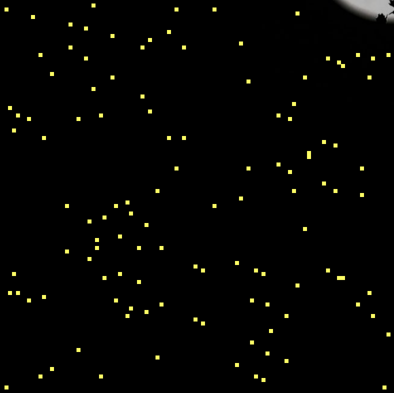
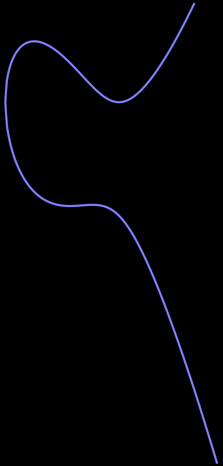


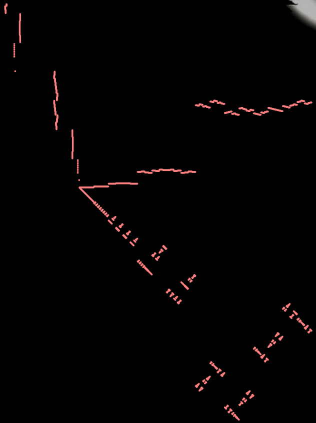


over $\mathbb{F}_{103}$

Elliptic curve

$y^2 + xy = x^3 + ax + b$

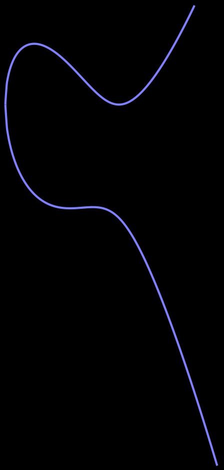




over $\mathbb{Q}_2$

Elliptic curve

$y^2 + xy = x^3 + ax + b$



Elliptic curve

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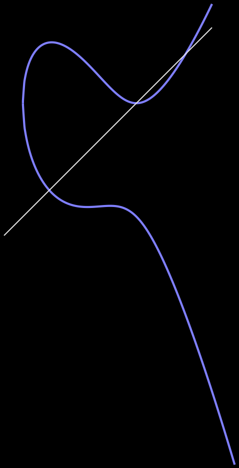

Elliptic curve

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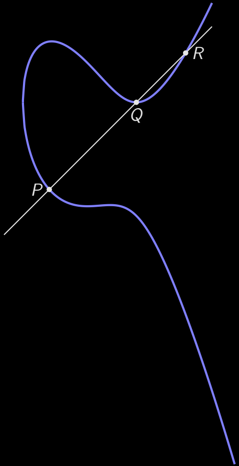

Elliptic curve

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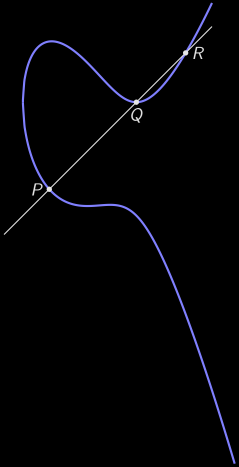
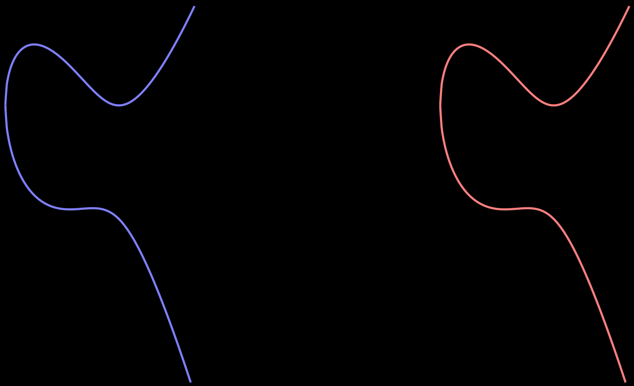


