LOZENGE TILINGS OF DOUBLY-INTRUDED HEXAGONS

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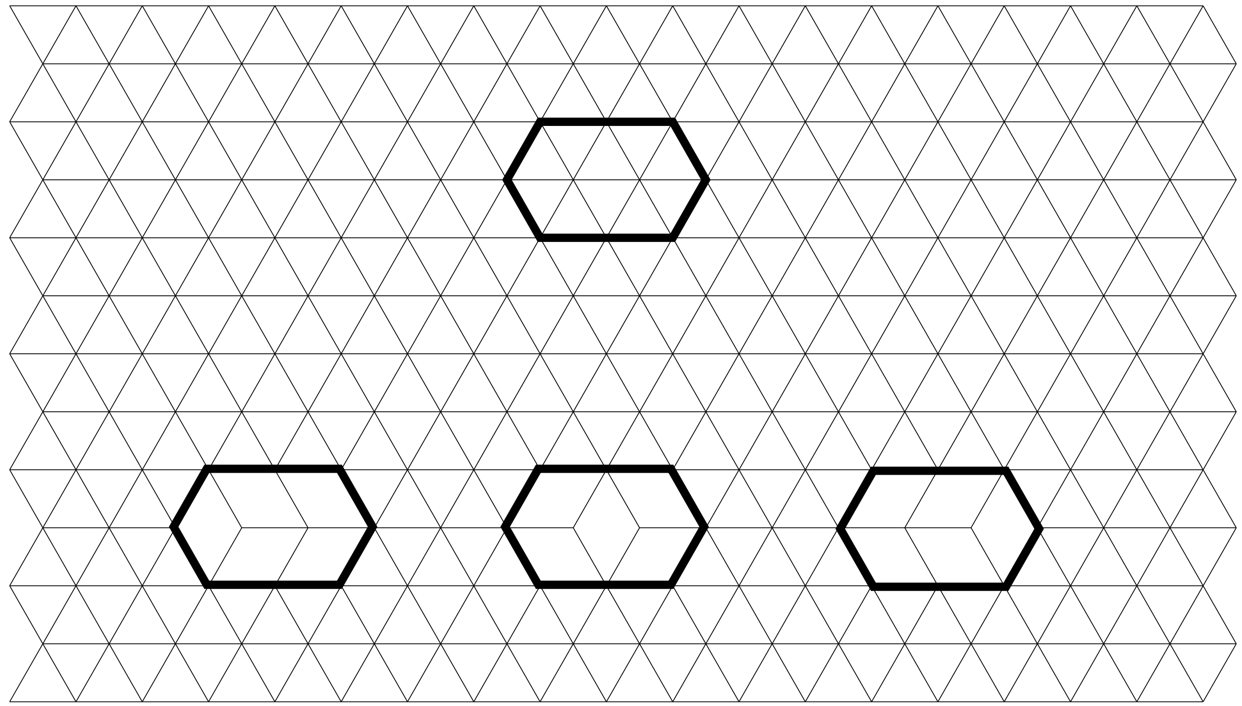
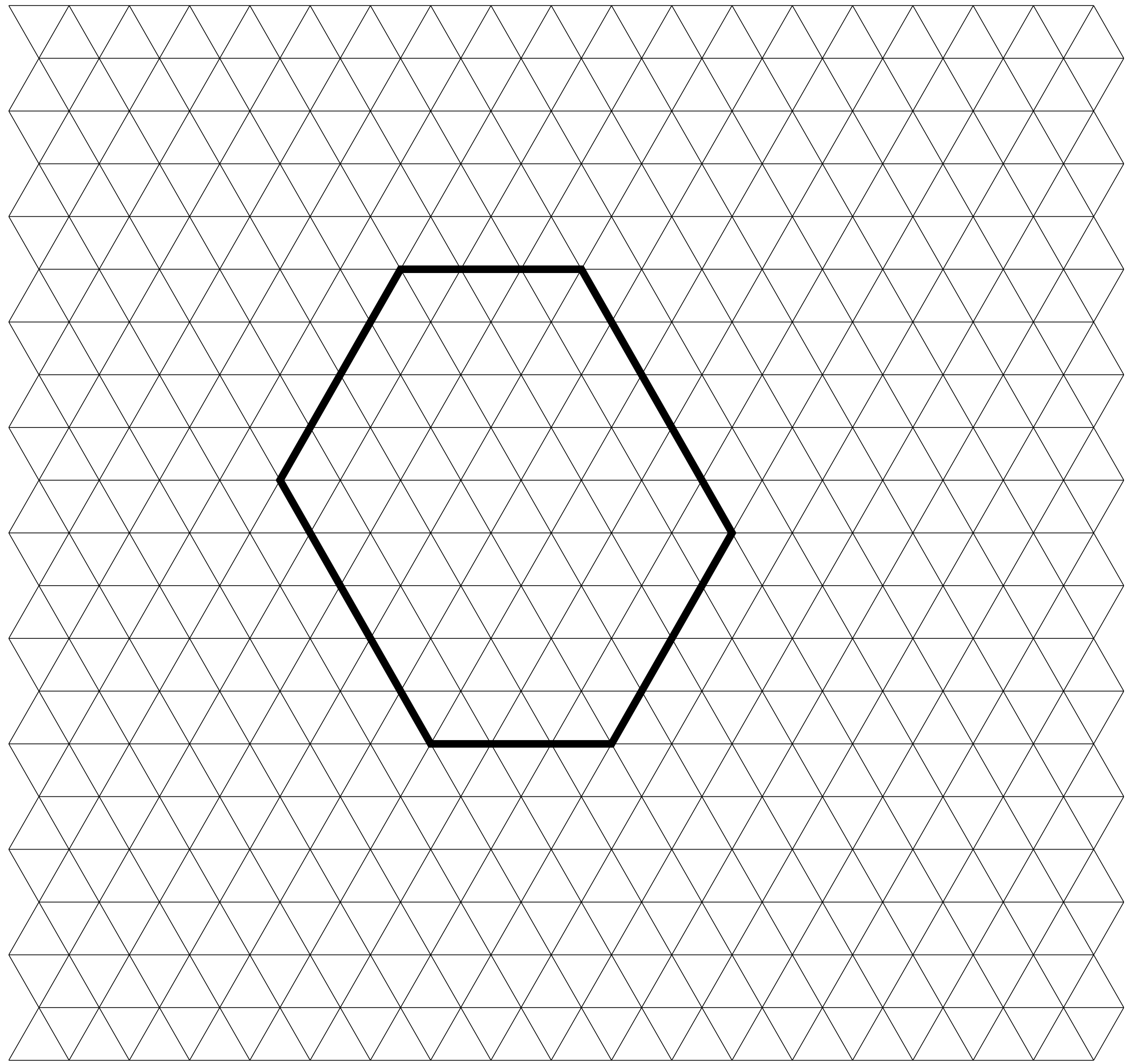


