Explicit Generating Series for Small-Step Walks in the Quarter Plane

Frédéric Chyzak

Inria — informatiques mathématiques

Transient Transcendence in Transylvania

Braşov, Romania, May 13–17, 2019

Joint work with A. Bostan, M. van Hoeij, M. Kauers, and L. Pech (2017)

Lattice Walks, Why?

Applications in many areas of science

- discrete mathematics (permutations, trees, words, urns, ...)
- statistical physics (Ising model, ...)
- probability theory (branching processes, games of chance, ...)
- operations research (queueing theory, ...)

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A history and a survey of lattice path enumeration

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ABSTRACT

In celebration of the Sixth International Conference on Lattice Path Counting and Applications, it is befitting to review the history of lattice path enumeration and to survey how the topic has progressed thus far.

We start the history with early games of chance specifically the ruin problem which later appears as the ballot problem. We discuss André's Reflection Principle and its misnomer, its relation with the method of images and possible origins from physics and Brownian motion, and the earliest evidence of lattice path techniques and solutions.

In the survey, we give representative articles on lattice path enumeration found in the literature in the last 35 years by the lattice, step set, boundary, characteristics counted, and solution method. Some of this work appears in the author's 2005 dissertation.

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- statistical physics (Ising model, ...)
- probability theory (branching processes, games of chance, ...)
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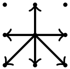
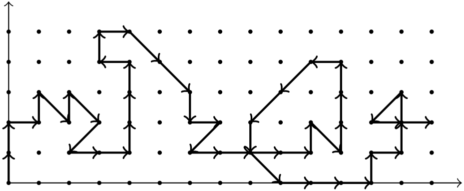
This talk:
Computer Algebra applied to Combinatorics

▷ Nearest-neighbor walks in the quarter plane = walks in $\mathbb{N}^2$ starting at $(0,0)$ and using steps in a *fixed* subset $\mathfrak{S}$ of

$$\{\swarrow, \leftarrow, \nwarrow, \uparrow, \nearrow, \rightarrow, \searrow, \downarrow\}.$$

▷ Example with $n = 45$, $i = 14$, $j = 2$ for:

$\mathfrak{S} =$ 



Enumerative Combinatorics of Lattice Walks

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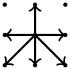
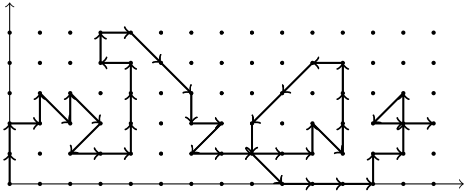
$\mathfrak{S} =$ 



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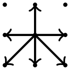
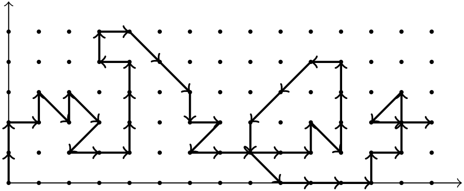
$\mathfrak{S} =$ 



▷ Counting sequence: $f_{n;i,j}$ = number of walks of length n ending at (i,j).

▷ Specializations:

- $f_{n;0,0}$ = number of walks of length n returning to origin ("excursions");
- $f_n = \sum_{i,j\geq 0} f_{n;i,j}$ = number of walks with prescribed length n.

▷ Complete generating series: $F(x,y;t) = \sum_{n=0}^{\infty} \left(\sum_{i,j=0}^{\infty} f_{n;i,j} x^i y^j \right) t^n \in \mathbb{Q}[x,y][[t]]$.

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Combinatorial questions: Given $\mathfrak{S}$, what can be said about $F(x,y;t)$, resp. $f_{n;i,j}$, and their variants?

- Algebraic nature of F: algebraic? transcendental?
- Explicit form: of F? of f?
- Asymptotics of f?

Generating Series and Combinatorial Problems

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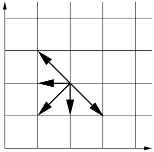
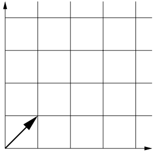
Our goal: Use computer algebra to give computational answers.

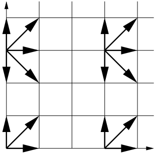
From the 2^8 step sets $\mathfrak{S} \subseteq \{-1, 0, 1\}^2 \setminus \{(0, 0)\}$, some are:



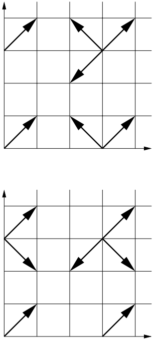
trivial,



too simple,



intrinsic to the half plane,



related by symmetries.

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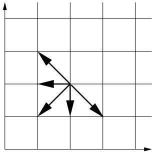

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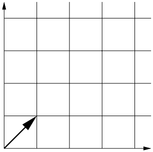

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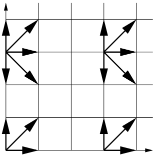

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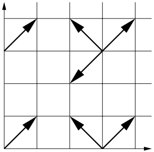

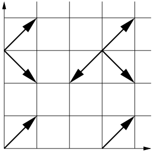

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One is left with 79 interesting distinct models.