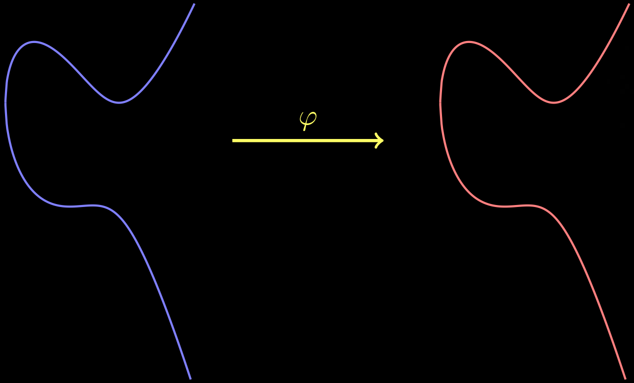
$$P + Q + R = 0$$

Isogenies

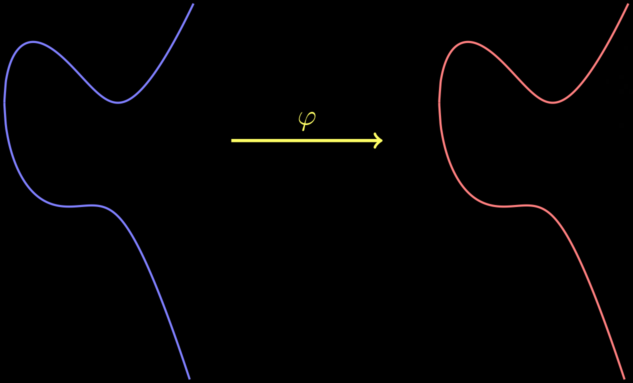
Isogenies



Isogenies

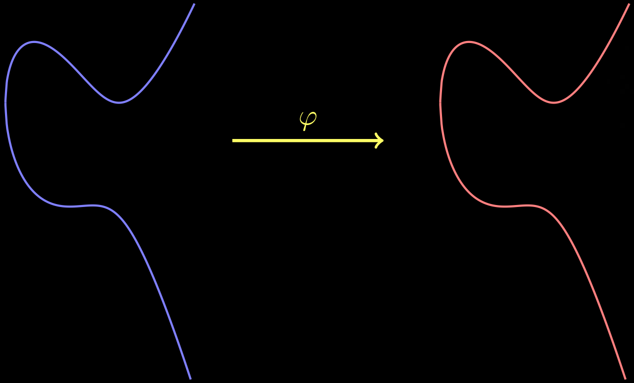


Isogenies



$$(x,\ y) \longmapsto (f(x,y),\ g(x,y))$$

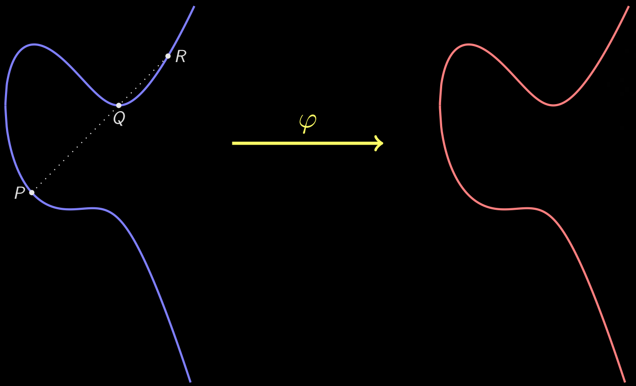
Isogenies



$$(x,\ y) \longmapsto (f(x,y),\ g(x,y))$$

rational functions

Isogenies



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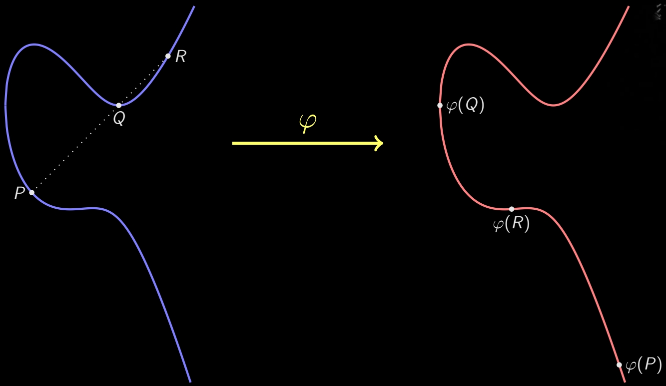
rational functions

