Explicit Generating Series for Small-Step Walks in the Quarter Plane

Frédéric Chyzak

Transient Transcendence in Transylvania
Brașov, Romania, May 13–17, 2019
Joint work with A. Bostan, M. van Hoeij, M. Kauers, and L. Pech (2017)
Applications in many areas of science

- discrete mathematics (permutations, trees, words, urns, ...)
- statistical physics (Ising model, ...)
- probability theory (branching processes, games of chance, ...)
- operations research (queueing theory, ...)

A history and a survey of lattice path enumeration

Katherine Humphreys

Department of Mathematical Sciences, Florida Atlantic University, Boca Raton, FL 33431, USA

**Abstract**

In celebration of the Sixth International Conference on Lattice Path Counting and Applications, it is befitting to review the history of lattice path enumeration and to survey how the topic has progressed thus far.

We start the history with early games of chance specifically the ruin problem which later appears as the ballot problem. We discuss André's Reflection Principle and its misnomer, its relation with the method of images and possible origins from physics and Brownian motion, and the earliest evidence of lattice path techniques and solutions.

In the survey, we give representative articles on lattice path enumeration found in the literature in the last 35 years by the lattice, step set, boundary, characteristics counted, and solution method. Some of this work appears in the author’s 2005 dissertation.

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Lattice Walks, Why?

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- discrete mathematics (permutations, trees, words, urns, …)
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This talk:

Computer Algebra applied to Combinatorics
Nearest-neighbor walks in the quarter plane = walks in $\mathbb{N}^2$ starting at $(0,0)$ and using steps in a fixed subset $\mathcal{S}$ of

$$\{\searrow, \leftarrow, \nwarrow, \uparrow, \nearrow, \rightarrow, \swarrow, \downarrow\}.$$

Example with $n = 45$, $i = 14$, $j = 2$ for:

$\mathcal{S} = \begin{array}{c}
\searrow \\
\leftarrow \\
\nwarrow \\
\uparrow \\
\nearrow \\
\rightarrow \\
\swarrow \\
\downarrow
\end{array}$

$$\begin{array}{c}
\searrow \\
\leftarrow \\
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\nearrow \\
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Example with $n = 45$, $i = 14$, $j = 2$ for:

Counting sequence: $f_{n;i,j} =$ number of walks of length $n$ ending at $(i,j)$.
Nearest-neighbor walks in the quarter plane = walks in $\mathbb{N}^2$ starting at $(0, 0)$ and using steps in a fixed subset $\mathcal{S}$ of

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Example with $n = 45$, $i = 14$, $j = 2$ for:

Counting sequence: $f_{n;i,j}$ = number of walks of length $n$ ending at $(i, j)$.

Specializations:

- $f_{n;0,0}$ = number of walks of length $n$ returning to origin (“excursions”);
- $f_n = \sum_{i,j \geq 0} f_{n,i,j}$ = number of walks with prescribed length $n$. 

Complete generating series: $F(x, y; t) = \sum_{n=0}^{\infty} \left( \sum_{i,j=0}^{\infty} f_{n,i,j} x^i y^j \right) t^n \in \mathbb{Q}[x, y][[t]]$. 
Complete generating series: \( F(x, y; t) = \sum_{n=0}^{\infty} \left( \sum_{i,j=0}^{\infty} f_{n,i,j} x^i y^j \right) t^n \in \mathbb{Q}[x, y][[t]]. \)

Specializations:
- Walks returning to the origin ("excursions"): \( F(0, 0; t); \)
- Walks with prescribed length: \( F(1, 1; t) = \sum_{n \geq 0} f_n t^n. \)
Generating Series and Combinatorial Problems

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Specializations:
- Walks returning to the origin ("excursions"): $F(0,0; t)$;
- Walks with prescribed length: $F(1,1; t) = \sum_{n \geq 0} f_n t^n$.

Combinatorial questions: Given $\mathcal{G}$, what can be said about $F(x, y; t)$, resp. $f_{n;i,j}$, and their variants?

- **Algebraic nature** of $F$: algebraic? transcendental?
- **Explicit form**: of $F$? of $f$?
- **Asymptotics** of $f$?
Generating Series and Combinatorial Problems

▷ Complete generating series: 
\[ F(x, y; t) = \sum_{n=0}^{\infty} \left( \sum_{i,j=0}^{\infty} f_{n;i,j} x^i y^j \right) t^n \in \mathbb{Q}[x, y][[t]]. \]

▷ Specializations:
- Walks returning to the origin ("excursions"): \[ F(0, 0; t); \]
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Our goal: Use computer algebra to give computational answers.
Small-Step Models of Interest

From the $2^8$ step sets $\mathcal{S} \subseteq \{-1, 0, 1\}^2 \setminus \{(0, 0)\}$, some are:

- trivial,
- too simple,
- intrinsic to the half plane,
- related by symmetries.
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One is left with 79 interesting distinct models.
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One is left with 79 interesting distinct models.

Is any further classification possible?
Classification of Univariate Power Series

- **D-finite power series**
- **algebraic**
- **hypergeom**

- **Algebraic**: \( S(t) \in \mathbb{Q}[[t]] \) root of a polynomial \( P \in \mathbb{Q}[t] \), i.e., \( P(t, S(t)) = 0 \).
- **D-finite**: \( S(t) \in \mathbb{Q}[[t]] \) satisfying a linear differential equation with polynomial coefficients.
- **Hypergeometric**: \( S(t) = \sum_{n=0}^{\infty} s_n t^n \) such that \( s_{n+1}s_n \in \mathbb{Q}(n) \). E.g., Gauss' \( 2F_1 \).