Small-Step Models of Interest

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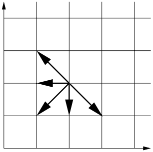
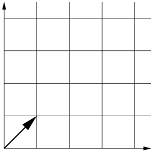


trivial,



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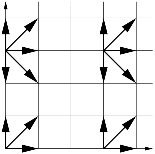

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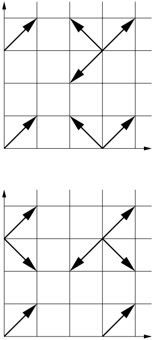

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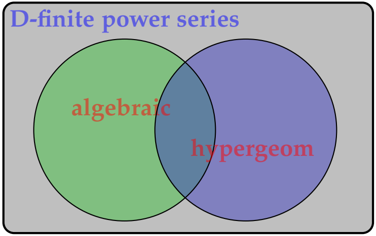
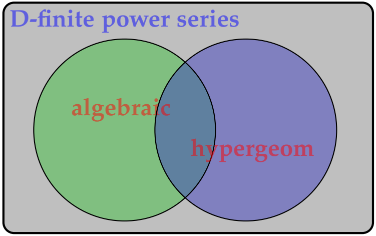
One is left with 79 interesting distinct models.

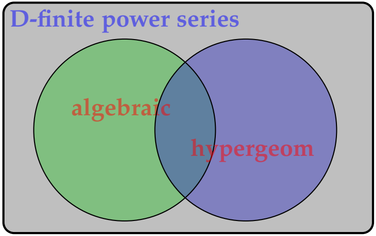
Is any further classification possible?

Classification of Univariate Power Series





▷ *Algebraic*: $S(t) \in \mathbb{Q}[[t]]$ root of a polynomial $P \in \mathbb{Q}[t, T]$, i.e., $P\big(t, S(t)\big) = 0$.



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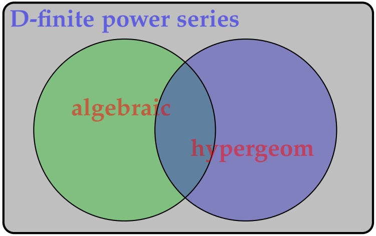
▹ *D-finite*: $S(t) \in \mathbb{Q}[[t]]$ satisfying a linear differential equation with polynomial coefficients $c_r(t)S^{(r)}(t) + \cdots + c_0(t)S(t) = 0$.

Classification of Univariate Power Series



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▷ *Hypergeometric*: $S(t) = \sum_{n=0}^{\infty} s_n t^n$ such that $\frac{s_{n+1}}{s_n} \in \mathbb{Q}(n)$. E.g., Gauss'

$$ {}_2F_1\left(\begin{matrix} a\ b \\ c \end{matrix} \,\middle|\, t\right) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{t^n}{n!}, \qquad (a)_n = a(a+1)\cdots(a+n-1), $$

$$ t(1-t)S''(t) + \big(c - (a+b+1)t\big)S'(t) - abS(t) = 0. $$

Table of All Conjectured D-Finite $F(1,1;t)$ [Bostan & Kauers, 2009]

	OEIS	$\mathfrak{S}$	alg	ord	equiv		OEIS	$\mathfrak{S}$	alg	ord	equiv
1	A005566		N	3	$\frac{4}{\pi}\frac{4^n}{n}$	13	A151275		N	5	$\frac{12\sqrt{30}}{\pi}\frac{(2\sqrt{6})^n}{n^2}$
2	A018224		N	3	$\frac{2}{\pi}\frac{4^n}{n}$	14	A151314		N	5	$\frac{\sqrt{6}\lambda\mu C^{5/2}}{5\pi}\frac{(2C)^n}{n^2}$
3	A151312		N	3	$\frac{\sqrt{6}}{\pi}\frac{6^n}{n}$	15	A151255		N	5	$\frac{24\sqrt{2}}{\pi}\frac{(2\sqrt{2})^n}{n^2}$
4	A151331		N	3	$\frac{8}{3\pi}\frac{8^n}{n}$	16	A151287		N	5	$\frac{2\sqrt{2}A^{7/2}}{\pi}\frac{(2A)^n}{n^2}$
5	A151266		N	5	$\frac{1}{2}\sqrt{\frac{3}{\pi}}\frac{3^n}{n^{1/2}}$	17	A001006		Y	3	$\frac{3}{2}\sqrt{\frac{3}{\pi}}\frac{3^n}{n^{3/2}}$
6	A151307		N	5	$\frac{1}{2}\sqrt{\frac{5}{2\pi}}\frac{5^n}{n^{1/2}}$	18	A129400		Y	3	$\frac{3}{2}\sqrt{\frac{3}{\pi}}\frac{6^n}{n^{3/2}}$
7	A151291		N	5	$\frac{4}{3\sqrt{\pi}}\frac{4^n}{n^{1/2}}$	19	A005558		N	4	$\frac{8}{\pi}\frac{4^n}{n^2}$
8	A151326		N	5	$\frac{2}{\sqrt{3\pi}}\frac{6^n}{n^{1/2}}$						
9	A151302		N	5	$\frac{1}{3}\sqrt{\frac{5}{2\pi}}\frac{5^n}{n^{1/2}}$	20	A151265		Y		$\frac{2\sqrt{2}}{\Gamma(1/4)}\frac{3^n}{n^{3/4}}$
10	A151329		N	5	$\frac{1}{3}\sqrt{\frac{7}{3\pi}}\frac{7^n}{n^{1/2}}$	21	A151278		Y		$\frac{3\sqrt{3}}{\sqrt{2}\Gamma(1/4)}\frac{3^n}{n^{3/4}}$
11	A151261		N	5	$\frac{12\sqrt{3}}{\pi}\frac{(2\sqrt{3})^n}{n^2}$	22	A151323		Y		$\frac{\sqrt{2}3^{3/4}}{\Gamma(1/4)}\frac{6^n}{n^{3/4}}$
12	A151297		N	5	$\frac{\sqrt{3}B^{7/2}}{2\pi}\frac{(2B)^n}{n^2}$	23	A060900		Y		$\frac{4\sqrt{3}}{3\Gamma(1/3)}\frac{4^n}{n^{2/3}}$

$A = 1+\sqrt{2},\ B = 1+\sqrt{3},\ C = 1+\sqrt{6},\ \lambda = 7+3\sqrt{6},\ \mu = \sqrt{\frac{4\sqrt{6}-1}{19}}$

▷ Computerized discovery of ODE by enumeration + Hermite–Padé.