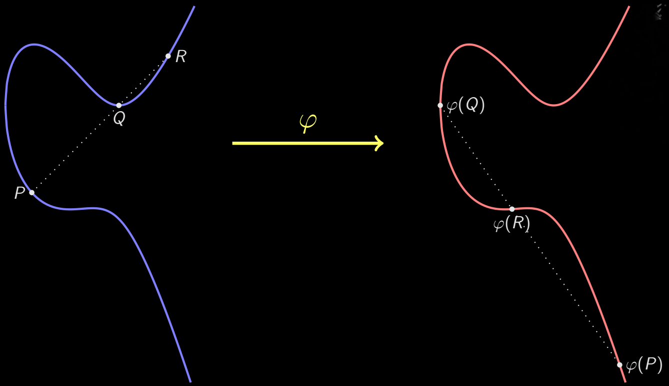
Isogenies



$$(x,\ y) \longmapsto \big(f(x,y),\ g(x,y)\big)$$

rational functions

Isogenies



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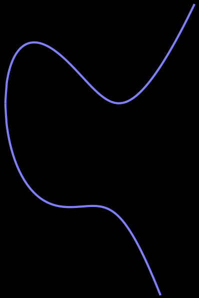
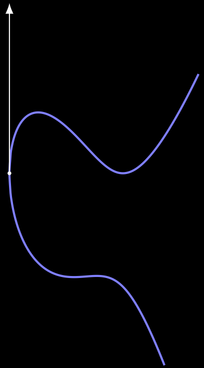
rational functions

Invariant vector fields

Invariant vector fields



Invariant vector fields



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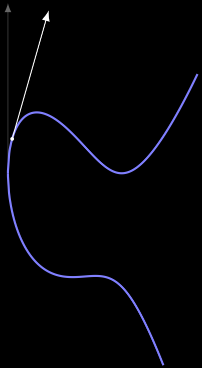

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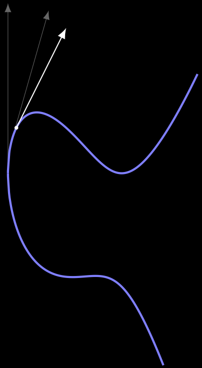

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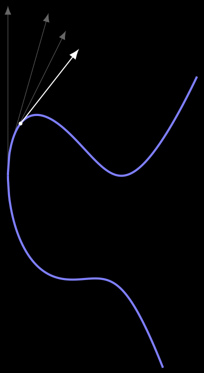

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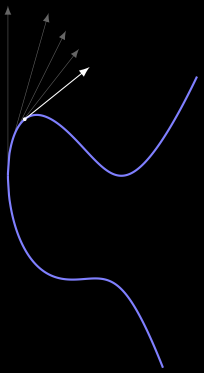

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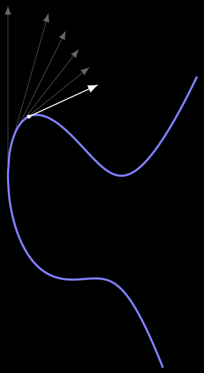

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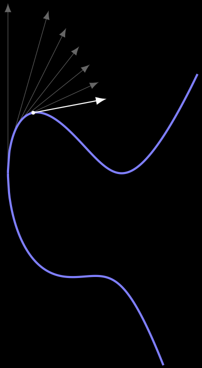

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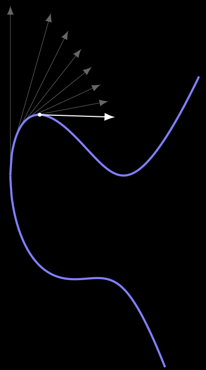

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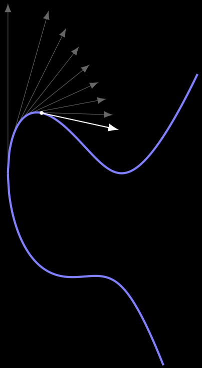

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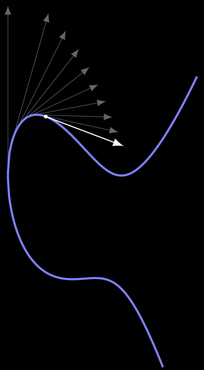

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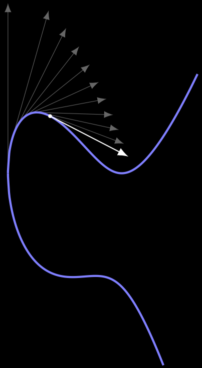