- Frédéric Chyzak
- Small-Step Walks
Classification of Univariate Power Series

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- **Hypergeometric:** $S(t) = \sum_{n=0}^{\infty} s_n t^n$ such that $\frac{s_{n+1}}{s_n} \in \mathbb{Q}(n)$. E.g., Gauss' hypergeometric function
  \[ _2 F_1 \left( \begin{array}{c} a \\ c \end{array} \middle| t \right) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{t^n}{n!}, \quad (a)_n = a(a+1) \cdots (a+n-1), \]
  \[ t(1-t)S''(t) + (c - (a+b+1)t)S'(t) - abS(t) = 0. \]
### Table of All Conjectured D-Finite $F(1,1;t)$ [Bostan & Kauers, 2009]

<table>
<thead>
<tr>
<th>OEIS</th>
<th>$\mathcal{G}$</th>
<th>alg ord</th>
<th>equiv</th>
<th>OEIS</th>
<th>$\mathcal{G}$</th>
<th>alg ord</th>
<th>equiv</th>
</tr>
</thead>
<tbody>
<tr>
<td>A005566</td>
<td>$\uparrow$</td>
<td>N</td>
<td>3</td>
<td>$\frac{4}{\pi} \frac{4^n}{n}$</td>
<td>A151275</td>
<td>$\uparrow$</td>
<td>N</td>
</tr>
<tr>
<td>A018224</td>
<td>$\times$</td>
<td>N</td>
<td>3</td>
<td>$\frac{2}{\pi} \frac{4^n}{n}$</td>
<td>A151314</td>
<td>$\times$</td>
<td>N</td>
</tr>
<tr>
<td>A151312</td>
<td>$\times$</td>
<td>N</td>
<td>3</td>
<td>$\frac{\sqrt{6}}{\pi} \frac{6^n}{n}$</td>
<td>A151255</td>
<td>$\times$</td>
<td>N</td>
</tr>
<tr>
<td>A151331</td>
<td>$\times$</td>
<td>N</td>
<td>3</td>
<td>$\frac{8}{3\pi} \frac{8^n}{n}$</td>
<td>A151287</td>
<td>$\times$</td>
<td>N</td>
</tr>
<tr>
<td>A151266</td>
<td>$\downarrow$</td>
<td>N</td>
<td>5</td>
<td>$\frac{1}{2} \sqrt{\frac{3}{\pi} \frac{3^n}{n^{1/2}}}$</td>
<td>A001006</td>
<td>$\downarrow$</td>
<td>Y</td>
</tr>
<tr>
<td>A151307</td>
<td>$\times$</td>
<td>N</td>
<td>5</td>
<td>$\frac{1}{2} \sqrt{\frac{5}{2\pi} \frac{5^n}{n^{1/2}}}$</td>
<td>A129400</td>
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<td>A151291</td>
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<td>5</td>
<td>$\frac{4}{3\sqrt{\pi} \frac{4^n}{n^{1/2}}}$</td>
<td>A0055558</td>
<td>$\downarrow$</td>
<td>N</td>
</tr>
<tr>
<td>A151326</td>
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<td>N</td>
<td>5</td>
<td>$\frac{2}{\sqrt{3\pi} \frac{6^n}{n^{1/2}}}$</td>
<td>A151265</td>
<td>$\downarrow$</td>
<td>Y</td>
</tr>
<tr>
<td>A151302</td>
<td>$\times$</td>
<td>N</td>
<td>5</td>
<td>$\frac{1}{3} \sqrt{\frac{5}{2\pi} \frac{5^n}{n^{1/2}}}$</td>
<td>A151278</td>
<td>$\times$</td>
<td>Y</td>
</tr>
<tr>
<td>A151329</td>
<td>$\times$</td>
<td>N</td>
<td>5</td>
<td>$\frac{1}{3} \sqrt{\frac{7}{3\pi} \frac{7^n}{n^{1/2}}}$</td>
<td>A151323</td>
<td>$\times$</td>
<td>Y</td>
</tr>
<tr>
<td>A151261</td>
<td>$\uparrow$</td>
<td>N</td>
<td>5</td>
<td>$\frac{12\sqrt{3}}{\pi} \frac{(2\sqrt{3})^n}{n^2}$</td>
<td>A060900</td>
<td>$\uparrow$</td>
<td>Y</td>
</tr>
<tr>
<td>A151297</td>
<td>$\times$</td>
<td>N</td>
<td>5</td>
<td>$\frac{\sqrt{3B^{7/2}}}{2\pi} \frac{(2B)^n}{n^2}$</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

$A = 1 + \sqrt{2}, \ B = 1 + \sqrt{3}, \ C = 1 + \sqrt{6}, \ \lambda = 7 + 3\sqrt{6}, \ \mu = \sqrt{\frac{4\sqrt{6} - 1}{19}}$