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12	A151297		N	5	$\frac{\sqrt{3}B^{7/2}}{2\pi}\frac{(2B)^n}{n^2}$
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▷ Computerized discovery of asymptotics by enumeration + LLL/PSLQ.

Further Previous Work

Confirmation of D-finiteness

▷ Human proofs for cases 1–22 in [Bousquet-Mélou & Mishna, 2010], but method not adapted to exhibit ODEs.

▷ Computer proof for case 23 in [Bostan & Kauers, 2010].

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Fix of asymptotic formulas (first observed/proved by Melczer)

In fact:

	OEIS	$\mathfrak{S}$	equiv
11	A151261		$\begin{cases} \frac{12\sqrt{3}}{\pi} \frac{(2\sqrt{3})^n}{n^2} & (n = 2p) \\ \frac{18}{\pi} \frac{(2\sqrt{3})^n}{n^2} & (n = 2p+1) \end{cases}$
13	A151275		$\begin{cases} \frac{12\sqrt{30}}{\pi} \frac{(2\sqrt{6})^n}{n^2} & (n = 2p) \\ \frac{144}{\sqrt{5}\pi} \frac{(2\sqrt{6})^n}{n^2} & (n = 2p+1) \end{cases}$
15	A151255		$\begin{cases} \frac{24\sqrt{2}}{\pi} \frac{(2\sqrt{2})^n}{n^2} & (n = 2p) \\ \frac{32}{\pi} \frac{(2\sqrt{2})^n}{n^2} & (n = 2p+1) \end{cases}$

Contributions

▷ First proof of formerly guessed linear differential operators for $F(1,1;t)$.

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▷ Proof of algebricity, resp. transcendence, of those series.
▷ Similar proofs for $F(0,0;t)$, $F(0,1;t)$, and $F(1,0;t)$.

▷ Conjectured asymptotic formulas for the coefficients of $F(0,0;t)$, $F(0,1;t)$, $F(1,0;t)$, since then proved by Melczer and Wilson.

Table of D-Finite $F(x,y;t)$ at $x = y = 0$ [This work]

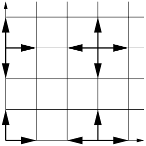
	OEIS	$\mathfrak{S}$	alg	conj'd equiv
1	A005568		N	$\begin{cases} \frac{32}{\pi}\frac{4^n}{n^3} & (n=2p) \\ 0 & (n=2p+1) \end{cases}$
2	A001246		N	$\begin{cases} \frac{8}{\pi}\frac{4^n}{n^3} & (n=2p) \\ 0 & (n=2p+1) \end{cases}$
3	A151362		N	$\begin{cases} \frac{3\sqrt{6}}{\pi}\frac{6^n}{n^3} & (n=2p) \\ 0 & (n=2p+1) \end{cases}$
4	A172361		N	$\frac{128}{27\pi}\frac{8^n}{n^3}$
5	A151332		N	$\begin{cases} \frac{16\sqrt{2}}{\pi}\frac{(2\sqrt{2})^n}{n^3} & (n=4p) \\ 0 & (n=4p+1,2,3) \end{cases}$
6	A151357		N	$\frac{2A^{3/2}}{\pi}\frac{(2A)^n}{n^3}$
7	A151341		N	$\begin{cases} \frac{12\sqrt{3}}{\pi}\frac{(2\sqrt{3})^n}{n^3} & (n=2p) \\ 0 & (n=2p+1) \end{cases}$
8	A151368		N	$\frac{2B^{3/2}}{\pi}\frac{(2B)^n}{n^3}$
9	A151345		N	$\begin{cases} \frac{24\sqrt{30}}{25\pi}\frac{(2\sqrt{6})^n}{n^3} & (n=2p) \\ 0 & (n=2p+1) \end{cases}$
10	A151370		N	$\frac{2\mu^3 C^{3/2}}{\pi}\frac{(2C)^n}{n^3}$
11	A151341		N	$\begin{cases} \frac{12\sqrt{3}}{\pi}\frac{(2\sqrt{3})^n}{n^3} & (n=2p) \\ 0 & (n=2p+1) \end{cases}$
12	A151368		N	$\frac{2B^{3/2}}{\pi}\frac{(2B)^n}{n^3}$
13	A151345		N	$\begin{cases} \frac{24\sqrt{30}}{25\pi}\frac{(2\sqrt{6})^n}{n^3} & (n=2p) \\ 0 & (n=2p+1) \end{cases}$
14	A151370		N	$\frac{2\mu^3 C^{3/2}}{\pi}\frac{(2C)^n}{n^3}$
15	A151332		N	$\begin{cases} \frac{16\sqrt{2}}{\pi}\frac{(2\sqrt{2})^n}{n^3} & (n=4p) \\ 0 & (n=4p+1,2,3) \end{cases}$
16	A151357		N	$\frac{2A^{3/2}}{\pi}\frac{(2A)^n}{n^3}$
17	A151334		N	$\begin{cases} \frac{81\sqrt{3}}{\pi}\frac{3^n}{n^4} & (n=3p) \\ 0 & (n=3p+1,2) \end{cases}$
18	A151366		N	$\frac{27\sqrt{3}}{\pi}\frac{6^n}{n^4}$
19	A138349		N	$\begin{cases} \frac{768}{\pi}\frac{4^n}{n^5} & (n=2p) \\ 0 & (n=2p+1) \end{cases}$