$H_{1,2,1}$ and its three lozenge tilings

$$\mathrm{M}(H_{1,2,1}) = 3$$



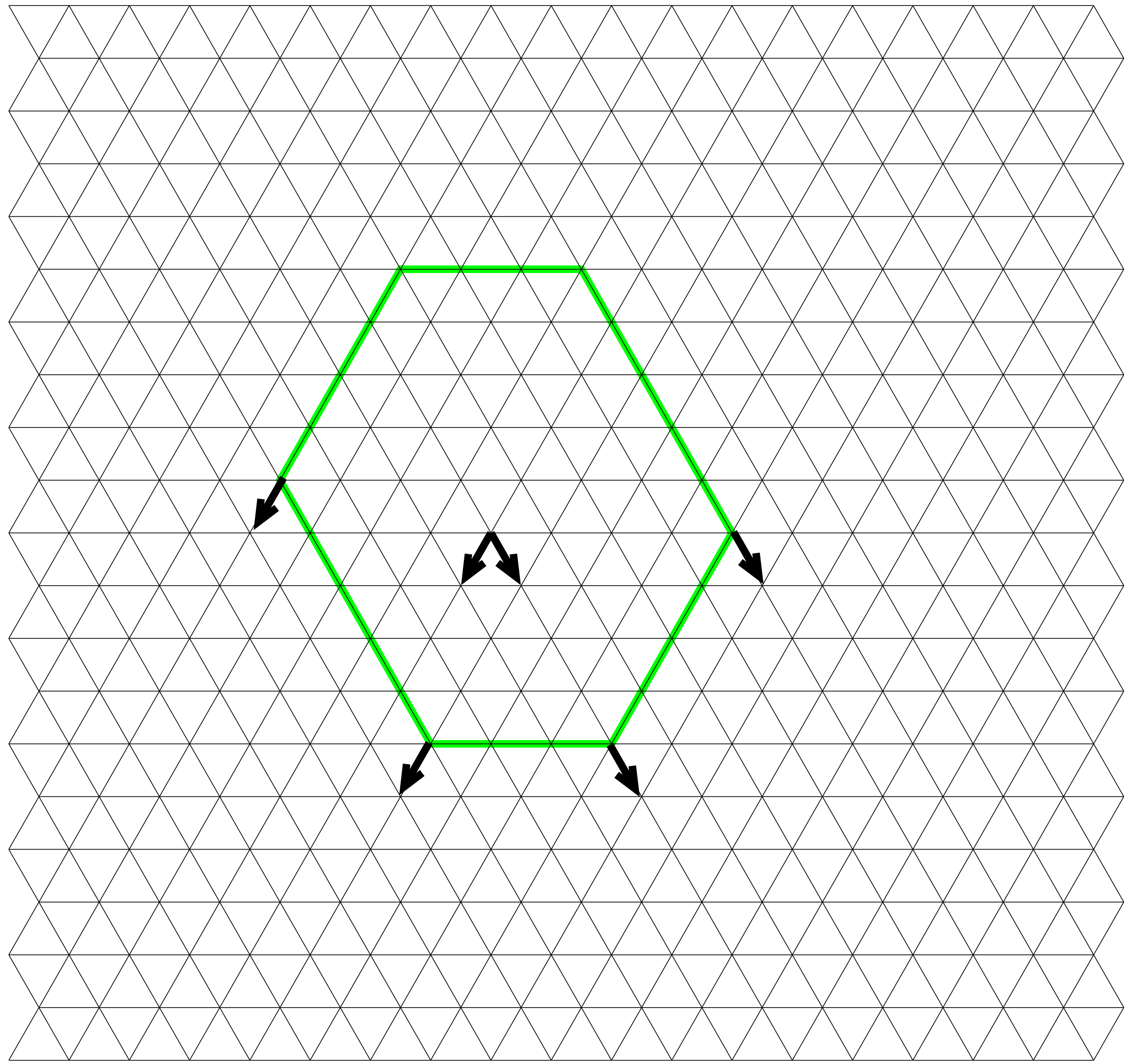