▷ Computerized discovery of ODE by enumeration + Hermite–Padé.
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<tbody>
<tr>
<td>A005566</td>
<td>N 3</td>
<td>$\frac{4}{\pi} \frac{4^n}{n}$</td>
<td>A151275</td>
<td>N 5</td>
<td>$\frac{12\sqrt{30}}{\pi} \frac{(2\sqrt{6})^n}{n^2}$</td>
</tr>
<tr>
<td>A018224</td>
<td>N 3</td>
<td>$\frac{2}{\pi} \frac{4^n}{n}$</td>
<td>A151314</td>
<td>N 5</td>
<td>$\frac{\sqrt{6}\lambda\mu C^{5/2}}{5\pi} \frac{(2C)^n}{n^2}$</td>
</tr>
<tr>
<td>A151312</td>
<td>N 3</td>
<td>$\frac{\sqrt{6}}{\pi} \frac{6^n}{n}$</td>
<td>A151255</td>
<td>N 5</td>
<td>$\frac{24\sqrt{2}}{\pi} \frac{(2\sqrt{2})^n}{n^2}$</td>
</tr>
<tr>
<td>A151331</td>
<td>N 3</td>
<td>$\frac{8}{3\pi} \frac{8^n}{n}$</td>
<td>A151287</td>
<td>N 5</td>
<td>$\frac{2\sqrt{2} A^{7/2}}{\pi} \frac{(2A)^n}{n^2}$</td>
</tr>
<tr>
<td>A151266</td>
<td>N 5</td>
<td>$\frac{1}{2} \sqrt{\frac{3}{\pi} \frac{3^n}{n^{1/2}}}$</td>
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<td>A151291</td>
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<td>N 4</td>
<td>$\frac{8}{\pi} \frac{4^n}{n^2}$</td>
</tr>
<tr>
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<td>N 5</td>
<td>$\frac{2}{\sqrt{3}\pi} \frac{6^n}{n^{1/2}}$</td>
<td>A151265</td>
<td>Y</td>
<td>$\frac{2\sqrt{2}}{\Gamma(1/4)} \frac{3^n}{n^{3/4}}$</td>
</tr>
<tr>
<td>A151302</td>
<td>N 5</td>
<td>$\frac{1}{3} \sqrt{\frac{5}{2\pi} \frac{5^n}{n^{1/2}}}$</td>
<td>A151278</td>
<td>Y</td>
<td>$\frac{3\sqrt{3}}{2\Gamma(1/4)} \frac{3^n}{n^{3/4}}$</td>
</tr>
<tr>
<td>A151329</td>
<td>N 5</td>
<td>$\frac{1}{3} \sqrt{\frac{7}{3\pi} \frac{7^n}{n^{1/2}}}$</td>
<td>A151323</td>
<td>Y</td>
<td>$\frac{\sqrt{23}^{3/4}}{\Gamma(1/4)} \frac{6^n}{n^{3/4}}$</td>
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<tr>
<td>A151261</td>
<td>N 5</td>
<td>$\frac{12\sqrt{3}}{\pi} \frac{(2\sqrt{3})^n}{n^2}$</td>
<td>A060900</td>
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▶ Computerized discovery of asymptotics by enumeration + LLL/PSLQ.
Further Previous Work

**Confirmation of D-finiteness**

- Human proofs for cases 1–22 in [Bousquet-Mélou & Mishna, 2010], but method not adapted to exhibit ODEs.
- Computer proof for case 23 in [Bostan & Kauers, 2010].
Confirmation of D-finiteness

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Fix of asymptotic formulas (first observed/proved by Melczer)

In fact:

<table>
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<tr>
<th>OEIS</th>
<th>( \mathcal{S} )</th>
<th>equiv</th>
</tr>
</thead>
</table>
| A151261  | \[
\frac{12\sqrt{3}}{\pi} \frac{(2\sqrt{3})^n}{n^2} \]
|          | \[
\frac{18}{\pi} \frac{(2\sqrt{3})^n}{n^2} \] \((n = 2p + 1)\) |
| A151275  | \[
\frac{12\sqrt{30}}{\pi} \frac{(2\sqrt{6})^n}{n^2} \]
|          | \[
\frac{144}{\sqrt{5\pi}} \frac{(2\sqrt{6})^n}{n^2} \] \((n = 2p + 1)\) |
| A151255  | \[
\frac{24\sqrt{2}}{\pi} \frac{(2\sqrt{2})^n}{n^2} \]
|          | \[
\frac{32}{\pi} \frac{(2\sqrt{2})^n}{n^2} \] \((n = 2p + 1)\) |

Frédéric Chyzak
Small-Step Walks
First proof of formerly guessed linear differential operators for $F(1,1; t)$. 
Contributions

- First proof of formerly guessed linear differential operators for $F(1,1; t)$.
- Discovery and proof of explicit hypergeometric expressions for $F(x,y; t)$.
- Proof of algebricity, resp. transcendence, of those series.
- Similar proofs for $F(0,0; t)$, $F(0,1; t)$, and $F(1,0; t)$.
- Conjectured asymptotic formulas for the coefficients of $F(0,0; t)$, $F(0,1; t)$, $F(1,0; t)$, since then proved by Melczer and Wilson.
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Conjectured asymptotic formulas for the coefficients of $F(0,0;t)$, $F(0,1;t)$, $F(1,0;t)$, since then proved by Melczer and Wilson.
### Table of D-Finite $F(x, y; t)$ at $x = y = 0$ [This work]