Table of D-Finite $F(x,y;t)$ at $x=0$, $y=1$ [This work]

	OEIS	$\mathfrak{S}$	alg	conj'd equiv
1	A005558		N	$\frac{8}{\pi}\frac{4^n}{n^2}$
2	A151392		N	$\begin{cases} \frac{4}{\pi}\frac{4^n}{n^2} & (n=2p) \\ 0 & (n=2p+1) \end{cases}$
3	A151478		N	$\frac{3\sqrt{6}}{2\pi}\frac{6^n}{n^2}$
4	A151496		N	$\frac{32}{9\pi}\frac{8^n}{n^2}$
5	A151380		N	$\frac{3}{4}\sqrt{\frac{3}{\pi}}\frac{3^n}{n^{3/2}}$
6	A151450		N	$\frac{5}{16}\sqrt{\frac{10}{\pi}}\frac{5^n}{n^{3/2}}$
7	A148790		N	$\frac{8}{3\sqrt{\pi}}\frac{4^n}{n^{3/2}}$
8	A151485		N	$\sqrt{\frac{3}{\pi}}\frac{6^n}{n^{3/2}}$
9	A151440		N	$\frac{5}{24}\sqrt{\frac{10}{\pi}}\frac{5^n}{n^{3/2}}$
10	A151493		N	$\frac{7}{54}\sqrt{\frac{21}{\pi}}\frac{7^n}{n^{3/2}}$
11	A151394		N	$\begin{cases} \frac{36\sqrt{3}}{\pi}\frac{(2\sqrt{3})^n}{n^3} & (n=2p) \\ \frac{54}{\pi}\frac{(2\sqrt{3})^n}{n^3} & (n=2p+1) \end{cases}$
12	A151472		N	$\frac{3B^{7/2}}{2\pi}\frac{(2B)^n}{n^3}$
13	A151437		N	$\begin{cases} \frac{72\sqrt{30}}{5\pi}\frac{(2\sqrt{6})^n}{n^3} & (n=2p) \\ \frac{864\sqrt{5}}{25\pi}\frac{(2\sqrt{6})^n}{n^3} & (n=2p+1) \end{cases}$
14	A151492		N	$\frac{6\lambda\mu^3C^{5/2}}{5\pi}\frac{(2C)^n}{n^3}$
15	A151375		N	$\begin{cases} \frac{448\sqrt{2}}{9\pi}\frac{(2\sqrt{2})^n}{n^3} & (n=4p) \\ \frac{640}{9\pi}\frac{(2\sqrt{2})^n}{n^3} & (n=4p+1) \\ \frac{416\sqrt{2}}{9\pi}\frac{(2\sqrt{2})^n}{n^3} & (n=4p+2) \\ \frac{512}{9\pi}\frac{(2\sqrt{2})^n}{n^3} & (n=4p+3) \end{cases}$
16	A151430		N	$\frac{4A^{7/2}}{\pi}\frac{(2A)^n}{n^3}$
17	A151378		N	$\frac{27}{8}\sqrt{\frac{3}{\pi}}\frac{3^n}{n^{5/2}}$
18	A151483		Y	$\frac{27}{8}\sqrt{\frac{3}{\pi}}\frac{6^n}{n^{5/2}}$
19	A005568		N	$\begin{cases} \frac{32}{\pi}\frac{4^n}{n^3} & (n=2p) \\ 0 & (n=2p+1) \end{cases}$

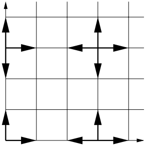
Table of D-Finite $F(x,y;t)$ at $x=1, y=0$ [This work]

	OEIS	𝔖	alg	conj'd equiv
1	A005558		N	$\frac{8}{\pi}\frac{4^n}{n^2}$
2	A151392		N	$\begin{cases} \frac{4}{\pi}\frac{4^n}{n^2} & (n=2p) \\ 0 & (n=2p+1) \end{cases}$
3	A151471		N	$\begin{cases} \frac{2\sqrt{6}}{\pi}\frac{6^n}{n^2} & (n=2p) \\ 0 & (n=2p+1) \end{cases}$
4	A151496		N	$\frac{32}{9\pi}\frac{8^n}{n^2}$
5	A151379		N	$\begin{cases} \frac{4\sqrt{2}}{\pi}\frac{(2\sqrt{2})^n}{n^2} & (n=2p) \\ 0 & (n=2p+1) \end{cases}$
6	A148934		N	$\frac{\sqrt{2}A^{3/2}}{\pi}\frac{(2A)^n}{n^2}$
7	A151410		N	$\begin{cases} \frac{4\sqrt{3}}{\pi}\frac{(2\sqrt{3})^n}{n^2} & (n=2p) \\ 0 & (n=2p+1) \end{cases}$
8	A151464		N	$\frac{2B^{3/2}\sqrt{3}}{3\pi}\frac{(2B)^n}{n^2}$
9	A151423		N	$\begin{cases} \frac{4\sqrt{30}}{5\pi}\frac{(2\sqrt{6})^n}{n^2} & (n=2p) \\ 0 & (n=2p+1) \end{cases}$
10	A151490		N	$\frac{\sqrt{6}\mu C^{3/2}}{3\pi}\frac{(2C)^n}{n^2}$
11	A151410		N	$\begin{cases} \frac{4\sqrt{3}}{\pi}\frac{(2\sqrt{3})^n}{n^2} & (n=2p) \\ 0 & (n=2p+1) \end{cases}$
12	A151464		N	$\frac{2B^{3/2}\sqrt{3}}{3\pi}\frac{(2B)^n}{n^2}$
13	A151423		N	$\begin{cases} \frac{4\sqrt{30}}{5\pi}\frac{(2\sqrt{6})^n}{n^2} & (n=2p) \\ 0 & (n=2p+1) \end{cases}$
14	A151490		N	$\frac{\sqrt{6}\mu C^{3/2}}{3\pi}\frac{(2C)^n}{n^2}$
15	A151379		N	$\begin{cases} \frac{4\sqrt{2}}{\pi}\frac{(2\sqrt{2})^n}{n^2} & (n=2p) \\ 0 & (n=2p+1) \end{cases}$
16	A148934		N	$\frac{\sqrt{2}A^{3/2}}{\pi}\frac{(2A)^n}{n^2}$
17	A151497		N	$\frac{27}{8}\sqrt{\frac{3}{\pi}}\frac{3^n}{n^{5/2}}$
18	A151483		Y	$\frac{27}{8}\sqrt{\frac{3}{\pi}}\frac{6^n}{n^{5/2}}$
19	A005817		N	$\frac{32}{\pi}\frac{4^n}{n^3}$