The hexagon $H_{3,5,4}$

$$H(n) := 0!\,1!\,\cdots\,(n-1)!$$

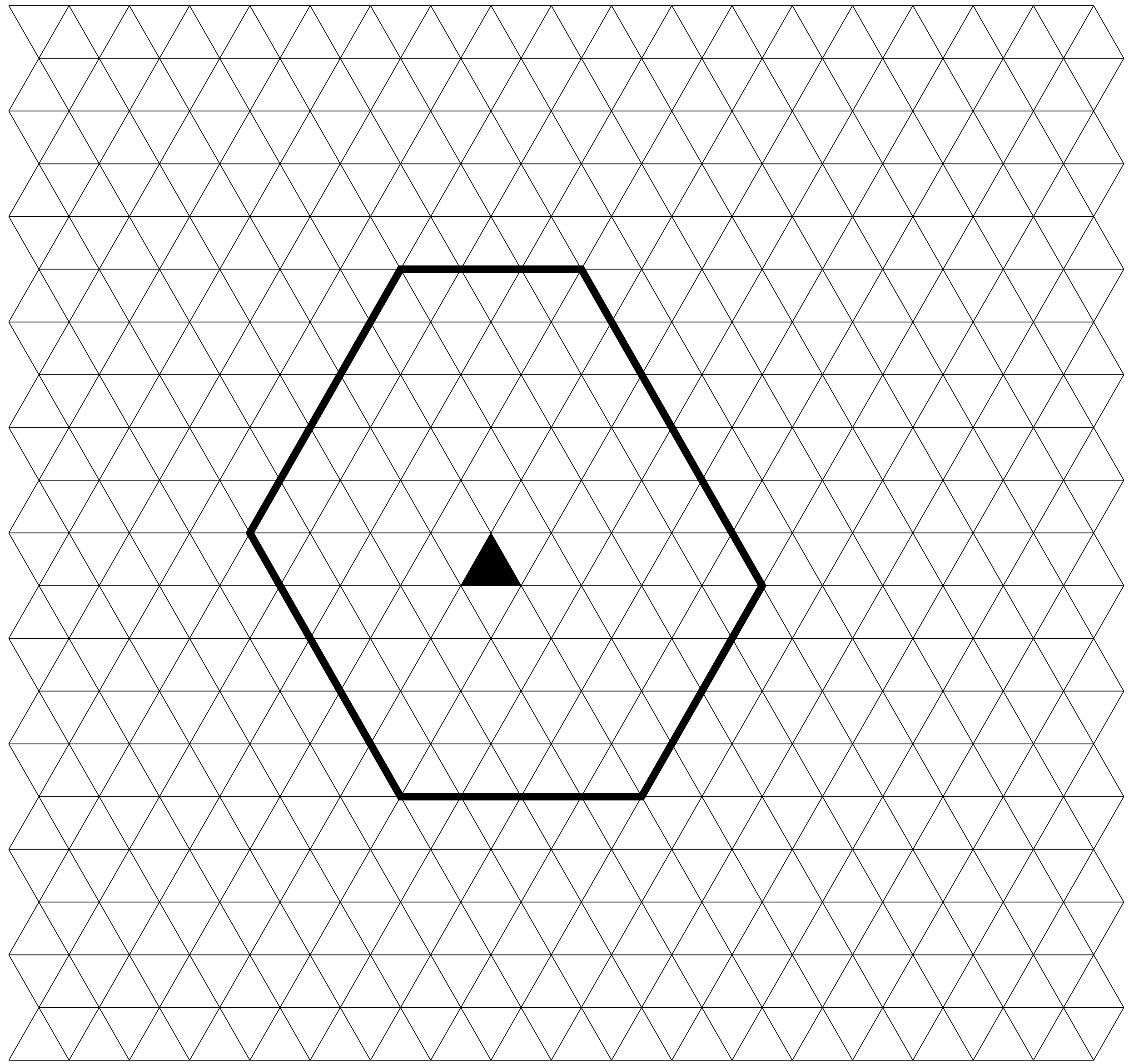
Macmahon's Theorem ($\sim$ 1900):