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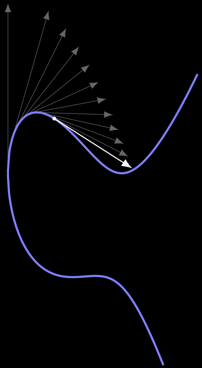

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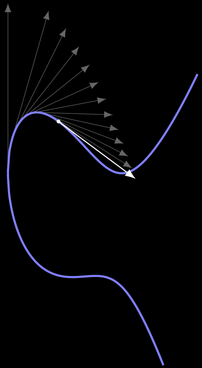

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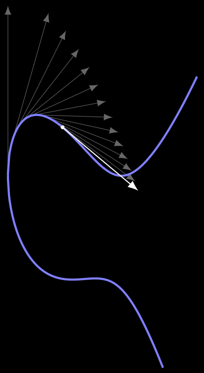

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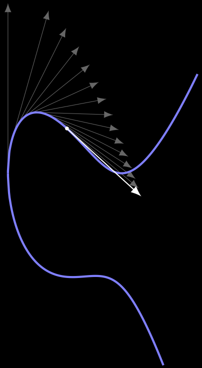

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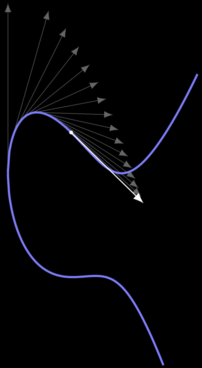

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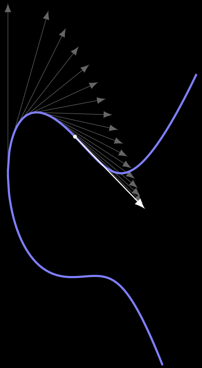

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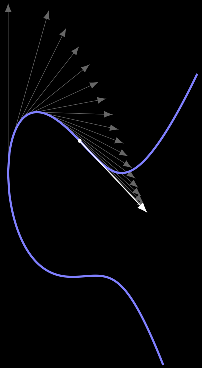

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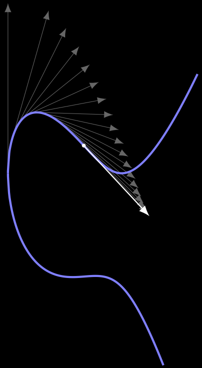

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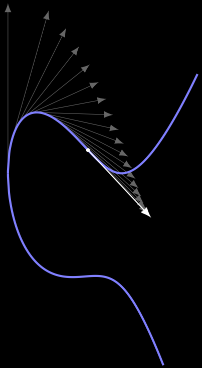

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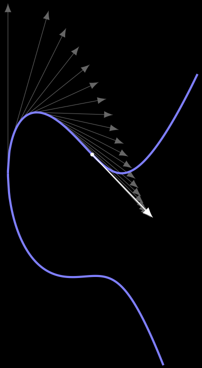

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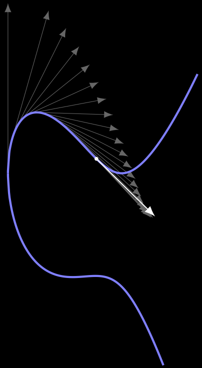

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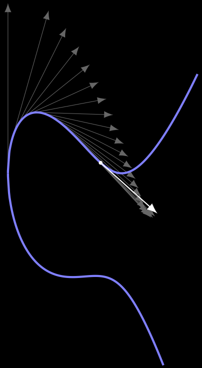

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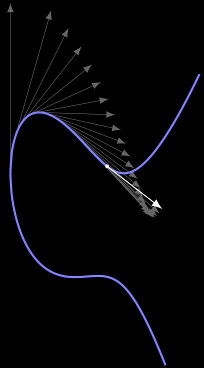

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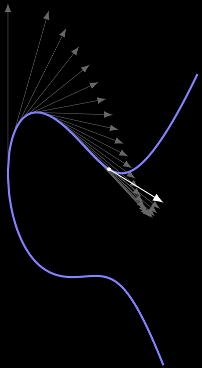

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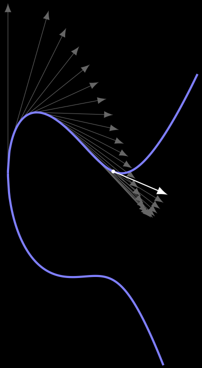