<table>
<thead>
<tr>
<th>OEIS</th>
<th>$\mathcal{S}$ alg</th>
<th>$\text{conj'd equiv}$</th>
<th>OEIS</th>
<th>$\mathcal{S}$ alg</th>
<th>$\text{conj'd equiv}$</th>
</tr>
</thead>
</table>
| 1 A005568 | $\mathcal{N}$ | \[
\begin{cases}
\frac{32}{\pi} \frac{4n}{n^3} & (n = 2p) \\
0 & (n = 2p + 1)
\end{cases}
\] | 13 A151345 | $\mathcal{N}$ | \[
\begin{cases}
\frac{24\sqrt{30}}{25\pi} \frac{(2\sqrt{6})^n}{n^3} & (n = 2p) \\
0 & (n = 2p + 1)
\end{cases}
\] |
| 2 A001246 | $\mathcal{N}$ | \[
\begin{cases}
\frac{8}{\pi} \frac{4n}{n^3} & (n = 2p) \\
0 & (n = 2p + 1)
\end{cases}
\] | 14 A151370 | $\mathcal{N}$ | \[
\frac{2\mu^3 C^{3/2}}{\pi} \frac{(2C)^n}{n^3}
\] |
| 3 A151362 | $\mathcal{N}$ | \[
\begin{cases}
\frac{3\sqrt{6}}{\pi} \frac{6n}{n^3} & (n = 2p) \\
0 & (n = 2p + 1)
\end{cases}
\] | 15 A151332 | $\mathcal{N}$ | \[
\begin{cases}
\frac{16\sqrt{2}}{\pi} \frac{(2\sqrt{2})^n}{n^3} & (n = 4p) \\
0 & (n = 4p + 1, 2, 3)
\end{cases}
\] |
| 4 A172361 | $\mathcal{N}$ | \[
\frac{128}{27\pi} \frac{8n}{n^3}
\] | 16 A151357 | $\mathcal{N}$ | \[
\frac{2\Lambda^{3/2}}{\pi} \frac{(2\Lambda)^n}{n^3}
\] |
| 5 A151332 | $\mathcal{N}$ | \[
\begin{cases}
\frac{16\sqrt{2}}{\pi} \frac{(2\sqrt{2})^n}{n^3} & (n = 4p) \\
0 & (n = 4p + 1, 2, 3)
\end{cases}
\] | 17 A151334 | $\mathcal{N}$ | \[
\begin{cases}
\frac{81\sqrt{3}}{\pi} \frac{3n}{n^4} & (n = 3p) \\
0 & (n = 3p + 1, 2)
\end{cases}
\] |
| 6 A151357 | $\mathcal{N}$ | \[
\frac{2\Lambda^{3/2}}{\pi} \frac{(2\Lambda)^n}{n^3}
\] | 18 A151366 | $\mathcal{N}$ | \[
\frac{27\sqrt{3}}{\pi} \frac{6n}{n^4}
\] |
| 7 A151341 | $\mathcal{N}$ | \[
\begin{cases}
\frac{12\sqrt{3}}{\pi} \frac{(2\sqrt{3})^n}{n^3} & (n = 2p) \\
0 & (n = 2p + 1)
\end{cases}
\] | 19 A138349 | $\mathcal{N}$ | \[
\begin{cases}
\frac{768}{\pi} \frac{4n}{n^5} & (n = 2p) \\
0 & (n = 2p + 1)
\end{cases}
\] |
Table of D-Finite $F(x, y; t)$ at $x = 0, y = 1$ [This work]

<table>
<thead>
<tr>
<th>OEIS</th>
<th>$\mathcal{G}$ alg</th>
<th>conj’d equiv</th>
<th>OEIS</th>
<th>$\mathcal{G}$ alg</th>
<th>conj’d equiv</th>
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<td>$\frac{8}{\pi} \frac{4^n}{n^2}$</td>
<td>12</td>
<td>A151472</td>
<td>$\frac{3b^{7/2}}{2\pi} \frac{(2B)^n}{n^3}$</td>
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<td>2</td>
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<td>$\left{ \begin{array}{ll} \frac{4}{\pi} \frac{4^n}{n^2} &amp; (n = 2p) \ 0 &amp; (n = 2p + 1) \end{array} \right.$</td>
<td>13</td>
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<td>$\left{ \begin{array}{ll} \frac{72\sqrt{30}}{5\pi} \frac{(2\sqrt{6})^n}{n^3} &amp; (n = 2p) \ \frac{864\sqrt{5}}{25\pi} \frac{(2\sqrt{6})^n}{n^3} &amp; (n = 2p + 1) \end{array} \right.$</td>
</tr>
<tr>
<td>3</td>
<td>A151478</td>
<td>$\frac{3\sqrt{6}}{2\pi} \frac{6^n}{n^2}$</td>
<td>14</td>
<td>A151492</td>
<td>$\frac{6\lambda^3 C^5/2}{5\pi} \frac{(2C)^n}{n^3}$</td>
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<tr>
<td>4</td>
<td>A151496</td>
<td>$\frac{32}{9\pi} \frac{8^n}{n^2}$</td>
<td>15</td>
<td>A151375</td>
<td>$\left{ \begin{array}{ll} \frac{448\sqrt{2}}{9\pi} \frac{(2\sqrt{2})^n}{n^3} &amp; (n = 4p) \ \frac{640}{9\pi} \frac{(2\sqrt{2})^n}{n^3} &amp; (n = 4p + 1) \ \frac{416\sqrt{2}}{9\pi} \frac{(2\sqrt{2})^n}{n^3} &amp; (n = 4p + 2) \ \frac{512}{9\pi} \frac{(2\sqrt{2})^n}{n^3} &amp; (n = 4p + 3) \end{array} \right.$</td>
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<td>A151394</td>
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<td>$\uparrow$ N</td>
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The Kernel Equation [≤ Knuth, 1968]: an Example, Frédéric Chyzak

walk of length $n + 1 = \text{walk of length } n$ followed by a step from $\{←, ↑, →, ↓\}$
The Kernel Equation [Knuth, 1968]: an Example.

walk of length $n + 1 =$
walk of length $n$ followed by a step from $\{←, ↑, →, ↓\}$,
provided this remains in the quarter plane!
walk of length \( n + 1 = \) walk of length \( n \) followed by a step from \( \{ ←, ↑, →, ↓\} \), provided this remains in the quarter plane!