walk of length $n+1$ =
walk of length n followed by a step from $\{\leftarrow, \uparrow, \rightarrow, \downarrow\}$

The Kernel Equation [≤ Knuth, 1968]: an Example,



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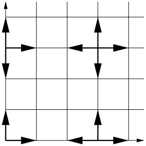

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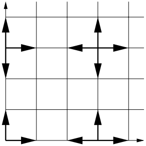
Recurrence relation:

$$f_{n+1;i,j} = f_{n;i+1,j} + [\![0 < j]\!]\, f_{n;i,j-1} + [\![0 < i]\!]\, f_{n;i-1,j} + f_{n;i,j+1}.$$



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$$f_{n+1;i,j}x^i y^j t^{n+1} = \left(f_{n;i+1,j}x^{i+1}y^j t^n\right) \times \bar{x}t + [\![0 < j]\!] \left(f_{n;i,j-1}x^i y^{j-1} t^n\right) \times yt + [\![0 < i]\!] \left(f_{n;i-1,j}x^{i-1}y^j t^n\right) \times xt + \left(f_{n;i,j+1}x^i y^{j+1} t^n\right) \times \bar{y}t,$$

Notation: $\bar{x} = \dfrac{1}{x}, \quad \bar{y} = \dfrac{1}{y}.$

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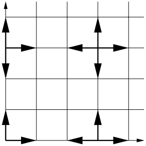

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$$F(x,y;t) - 1 = \big(F(x,y;t) - F(0,y;t)\big) \times \bar{x}t + F(x,y;t) \times yt + F(x,y;t) \times xt + \big(F(x,y;t) - F(x,0;t)\big) \times \bar{y}t,$$

Notation: $\bar{x} = \dfrac{1}{x}, \quad \bar{y} = \dfrac{1}{y}.$

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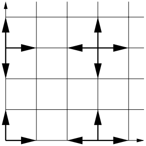

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Functional ("kernel") equation:

$$(1 - t\,(x + \bar{x} + y + \bar{y}))\, F(x, y; t) = -\bar{y}tF(x, 0; t) - \bar{x}tF(0, y; t) + 1.$$

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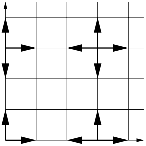

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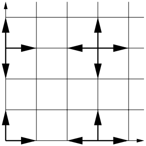
Remarks:

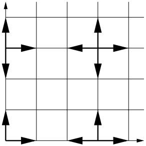
- Erasing the constraint leads to a rational generating series.
- Direct attempt to solve leads to tautologies.



$J = 1 - t\sum_{(i,j)\in\mathfrak{S}} x^i y^j = 1 - t(x + \bar{x} + y + \bar{y})$ is invariant under the change of (x, y) into, respectively:

$$(\bar{x}, y), (\bar{x}, \bar{y}), (x, \bar{y}) \quad .$$

D-Finiteness via the Finite Group: an Example, ✥



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$$J(x, y; t)xyF(x, y; t) = -txF(x, 0; t) - tyF(0, y; t) + xy,$$

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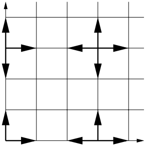

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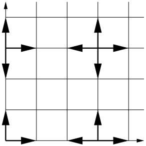

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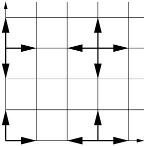
Kernel equation:

$$\begin{aligned} J(x,y;t)xyF(x,y;t) &= -txF(x,0;t) - tyF(0,y;t) + xy, \\ -\ J(x,y;t)\bar{x}yF(\bar{x},y;t) &= t\bar{x}F(\bar{x},0;t) + tyF(0,y;t) - \bar{x}y, \\ J(x,y;t)\bar{x}\bar{y}F(\bar{x},\bar{y};t) &= -t\bar{x}F(\bar{x},0;t) - t\bar{y}F(0,\bar{y};t) + \bar{x}\bar{y}, \end{aligned}$$

D-Finiteness via the Finite Group: an Example,



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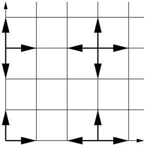
J(x,y;t)\bar{x}\bar{y}F(\bar{x},\bar{y};t) &= -t\bar{x}F(\bar{x},0;t) - t\bar{y}F(0,\bar{y};t) + \bar{x}\bar{y}, \\
-J(x,y;t)x\bar{y}F(x,\bar{y};t) &= txF(x,0;t) + t\bar{y}F(0,\bar{y};t) - x\bar{y}.
\end{aligned}$$