$$\mathrm{M}(H_{x,y,z}) = \frac{\mathrm{H}(x)\,\mathrm{H}(y)\,\mathrm{H}(z)\,\mathrm{H}(x+y+z)}{\mathrm{H}(x+y)\,\mathrm{H}(x+z)\,\mathrm{H}(y+z)}$$

$$H(n) := 0!\,1!\,\cdots\,(n-1)!$$

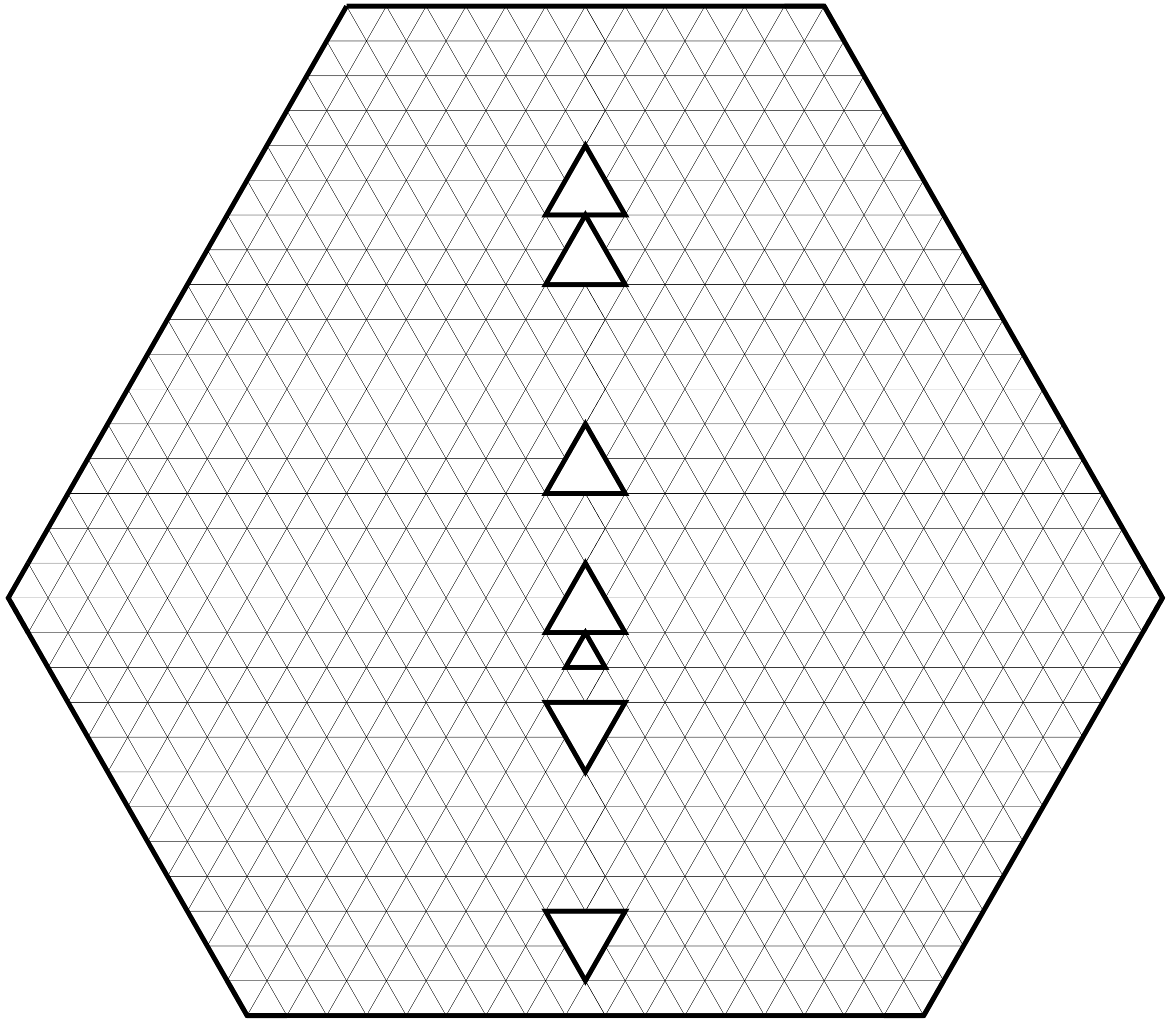


Creating a unit hole



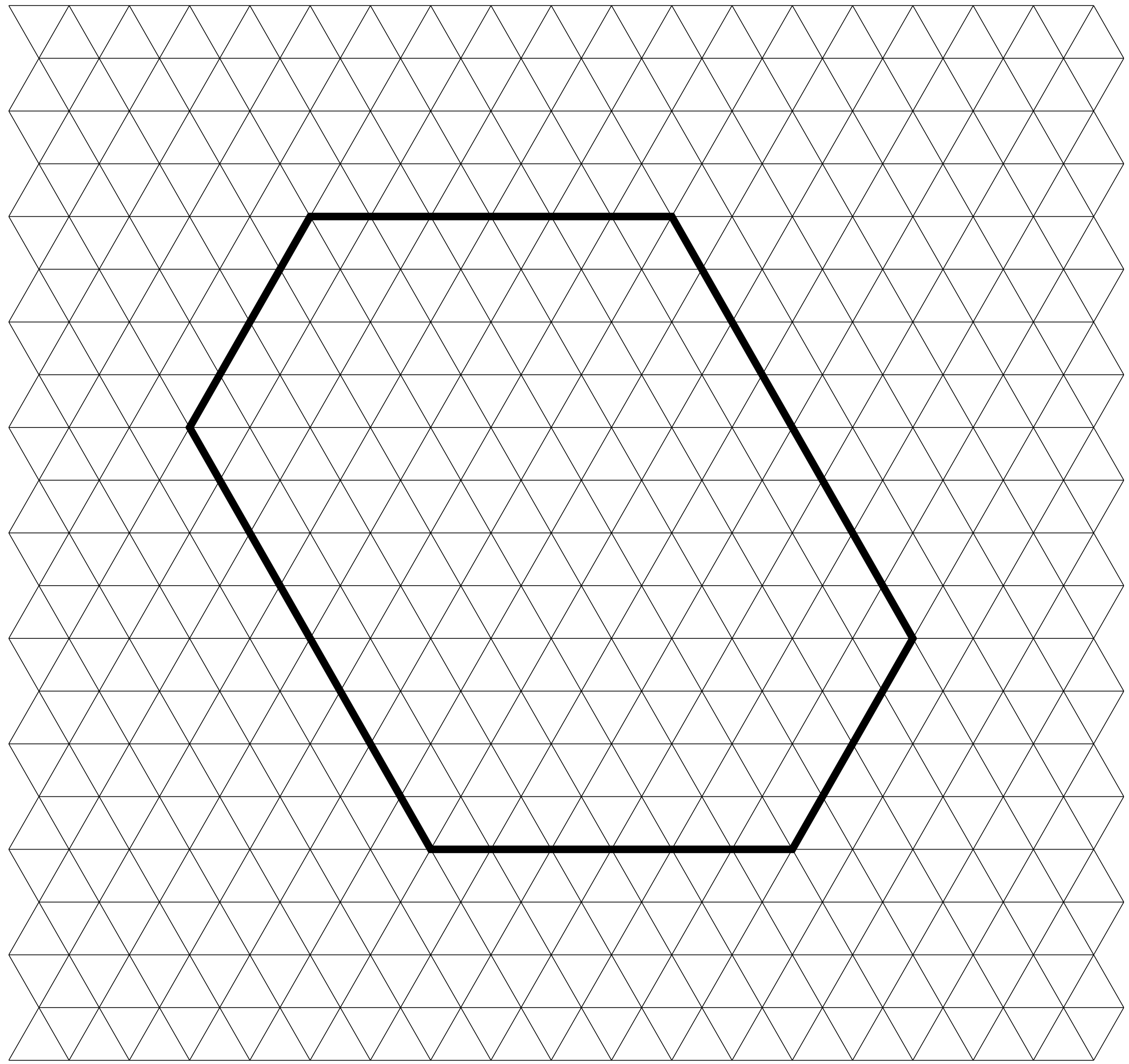
How many tilings?