Recurrence relation:

\[
f_{n+1;i,j} = f_{n;i+1,j} + [0 < j] f_{n;i,j-1} + [0 < i] f_{n;i-1,j} + f_{n;i,j+1}.
\]

Frédéric Chyzak
The Kernel Equation \([\leq \text{Knuth, 1968}]:\) an Example, $$
abla$$

walk of length \(n + 1 =\)
walk of length \(n\) followed by a step from \(\{\leftarrow, \uparrow, \rightarrow, \downarrow\}\),

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\[
f_{n+1;i,j} = f_{n;i+1,j} + [0 < j] \ f_{n;i,j-1} + [0 < i] \ f_{n;i-1,j} + f_{n;i,j+1}.
\]

\[
f_{n+1;i,j} x^i y^j t^{n+1} = \left( f_{n;i+1,j} x^i+1 y^j t^n \right) \times \bar{x} t + [0 < j] \left( f_{n;i,j-1} x^i y^{j-1} t^n \right) \times \bar{y} t +
\]

\[
[0 < i] \left( f_{n;i-1,j} x^{i-1} y^j t^n \right) \times x t + \left( f_{n;i,j+1} x^i y^{j+1} t^n \right) \times \bar{y} t,
\]

Notation: \(\bar{x} = \frac{1}{x}, \quad \bar{y} = \frac{1}{y}\).
The Kernel Equation \( \leq \text{Knuth, 1968} \): an Example, \( \mathbb{Z} \)

walk of length \( n + 1 = \) walk of length \( n \) followed by a step from \( \{ \leftarrow, \uparrow, \rightarrow, \downarrow \} \), provided this remains in the quarter plane!

Recurrence relation:

\[
fn+1;i,j = fn;i+1,j + [0 < j] fn;i,j−1 + [0 < i] fn;i−1,j + fn;i,j+1.
\]

\[
fn+1;i,j x^i y^j t^{n+1} = \left( fn;i+1,j x^{i+1} y^j t^n \right) \times \bar{x}t + [0 < j] \left( fn;i,j−1 x^i y^{j−1} t^n \right) \times yt + [0 < i] \left( fn;i−1,j x^{i−1} y^j t^n \right) \times xt + \left( fn;i,j+1 x^i y^{j+1} t^n \right) \times \bar{y}t,
\]

\[
F(x,y;t) − 1 = \left( F(x,y;t) − F(0,y;t) \right) \times \bar{x}t + F(x,y;t) \times yt + F(x,y;t) \times xt + \left( F(x,y;t) − F(x,0;t) \right) \times \bar{y}t,
\]

Notation: \( \bar{x} = \frac{1}{x}, \quad \bar{y} = \frac{1}{y} \).
The Kernel Equation [<i>Knuth, 1968</i>]: an Example, \[
\begin{align*}
\text{walk of length } n + 1 &= \text{walk of length } n \text{ followed by a step from } \{\leftarrow, \uparrow, \rightarrow, \downarrow\}, \\
&\text{provided this remains in the quarter plane!}
\end{align*}
\]

Recurrence relation:
\[
f_{n+1;i,j} = f_{n;i+1,j} + \mathbb{I}(0 < j)f_{n;i,j-1} + \mathbb{I}(0 < i)f_{n;i-1,j} + f_{n;i,j+1}.
\]

Functional (“kernel”) equation:
\[
(1 - t(x + \bar{x} + y + \bar{y})) F(x, y; t) = -\bar{y}tF(x, 0; t) - \bar{x}tF(0, y; t) + 1.
\]
The Kernel Equation \cite{Knuth,1968}: an Example, \( \otimes \)

Walk of length \( n + 1 \) =
walk of length \( n \) followed by a step from \( \{ \leftarrow, \uparrow, \rightarrow, \downarrow \} \),
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Functional (“kernel”) equation:

\[
(1 - t (x + \bar{x} + y + \bar{y})) F(x, y; t) = -\bar{y} t F(x, 0; t) - \bar{x} t F(0, y; t) + 1.
\]

Remarks:

- Erasing the constraint leads to a rational generating series.
- Direct attempt to solve leads to tautologies.
$J = 1 - t \sum_{(i,j) \in S} x^i y^j = 1 - t(x + \bar{x} + y + \bar{y})$ is invariant under the change of $(x, y)$ into, respectively:

$$(\bar{x}, y), (\bar{x}, \bar{y}), (x, \bar{y})$$
\[ J = 1 - t \sum_{(i,j) \in \mathcal{S}} x^i y^j = 1 - t(x + \bar{x} + y + \bar{y}) \] is invariant under the change of \((x, y)\) into, respectively:

\[(\bar{x}, y), (\bar{x}, \bar{y}), (x, \bar{y})\].

Kernel equation:

\[ J(x, y; t)xyF(x, y; t) = -txF(x, 0; t) - tyF(0, y; t) + xy, \]
$J = 1 - t \sum_{(i,j) \in \mathcal{S}} x^i y^j = 1 - t(x + \bar{x} + y + \bar{y})$ is invariant under the change of $(x, y)$ into, respectively:

$(\bar{x}, y), (\bar{x}, \bar{y}), (x, \bar{y})$.

Kernel equation:

$$J(x, y; t)xyF(x, y; t) = -txF(x, 0; t) - tyF(0, y; t) + xy,$$

$$-J(x, y; t)\bar{x}yF(\bar{x}, y; t) = t\bar{x}F(\bar{x}, 0; t) + tyF(0, y; t) - \bar{x}y,$$
J = 1 - t \sum_{(i,j) \in S} x^i y^j = 1 - t(x + \bar{x} + y + \bar{y}) \text{ is invariant under the change of } (x, y) \text{ into, respectively:}

\((\bar{x}, y), (\bar{x}, \bar{y}), (x, \bar{y})\).