D-Finiteness via the Finite Group: an Example, ✥



$J = 1 - t\sum_{(i,j)\in\mathfrak{S}} x^i y^j = 1 - t(x + \bar{x} + y + \bar{y})$ is invariant under the change of (x, y) into any element of

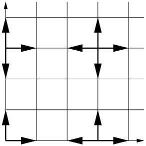
$$\mathcal{G} = \{(x,y), (\bar{x},y), (\bar{x},\bar{y}), (x,\bar{y})\}.$$

Kernel equation:

$$\begin{aligned} J(x,y;t)xyF(x,y;t) &= -txF(x,0;t) - tyF(0,y;t) + xy, \\ -J(x,y;t)\bar{x}yF(\bar{x},y;t) &= t\bar{x}F(\bar{x},0;t) + tyF(0,y;t) - \bar{x}y, \\ J(x,y;t)\bar{x}\bar{y}F(\bar{x},\bar{y};t) &= -t\bar{x}F(\bar{x},0;t) - t\bar{y}F(0,\bar{y};t) + \bar{x}\bar{y}, \\ -J(x,y;t)x\bar{y}F(x,\bar{y};t) &= txF(x,0;t) + t\bar{y}F(0,\bar{y};t) - x\bar{y}. \end{aligned}$$

Adding together yields:

$$J(x,y;t)\sum_{g\in\mathcal{G}} \operatorname{sign}(g)\, g\big(xy\,F(x,y;t)\big) = \qquad xy - \bar{x}y + \bar{x}\bar{y} - x\bar{y}.$$



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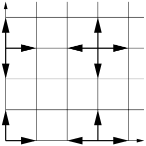
-J(x,y;t)\bar{x}yF(\bar{x},y;t) &= t\bar{x}F(\bar{x},0;t) + tyF(0,y;t) - \bar{x}y,\\
J(x,y;t)\bar{x}\bar{y}F(\bar{x},\bar{y};t) &= -t\bar{x}F(\bar{x},0;t) - t\bar{y}F(0,\bar{y};t) + \bar{x}\bar{y},\\
-J(x,y;t)x\bar{y}F(x,\bar{y};t) &= txF(x,0;t) + t\bar{y}F(0,\bar{y};t) - x\bar{y}.
\end{aligned}$$

Adding together yields:

$$\sum_{g\in\mathcal{G}} \operatorname{sign}(g)\, g\big(xy\,F(x,y;t)\big) = \frac{xy - \bar{x}y + \bar{x}\bar{y} - x\bar{y}}{J(x,y;t)}.$$

D-Finiteness via the Finite Group: an Example,



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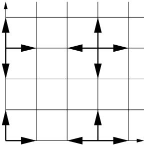
$$\begin{aligned} J(x,y;t)xyF(x,y;t) &= -txF(x,0;t) - tyF(0,y;t) + xy, \\ -J(x,y;t)\bar{x}yF(\bar{x},y;t) &= t\bar{x}F(\bar{x},0;t) + tyF(0,y;t) - \bar{x}y, \\ J(x,y;t)\bar{x}\bar{y}F(\bar{x},\bar{y};t) &= -t\bar{x}F(\bar{x},0;t) - t\bar{y}F(0,\bar{y};t) + \bar{x}\bar{y}, \\ -J(x,y;t)x\bar{y}F(x,\bar{y};t) &= txF(x,0;t) + t\bar{y}F(0,\bar{y};t) - x\bar{y}. \end{aligned}$$

Adding together yields:

$$[x^>][y^>] \sum_{g\in\mathcal{G}} \operatorname{sign}(g)\, g\big(xy\, F(x,y;t)\big) = [x^>][y^>] \frac{xy - \bar{x}y + \bar{x}\bar{y} - x\bar{y}}{J(x,y;t)}.$$

D-Finiteness via the Finite Group: an Example,



$J = 1 - t\sum_{(i,j)\in\mathfrak{S}} x^i y^j = 1 - t(x + \bar{x} + y + \bar{y})$ is invariant under the change of (x, y) into any element of

$$\mathcal{G} = \{(x, y), (\bar{x}, y), (\bar{x}, \bar{y}), (x, \bar{y})\}.$$

Kernel equation:

$$\begin{aligned}
J(x, y; t)xyF(x, y; t) &= -txF(x, 0; t) - tyF(0, y; t) + xy,\\
-J(x, y; t)\bar{x}yF(\bar{x}, y; t) &= t\bar{x}F(\bar{x}, 0; t) + tyF(0, y; t) - \bar{x}y,\\
J(x, y; t)\bar{x}\bar{y}F(\bar{x}, \bar{y}; t) &= -t\bar{x}F(\bar{x}, 0; t) - t\bar{y}F(0, \bar{y}; t) + \bar{x}\bar{y},\\
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Cases 1–19 are D-Finite

$J = 1 - t\sum_{(i,j)\in\mathfrak{S}} x^i y^j \qquad \longrightarrow \qquad$ a group $\mathcal{G}$ of birational transformations

Theorem [Bousquet-Mélou & Mishna, 2010]

Let $\mathfrak{S}$ be one of the step sets 1–19. Then, the group $\mathcal{G}$ is finite and:

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▷ Remark: The formula provides no direct information for $x = y = 1$.

Explicit Expressions for the Cases 1–19

Theorem [This work]

Let $\mathfrak{S}$ be one of the step sets 1–19. Then, the generating series $F(x, y; t)$ is expressible using iterated integrals of ${}_2F_1$ functions.