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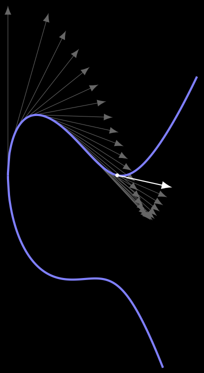

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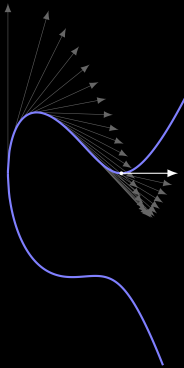

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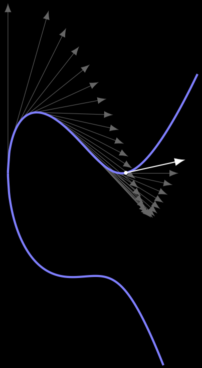

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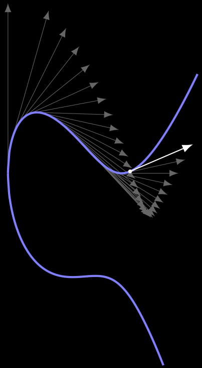

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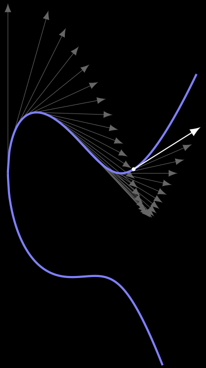

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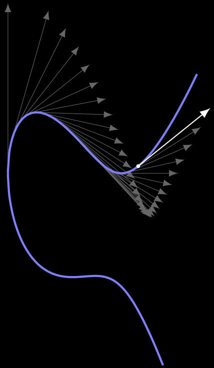

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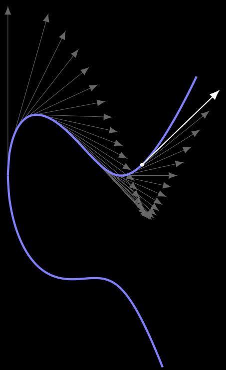
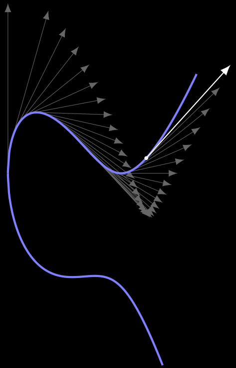


Invariant vector fields



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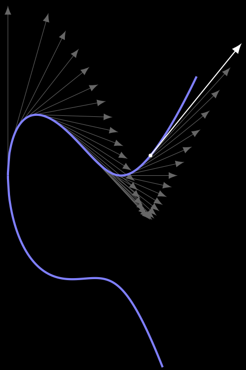

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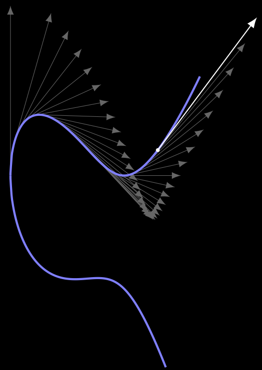

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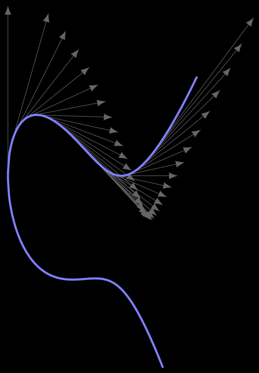
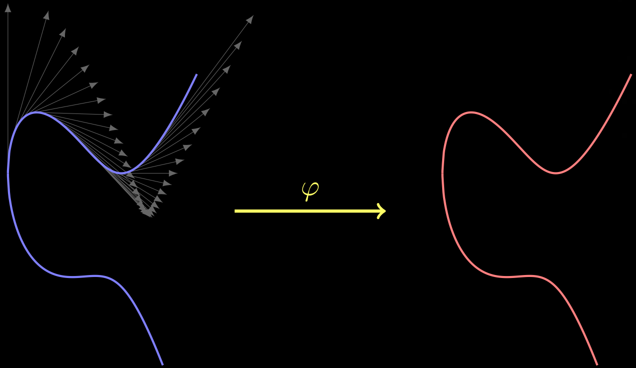
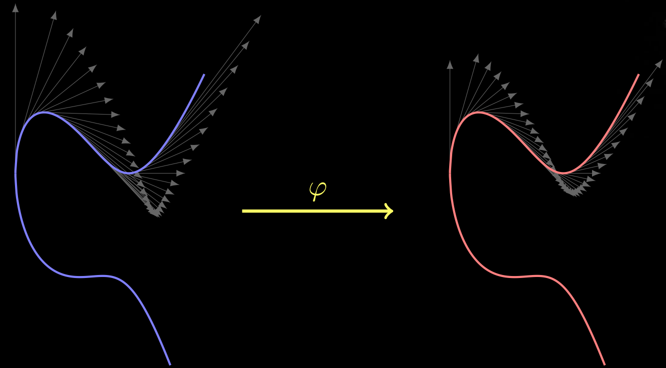