1000000

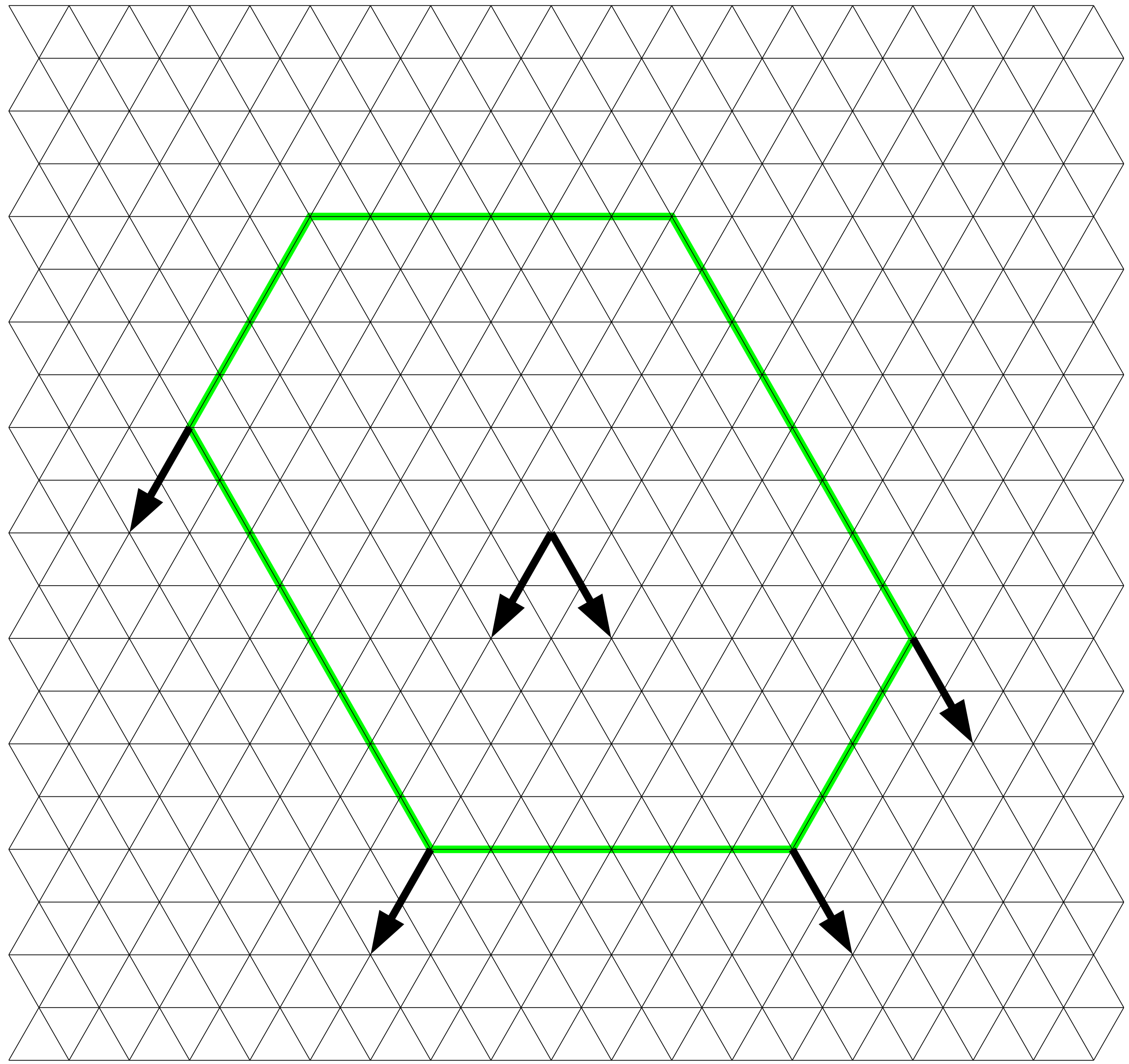


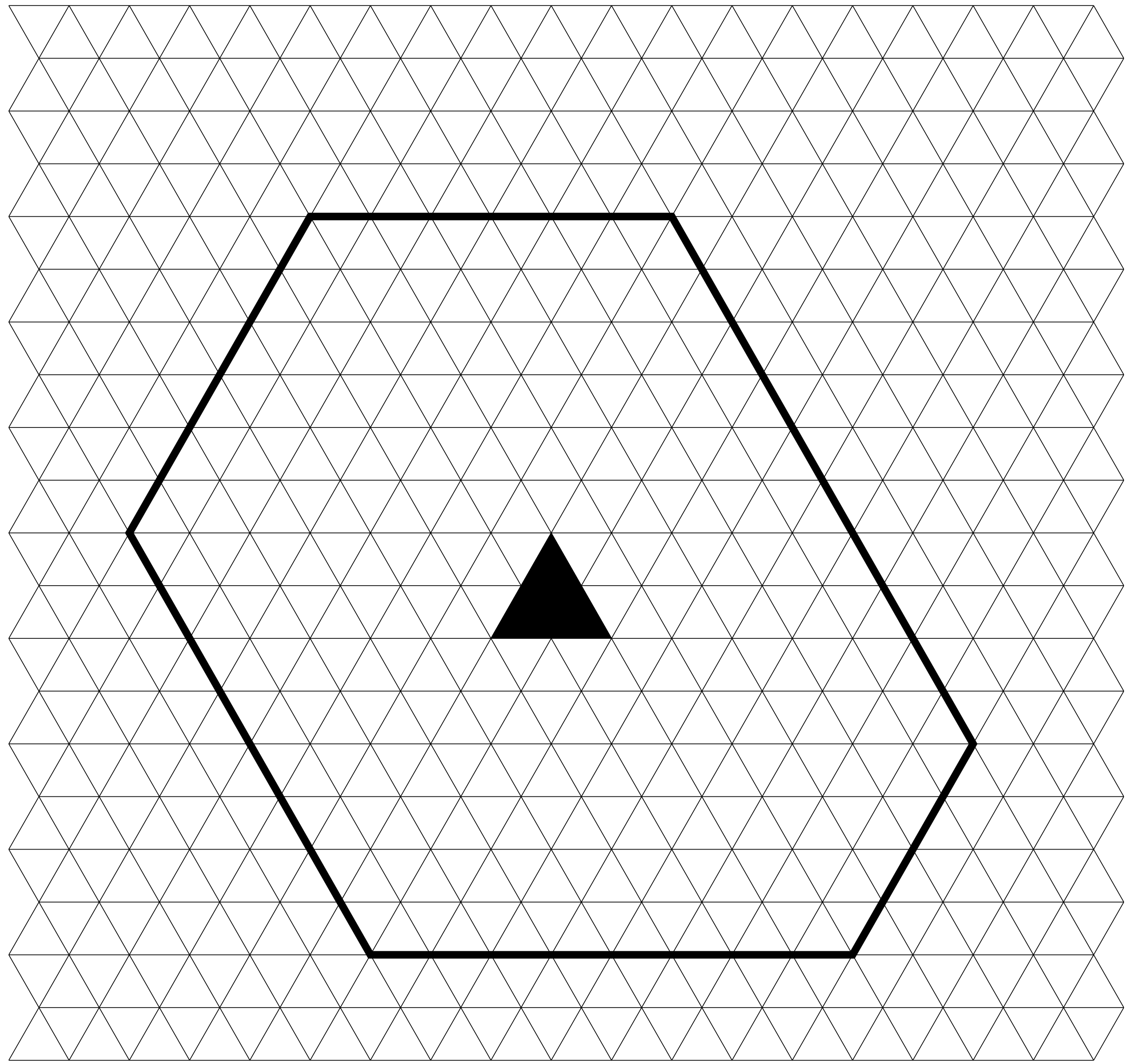
Collinear holes in a symmetric hexagon

Theorem (C., 1998). The number of lozenge tilings is given by an explicit product formula.