Kernel equation:

\[ J(x, y; t)xyF(x, y; t) = -txF(x, 0; t) - tyF(0, y; t) + xy, \]
\[ - J(x, y; t)\bar{x}yF(\bar{x}, y; t) = t\bar{x}F(\bar{x}, 0; t) + tyF(0, y; t) - \bar{x}y, \]
\[ J(x, y; t)\bar{x}\bar{y}F(\bar{x}, \bar{y}; t) = -t\bar{x}F(\bar{x}, 0; t) - t\bar{y}F(0, \bar{y}; t) + \bar{x}\bar{y}, \]
$J = 1 - t \sum_{(i,j) \in S} x^i y^j = 1 - t(x + \bar{x} + y + \bar{y})$ is invariant under the change of $(x, y)$ into, respectively:

$(\bar{x}, y), (\bar{x}, \bar{y}), (x, \bar{y})$.

Kernel equation:

\[
J(x, y; t)xyF(x, y; t) = -txF(x, 0; t) - tyF(0, y; t) + xy,
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\[
- J(x, y; t)\bar{x}yF(\bar{x}, y; t) = t\bar{x}F(\bar{x}, 0; t) + tyF(0, y; t) - \bar{x}y,
\]
\[
J(x, y; t)\bar{x}\bar{y}F(\bar{x}, \bar{y}; t) = -t\bar{x}F(\bar{x}, 0; t) - t\bar{y}F(0, \bar{y}; t) + \bar{x}\bar{y},
\]
\[
- J(x, y; t)x\bar{y}F(x, \bar{y}; t) = txF(x, 0; t) + t\bar{y}F(0, \bar{y}; t) - x\bar{y}.
\]
J = 1 - t \sum_{(i,j) \in \mathcal{G}} x^i y^j = 1 - t(x + \bar{x} + y + \bar{y}) \text{ is invariant under the change of (x, y) into any element of } \\
\mathcal{G} = \{(x, y), (\bar{x}, y), (\bar{x}, \bar{y}), (x, \bar{y})\}.

Kernel equation:

\[
J(x, y; t)xyF(x, y; t) = -txF(x, 0; t) - tyF(0, y; t) + xy, \\
-J(x, y; t)x\bar{y}F(\bar{x}, y; t) = t\bar{x}F(\bar{x}, 0; t) + tyF(0, y; t) - \bar{x}y, \\
J(x, y; t)x\bar{x}F(\bar{x}, \bar{y}; t) = -t\bar{x}F(\bar{x}, 0; t) - t\bar{y}F(0, \bar{y}; t) + \bar{x}\bar{y}, \\
-J(x, y; t)x\bar{y}F(x, \bar{y}; t) = txF(x, 0; t) + t\bar{y}F(0, \bar{y}; t) - x\bar{y}.
\]

Adding together yields:

\[
J(x, y; t) \sum_{g \in \mathcal{G}} \text{sign}(g) g(xy F(x, y; t)) = xy - \bar{x}y + \bar{x}\bar{y} - x\bar{y}.
\]
D-Finiteness via the Finite Group: an Example, Frédéric Chyzak

\[ J = 1 - t \sum_{(i,j) \in S} x^i y^j = 1 - t(x + \bar{x} + y + \bar{y}) \] is invariant under the change of \((x, y)\) into any element of \(G = \{(x, y), (\bar{x}, y), (\bar{x}, \bar{y}), (x, \bar{y})\}\).

Kernel equation:

\[
J(x, y; t)xyF(x, y; t) = -txF(x, 0; t) - tyF(0, y; t) + xy,
- J(x, y; t)\bar{x}yF(\bar{x}, y; t) = t\bar{x}F(\bar{x}, 0; t) + tyF(0, y; t) - \bar{x}y,
J(x, y; t)\bar{x}\bar{y}F(\bar{x}, \bar{y}; t) = -t\bar{x}F(\bar{x}, 0; t) - \bar{y}F(0, \bar{y}; t) + \bar{x}\bar{y},
- J(x, y; t)x\bar{y}F(x, \bar{y}; t) = txF(x, 0; t) + t\bar{y}F(0, \bar{y}; t) - x\bar{y}.
\]

Adding together yields:

\[
\sum_{g \in G} \text{sign}(g) g(xy F(x, y; t)) = \frac{xy - \bar{x}y + \bar{x}\bar{y} - x\bar{y}}{J(x, y; t)}.
\]
$J = 1 - t \sum_{(i,j) \in \mathcal{S}} x^i y^j = 1 - t(x + \bar{x} + y + \bar{y})$ is invariant under the change of $(x, y)$ into any element of $G = \{(x, y), (\bar{x}, y), (\bar{x}, \bar{y}), (x, \bar{y})\}$.

Kernel equation:

$$J(x, y; t)xyF(x, y; t) = -txF(x, 0; t) - tyF(0, y; t) + xy,$$

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Adding together yields:

$$[x^>][y^>] \sum_{g \in G} \text{sign}(g) g(xy F(x, y; t)) = [x^>][y^>] \frac{xy - \bar{x}y + \bar{x}\bar{y} - x\bar{y}}{J(x, y; t)}.$$
$J = 1 - t \sum_{(i,j) \in \mathcal{S}} x^i y^j = 1 - t(x + \bar{x} + y + \bar{y})$ is invariant under the change of $(x, y)$ into any element of

$\mathcal{G} = \{(x, y), (\bar{x}, y), (\bar{x}, \bar{y}), (x, \bar{y})\}$.