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Theorem [This work]

Let $\mathfrak{S}$ be one of the step sets 1–19. Then, the generating series $F(1,1;t)$ is expressible using iterated integrals of ${}_2F_1$ functions.

Example: King walks in the quarter plane (A025595, )

$$F(1,1;t) = \frac{1}{t}\int_0^t \frac{1}{(1+4x)^3} \cdot {}_2F_1\left(\begin{matrix}\frac{3}{2} & \frac{3}{2}\\ & 2\end{matrix} \,\middle|\, \frac{16x(1+x)}{(1+4x)^2}\right) dx$$

$$= 1 + 3t + 18t^2 + 105t^3 + 684t^4 + 4550t^5 + 31340t^6 + 219555t^7 + \cdots$$

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Proved by deriving and solving:

$$t^2(4t+1)(8t-1)(2t-1)(t+1)y''' + t(576t^4 + 200t^3 - 252t^2 - 33t + 5)y'' + (1152t^4 + 88t^3 - 468t^2 - 48t + 4)y' + (384t^3 - 72t^2 - 144t - 12)y = 0.$$

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▷ Proof uses Creative telescoping, ODE factorization, ODE solving:

1. If $R = \sum_g \frac{\operatorname{sign}(g)\, g(xy)}{J(x,y;t)}$, then $F = \frac{1}{xy}[x^> y^>]R = \operatorname{Res}_{u,v} H$, for $H = \frac{R(1/u,1/v;t)}{(1-xu)(1-yv)}$.
2. If $L \in \mathbb{Q}(x,y)[t]\langle\partial_t\rangle$ and $U, V \in \mathbb{Q}(x,y,u,v,t)$ such that $L(H) = \partial_u U + \partial_v V$, then $L(F(x,y;t)) = 0$ after integration over closed contours.
   Use creative telescoping to find L (as well as U and V).
3. Factor L as $L_2 \cdot P_1 \cdots P_t$, where L_2 has order ≤ 2 and the P_i have order 1.
4. Solve L_2 in terms of ${}_2F_1$s and deduce F.

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   Use creative telescoping to find L (as well as U and V).
   Works in practice with early evaluation $(x,y) = (1,1)$, but not for symbolic (x,y).
   Works also for $(0,0)$, $(x,0)$, and $(0,y)$!
3. Factor L as $L_2 \cdot P_1 \cdots P_t$, where L_2 has order ≤ 2 and the P_i have order 1.
4. Solve L_2 in terms of ${}_2F_1$s and deduce F.
5. For $F(x,y;t)$, run whole process for $F(0,0;t)$, $F(x,0;t)$, and $F(0,y;t)$, then substitute into kernel equation!

Hypergeometric Series Occurring in Explicit Expressions for $F(x,y;t)$

	$\mathfrak{S}$	occurring ${}_2F_1$	w		$\mathfrak{S}$	occurring ${}_2F_1$	w
1		${}_2F_1\left(\left.\begin{matrix}\frac{1}{2}\ \frac{1}{2}\\ 1\end{matrix}\right\| w\right)$	$16t^2$	11		${}_2F_1\left(\left.\begin{matrix}\frac{1}{2}\ \frac{1}{2}\\ 1\end{matrix}\right\| w\right)$	$\frac{16t^2}{4t^2+1}$
2		${}_2F_1\left(\left.\begin{matrix}\frac{1}{2}\ \frac{1}{2}\\ 1\end{matrix}\right\| w\right)$	$16t^2$	12		${}_2F_1\left(\left.\begin{matrix}\frac{1}{4}\ \frac{3}{4}\\ 1\end{matrix}\right\| w\right)$	$\frac{64t^3(2t+1)}{(8t^2-1)^2}$
3		${}_2F_1\left(\left.\begin{matrix}\frac{1}{4}\ \frac{3}{4}\\ 1\end{matrix}\right\| w\right)$	$\frac{64t^2}{(12t^2+1)^2}$	13		${}_2F_1\left(\left.\begin{matrix}\frac{1}{4}\ \frac{3}{4}\\ 1\end{matrix}\right\| w\right)$	$\frac{64t^2(t^2+1)}{(16t^2+1)^2}$
4		${}_2F_1\left(\left.\begin{matrix}\frac{1}{2}\ \frac{1}{2}\\ 1\end{matrix}\right\| w\right)$	$\frac{16t(t+1)}{(4t+1)^2}$	14		${}_2F_1\left(\left.\begin{matrix}\frac{1}{4}\ \frac{3}{4}\\ 1\end{matrix}\right\| w\right)$	$\frac{64t^2(t^2+t+1)}{(12t^2+1)^2}$
5		${}_2F_1\left(\left.\begin{matrix}\frac{1}{4}\ \frac{3}{4}\\ 1\end{matrix}\right\| w\right)$	$64t^4$	15		${}_2F_1\left(\left.\begin{matrix}\frac{1}{4}\ \frac{3}{4}\\ 1\end{matrix}\right\| w\right)$	$64t^4$
6		${}_2F_1\left(\left.\begin{matrix}\frac{1}{4}\ \frac{3}{4}\\ 1\end{matrix}\right\| w\right)$	$\frac{64t^3(t+1)}{(1-4t^2)^2}$	16		${}_2F_1\left(\left.\begin{matrix}\frac{1}{4}\ \frac{3}{4}\\ 1\end{matrix}\right\| w\right)$	$\frac{64t^3(t+1)}{(1-4t^2)^2}$
7		${}_2F_1\left(\left.\begin{matrix}\frac{1}{2}\ \frac{1}{2}\\ 1\end{matrix}\right\| w\right)$	$\frac{16t^2}{4t^2+1}$	17		${}_2F_1\left(\left.\begin{matrix}\frac{1}{3}\ \frac{2}{3}\\ 1\end{matrix}\right\| w\right)$	$27t^3$
8		${}_2F_1\left(\left.\begin{matrix}\frac{1}{4}\ \frac{3}{4}\\ 1\end{matrix}\right\| w\right)$	$\frac{64t^3(2t+1)}{(8t^2-1)^2}$	18		${}_2F_1\left(\left.\begin{matrix}\frac{1}{3}\ \frac{2}{3}\\ 1\end{matrix}\right\| w\right)$	$27t^2(2t+1)$
9		${}_2F_1\left(\left.\begin{matrix}\frac{1}{4}\ \frac{3}{4}\\ 1\end{matrix}\right\| w\right)$	$\frac{64t^2(t^2+1)}{(16t^2+1)^2}$	19		${}_2F_1\left(\left.\begin{matrix}\frac{1}{2}\ \frac{1}{2}\\ 1\end{matrix}\right\| w\right)$	$16t^2$
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Observation: Related to complete elliptic integrals, $E(\sqrt{w})$ and $K(\sqrt{w})$.