Invariant vector fields



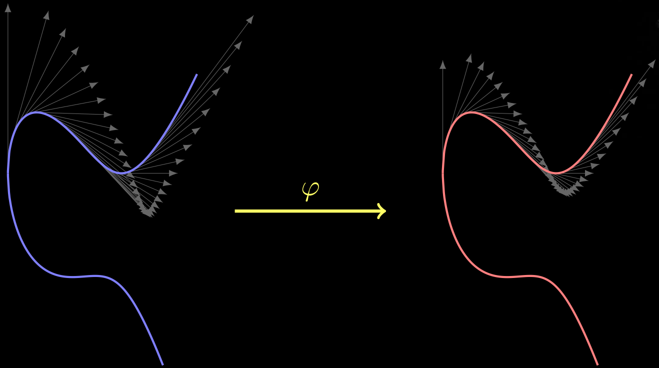
$$(x,\ y) \longmapsto (f(x,y),\ g(x,y))$$

Invariant vector fields



$$(x,\ y) \longmapsto (f(x,y),\ g(x,y))$$

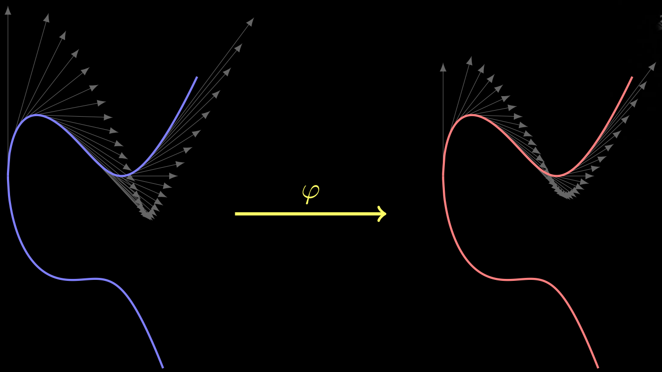
Invariant vector fields



$$(x,\ y) \longmapsto \text{~~}(f(x,y),\, g(x,y))\text{~~}$$

$$\left(f(x),\ c\left(\tfrac{x}{2}+y\right) f'(x) - \tfrac{1}{2} f(x)\right)$$

Invariant vector fields



$$(x,\, y) \longmapsto \cancel{(f(x,y),\, g(x,y))}$$

$$\left(f(x),\, c\left(\tfrac{x}{2}+y\right) f'(x) - \tfrac{1}{2} f(x)\right)$$

$$c^2\left(x^3 + \frac{x^2}{4} + ax + b\right)\cdot f'^2 = f^3 + \frac{f^2}{4} + a'f + b'$$

Linearization

Linearization

$$U T'^2 = V \circ T$$

$$U(x) = 4x + x^2 + 4ax^3 + 4bx^4$$

$$V(x) = 4x + x^2 + 4a'x^3 + 4b'x^4$$

Linearization

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Linearization

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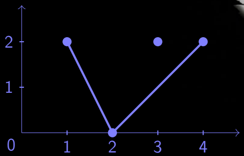

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$$UT'^2 = V \circ T$$

$$U(x) = 4x + x^2 + 4ax^3 + 4bx^4$$
$$= x \cdot (x - 4u) \cdot Q(x)$$
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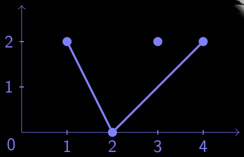

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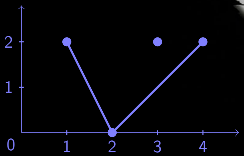

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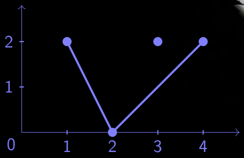