Kernel equation:

$J(x, y; t)xyF(x, y; t) = -txF(x, 0; t) - tyF(0, y; t) + xy,$

$- J(x, y; t)\bar{x}yF(\bar{x}, y; t) = t\bar{x}F(\bar{x}, 0; t) + tyF(0, y; t) - \bar{x}y,$

$J(x, y; t)\bar{x}\bar{y}F(\bar{x}, \bar{y}; t) = -t\bar{x}F(\bar{x}, 0; t) - t\bar{y}F(0, \bar{y}; t) + \bar{x}\bar{y},$

$- J(x, y; t)x\bar{y}F(x, \bar{y}; t) = txF(x, 0; t) + t\bar{y}F(0, \bar{y}; t) - x\bar{y}.$

Adding together yields:

$xy F(x, y; t) = [x^>][y^>] \frac{xy - \bar{x}y + \bar{x}\bar{y} - x\bar{y}}{J(x, y; t)}.$
Cases 1–19 are D-Finite

\[ J = 1 - t \sum_{(i,j) \in \mathcal{G}} x^i y^j \longrightarrow \text{a group } \mathcal{G} \text{ of birational transformations} \]

**Theorem** [Bousquet-Mélou & Mishna, 2010]

Let \( \mathcal{G} \) be one of the step sets 1–19. Then, the group \( \mathcal{G} \) is finite and:

\[ xy F(x, y; t) = [x^>] [y^>] \frac{\sum_{g \in \mathcal{G}} \text{sign}(g) g(xy)}{J(x, y; t)} \]

In particular, \( F(x, y; t) \) is D-finite.


\[ \text{Constructive proof, but impractical to get an ODE for } F(x, y; t) \]

\[ \text{Remark: The formula provides no direct information for } x = y = 1. \]
Cases 1–19 are D-Finite

\[ J = 1 - t \sum_{(i,j) \in \mathcal{S}} x^i y^j \rightarrow \text{a group } \mathcal{G} \text{ of birational transformations} \]

**Theorem** [Bousquet-Mélou & Mishna, 2010]

Let \( \mathcal{S} \) be one of the step sets 1–19. Then, the group \( \mathcal{G} \) is finite and:

\[ xy F(x, y; t) = [x^>] [y^>] \frac{\sum_{g \in \mathcal{G}} \text{sign}(g) \, g(xy)}{J(x, y; t)}. \]

In particular, \( F(x, y; t) \) is D-finite.

**Proof:** Use [Lipshitz, 1988] (“The diagonal of a D-finite power series is D-finite”) for positive parts of D-finite series.

▷ Constructive proof, but **impractical** to get an ODE for \( F(x, y; t) \).
Cases 1–19 are D-Finite

\[ J = 1 - t \sum_{(i,j) \in \mathcal{S}} x^i y^j \rightarrow \text{a group } \mathcal{G} \text{ of birational transformations} \]

**Theorem** [Bousquet-Mélou & Mishna, 2010]

Let \( \mathcal{S} \) be one of the step sets 1–19. Then, the group \( \mathcal{G} \) is finite and:

\[ xy F(x, y; t) = [x^>] [y^>] \sum_{g \in \mathcal{G}} \text{sign}(g) g(xy) \frac{J(x, y; t)}{J(x, y; t)}. \]

In particular, \( F(x, y; t) \) is D-finite.

**Proof**: Use [Lipshitz, 1988] (”The diagonal of a D-finite power series is D-finite”) for positive parts of D-finite series.

▷ Constructive proof, but impractical to get an ODE for \( F(x, y; t) \) by any algorithm; in fact, any such ODE is probably TOO LARGE TO BE MERELY WRITTEN!
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▷ Remark: The formula provides no direct information for \( x = y = 1 \).
Theorem [This work]
Let $\mathcal{S}$ be one of the step sets 1–19. Then, the generating series $F(x,y;t)$ is expressible using iterated integrals of $\,_2F_1$ functions.
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Let $\mathcal{S}$ be one of the step sets 1–19. Then, the generating series $F(1,1;t)$ is expressible using iterated integrals of $\,_{2}F_{1}$ functions.

**Example: King walks in the quarter plane (A025595, \[X\])**

\[
F(1,1;t) = \frac{1}{t} \int_{0}^{t} \frac{1}{(1+4x)^3} \cdot \,_{2}F_{1}\left(\begin{array}{c}
\frac{3}{2} \\
\frac{3}{2}
\end{array}\bigg| \frac{16x(1+x)}{(1+4x)^2}\right) \, dx
\]

\[
= 1 + 3t + 18t^2 + 105t^3 + 684t^4 + 4550t^5 + 31340t^6 + 219555t^7 + \cdots
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Explicit Expressions for the Cases 1–19

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Proved by deriving and solving:

$$t^2(4t + 1)(8t - 1)(2t - 1)(t + 1)y''' + t(576t^4 + 200t^3 - 252t^2 - 33t + 5)y'' +$$

$$(1152t^4 + 88t^3 - 468t^2 - 48t + 4)y' + (384t^3 - 72t^2 - 144t - 12)y = 0.$$
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▷ **Proof uses Creative telescoping, ODE factorization, ODE solving:**

1. If $R = \sum g \frac{\text{sign}(g) g(xy)}{f(x,y,t)}$, then $F = \frac{1}{xy}[x>y]R = \text{Res}_{u,v} H$, for $H = \frac{R(1/u,1/v,t)}{(1-xu)(1-yv)}$.

2. If $L \in \mathbb{Q}(x,y)[t]\langle \partial_t \rangle$ and $U, V \in \mathbb{Q}(x,y,u,v,t)$ such that $L(H) = \partial_u U + \partial_v V$, then $L(F(x,y;t)) = 0$ after integration over closed contours.

   Use creative telescoping to find $L$ (as well as $U$ and $V$).