Computer Algebra Ingredients (Steps 2 to 4)

Well-studied algorithms

- Creative telescoping: [Zeilberger, 1990], [Lipshitz, 1988], [Almkvist & Zeilberger, 1990], [Takayama, 1990], [Wilf & Zeilberger, 1990] [Chyzak, 2000], [Koutschan, 2010], [Chen, Kauers, & Singer, 2012], [Bostan, Lairez, & Salvy, 2013], [Lairez, 2015]
- Factorization of ODE: [Beke, 1894], [Schwarz, 1989], [Grigor'ev, 1990], [Singer, 1996], [van Hoeij, 1997]
- Solving with 2F1: [Bostan, Chyzak, van Hoeij, & Pech, 2011], [Fang, van Hoeij, 2011], [Kunwar, van Hoeij, 2013], [Kunwar, 2014], [van Hoeij, Vidunas, 2015], [van Hoeij, Imamoglu, 2015]

Already combined for a simpler problem: Diagonal 3D Rook Paths [Bostan, Chyzak, van Hoeij, & Pech, 2011]

Problem: Determine the number a_n of paths from $(0,0,0)$ to (n,n,n) that use positive multiples of $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$.

Solution: $$G(x) = 1 + 6 \cdot \int_0^x \frac{{}_2F_1\left(\begin{matrix}1/3\ 2/3\\ 2\end{matrix}\,\middle|\, \frac{27w(2-3w)}{(1-4w)^3}\right)}{(1-4w)(1-64w)}\,dw.$$

Key Idea (Step 1): Encoding Positive Parts as Algebraic Residues

Problem: Definitions of residues and positive parts of rational functions?

$$\cdots - \frac{1}{w^3} - \frac{1}{w^2} - \frac{1}{w} \stackrel{?}{=} \frac{1}{1-w} \stackrel{?}{=} 1 + w + w^2 + \cdots$$

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$$-1 \stackrel{?}{=} \mathrm{Res}_w \frac{1}{1-w} \stackrel{?}{=} 0$$

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$$0 \stackrel{?}{=} [w^>] \frac{1}{1-w} \stackrel{?}{=} w + w^2 + \cdots$$

Key Idea (Step 1): Encoding Positive Parts as Algebraic Residues

New formula

$$F(a,b;t) = \operatorname{Res}_{x,y}\left[\frac{\bar{x}\bar{y}R(x,y;t)}{(x-a)(y-b)}\right]_{\Gamma_1} = \operatorname{Res}_{x,y}\left[\frac{R(\bar{x},\bar{y};t)}{(1-ax)(1-by)}\right]_{\Gamma_2}.$$

Interpretation [Aparicio-Monforte & Kauers, 2013]

- $\operatorname{Res}_{x,y}$ is linear on the vector space $\mathbb{Q}^{\mathbb{Z}^2}$;
- the rational functions $R(x,y;t)$ and $(x-a)^{-1}(y-b)^{-1}$ are expanded as a series with support in the cone $\Gamma_1 = \{x^i y^j t^n : i, |j| \leq n \geq 0\}$;
- the rational functions $R(\bar{x},\bar{y};t)$ and $(1-ax)^{-1}(1-by)^{-1}$ are expanded as a series with support the cone $\Gamma_2 = \{x^i y^j t^n : -i, |j| \leq n \geq 0\}$;
- a theory of series with support in a cone legitimates the product.

Link with creative telescoping [This work]

$$L(H) = \partial_u U + \partial_v V \implies L([H]_\Gamma) = 0$$

provided H, U, V admit expansions with respect to the same cone Γ.

Theorem

- In cases 1–19, $F(x,y;t)$ is transcendental since $F(0,0;t)$ is.
- In cases 1–16 and 19, $F(1,1;t)$ is transcendental.
- Specific simplifications prove algebraicity of $F(1,1;t)$ in cases 17–18.

Proof: Define $G = (P_1 \cdots P_t)(F)$ so that $L_2(G) = 0$.

- F is algebraic $\Longrightarrow$ G is algebraic.
- Computing a few coefficients of G shows that this is not 0 on all cases of interest.
- Applying Kovacic's algorithm to L_2 (order 2) or just computing exponential solutions (order 1) decides whether L_2 has nonzero algebraic solutions.

Conclusions

A succession of functional equations of several types

rec. relation on $f_{n;i,j}$ $\rightarrow$ kernel equation on $F(x,y;t)$ $\rightarrow$ ODE on $F(1,1;t)$

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- Three kinds of conjectures now proved:
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Wanted

Better understanding of the systematic emergence of elliptic integrals