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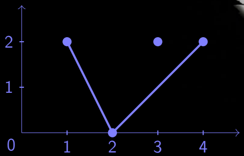

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$$(x^2 - 4x)\,y' - (x-2)\,y = g(x)$$

Algorithm DiffSolve

Algorithm `DiffSolve`

Input: N

Output: T at precision $O(x^N)$
$C = 1/T'$ at precision $O(x^{N/2})$

1. **if** $N \leq 2$ **then return** $x+O(x^2),\ 1+O(x)$
2. $n \leftarrow \text{floor}(N/2)$
3. $T, C \leftarrow$ `DiffSolve`(n)
4. **lift** T at precision $O(x^{N+1})$
   lift C at precision $O(x^{N-n+1})$
5. $T_2 \leftarrow T'^2$
   $G \leftarrow UT_2 - V \circ T$
   $C \leftarrow C \cdot (2 - CT_2)$
   $F \leftarrow G\, C\, A^{-3/2}$
6. $y \leftarrow$ `LinearDiffSolve`$\left((x^2-4x)y' - (x-2)y = 2\,F(ux)\right)$
7. $T \leftarrow T + y\, T'\, A^{1/2}$
8. **return** $T,\ C$

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   lift C at precision $O(x^{N-n+1})$
5. $T_2 \leftarrow T'^2$ // 1 mult. at prec. $O(x^N)$
   $G \leftarrow UT_2 - V \circ T$ // 3 mult. at prec. $O(x^N)$
   $C \leftarrow C \cdot (2 - CT_2)$ // 2 mult. at prec. $O(x^{N-n+1})$
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7. $T \leftarrow T + y\, T'\, A^{1/2}$ // 2 mult. at prec. $O(x^{N-n+1})$
8. return $T,\ C$

Algorithm LinearDiffSolve

Algorithm `LinearDiffSolve`

Input: A linear differential equation of the form

$$(x^2 - 4x)y' - (x-2)y = g(x) + O(x^N)$$

Output: y solution with $y(0) = 0$

1. $y_0 \leftarrow 0$
2. for i in $\{1, 2, \ldots, N-1\}$: $y_i \leftarrow \dfrac{(i-2)y_{i-1} - g_i}{4i - 2}$
3. return $\displaystyle\sum_{i=1}^{N-1} y_i x^i + O(x^N)$

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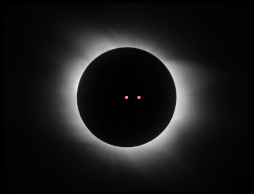

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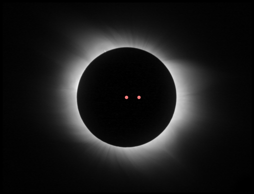

$\Longrightarrow$ 2-adic stability of LinearDiffSolve

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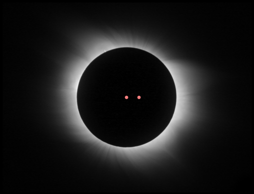

$\Longrightarrow$ 2-adic stability of `LinearDiffSolve`

$\Longrightarrow$ 2-adic stability of `DiffSolve`

Timings

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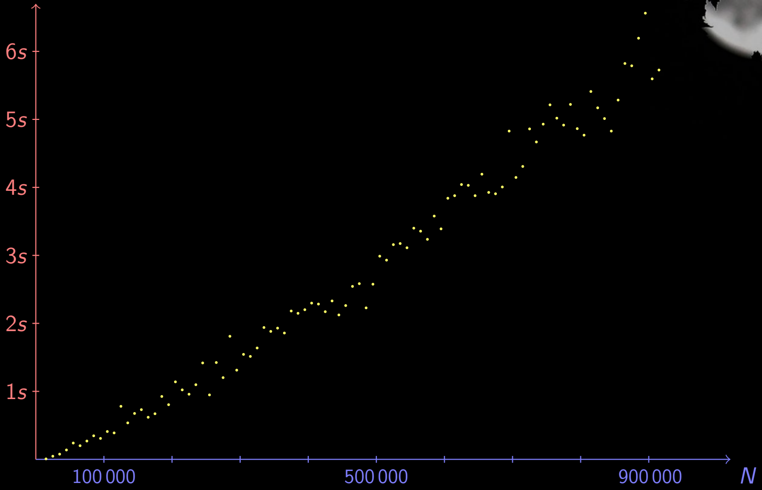

Thanks

for your attention