3. **Factor $L$ as $L_2 \cdot P_1 \cdots P_t$, where $L_2$ has order $\leq 2$ and the $P_i$ have order 1.**

4. **Solve $L_2$ in terms of $\,_{2}F_{1}$s and deduce $F$.**
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   Works also for $(0, 0)$, $(x, 0)$, and $(0, y)$!

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4. Solve $L_2$ in terms of $_2F_1$s and deduce $F$.

5. For $F(x,y;t)$, run whole process for $F(0,0;t)$, $F(x,0;t)$, and $F(0,y;t)$, then substitute into kernel equation!
Hypergeometric Series Occurring in Explicit Expressions for $F(x, y; t)$

<table>
<thead>
<tr>
<th>$\mathcal{G}$ occurring $2F_1$</th>
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Observation: Related to complete elliptic integrals, $E(\sqrt{w})$ and $K(\sqrt{w})$. 
Well-studied algorithms


- **Factorization of ODE:** [Beke, 1894], [Schwarz, 1989], [Grigor’ev, 1990], [Singer, 1996], [van Hoeij, 1997]

- **Solving with 2F1:** [Bostan, Chyzak, van Hoeij, & Pech, 2011], [Fang, van Hoeij, 2011], [Kunwar, van Hoeij, 2013], [Kunwar, 2014], [van Hoeij, Vidunas, 2015], [van Hoeij, Imamoglu, 2015]

Already combined for a simpler problem: Diagonal 3D Rook Paths
[Bostan, Chyzak, van Hoeij, & Pech, 2011]

Problem: Determine the number \( a_n \) of paths from \((0,0,0)\) to \((n,n,n)\) that use positive multiples of \((1,0,0)\), \((0,1,0)\), and \((0,0,1)\).

Solution: 
\[
G(x) = 1 + 6 \cdot \int_0^x 2F1 \left( \begin{array}{c} 1/3, 2/3 \\ 2 \end{array} \right| \frac{27w(2-3w)(1-4w)^3}{(1-64w)} \right) \frac{dw}{(1-4w)(1-64w)}.
\]
Problem: Definitions of residues and positive parts of rational functions?

\[
\cdots - \frac{1}{w^3} - \frac{1}{w^2} - \frac{1}{w} \equiv \frac{1}{1 - w} \equiv 1 + w + w^2 + \cdots
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Key Idea (Step 1): Encoding Positive Parts as Algebraic Residues

Problem: Definitions of residues and positive parts of rational functions?

\[ \cdots - \frac{1}{w^3} - \frac{1}{w^2} - \frac{1}{w} \overset{?}{=} \frac{1}{1-w} \overset{?}{=} 1 + w + w^2 + \cdots \]

\[ -1 \overset{?}{=} \text{Res}_w \frac{1}{1-w} \overset{?}{=} 0 \]
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\[
0 \equiv [w>] \frac{1}{1-w} \equiv w + w^2 + \cdots
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Key Idea (Step 1): Encoding Positive Parts as Algebraic Residues

New formula

\[ F(a, b; t) = \text{Res}_{x,y} \left[ \frac{\bar{x}\bar{y}R(x, y; t)}{(x - a)(y - b)} \right]_{\Gamma_1} = \text{Res}_{x,y} \left[ \frac{R(\bar{x}, \bar{y}; t)}{(1 - ax)(1 - by)} \right]_{\Gamma_2}. \]

Interpretation [Aparicio-Monforte & Kauers, 2013]

- \( \text{Res}_{x,y} \) is linear on the vector space \( \mathbb{Q}^\mathbb{Z}^2 \);
- the rational functions \( R(x, y; t) \) and \( (x - a)^{-1}(y - b)^{-1} \) are expanded as a series with support in the cone \( \Gamma_1 = \{ x^i y^j t^n : |i|, |j| \leq n \geq 0 \} \);
- the rational functions \( R(\bar{x}, \bar{y}; t) \) and \( (1 - ax)^{-1}(1 - by)^{-1} \) are expanded as a series with support the cone \( \Gamma_2 = \{ x^i y^j t^n : -i, |j| \leq n \geq 0 \} \);
- a theory of series with support in a cone legitimates the product.

Link with creative telescoping [This work]

\[ L(H) = \partial_u U + \partial_v V \implies L([H]_{\Gamma}) = 0 \]

provided \( H, U, V \) admit expansions with respect to the same cone \( \Gamma \).
Proofs of Algebraicity/Transcendence of $F(x,y;t)$ and $F(1,1;t)$

**Theorem**

- In cases 1–19, $F(x,y;t)$ is transcendental since $F(0,0;t)$ is.
- In cases 1–16 and 19, $F(1,1;t)$ is transcendental.
- Specific simplifications prove algebraicity of $F(1,1;t)$ in cases 17–18.

**Proof:** Define $G = (P_1 \cdots P_t)(F)$ so that $L_2(G) = 0$.

- $F$ is algebraic $\implies G$ is algebraic.
- Computing a few coefficients of $G$ shows that this is not 0 on all cases of interest.
- Applying Kovacic’s algorithm to $L_2$ (order 2) or just computing exponential solutions (order 1) decides whether $L_2$ has nonzero algebraic solutions.
Conclusions

A succession of functional equations of several types

rec. relation on \( f_{n;i,j} \) \( \rightarrow \) kernel equation on \( F(x,y;t) \) \( \rightarrow \) ODE on \( F(1,1;t) \)
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A succession of computer-algebra algorithms
creative telescoping → ODE factorization → ODE solving
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- Three kinds of conjectures now proved:
  - differential operators that witness D-finiteness,
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Wanted
Better understanding of the systematic emergence of elliptic integrals