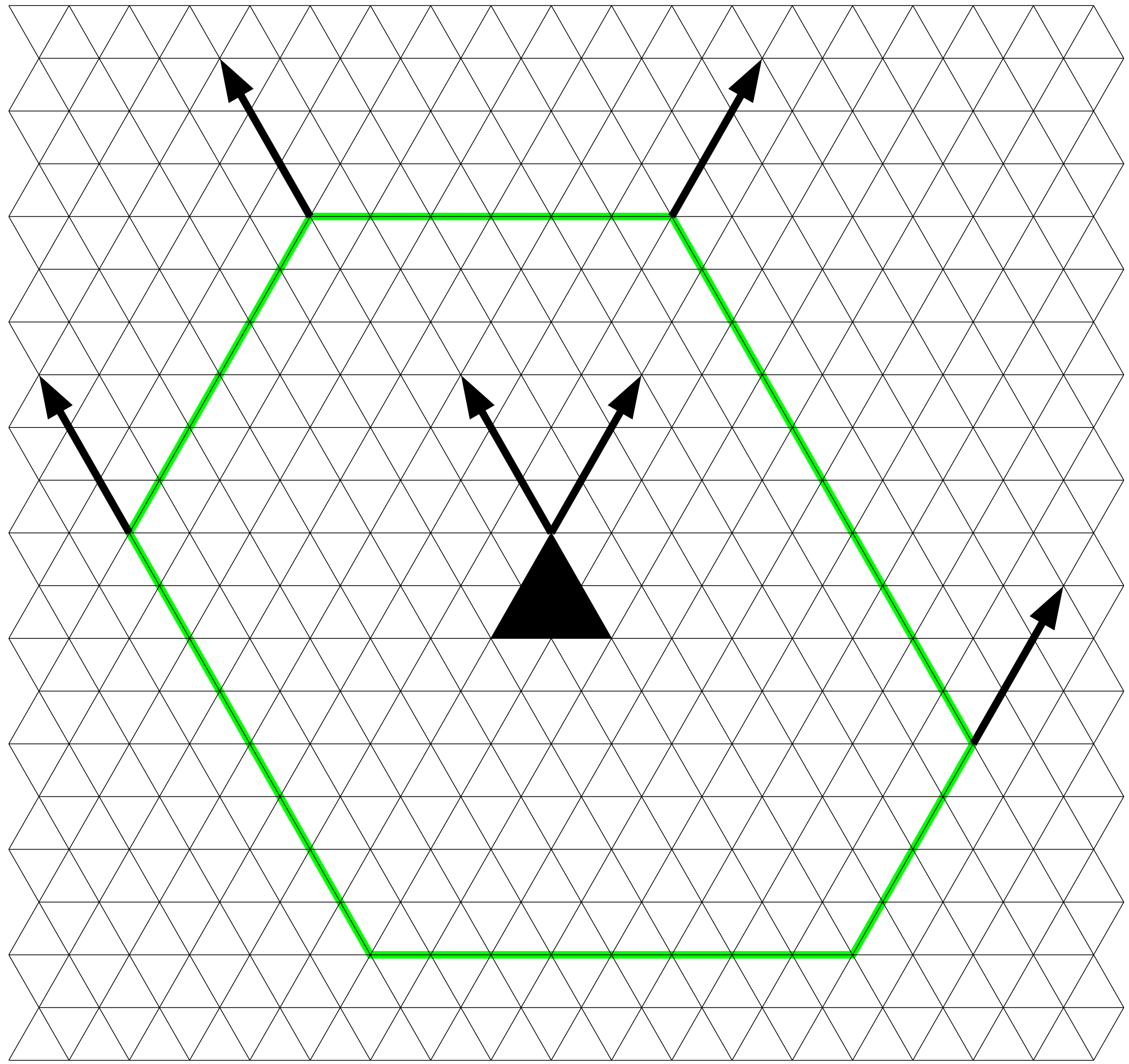


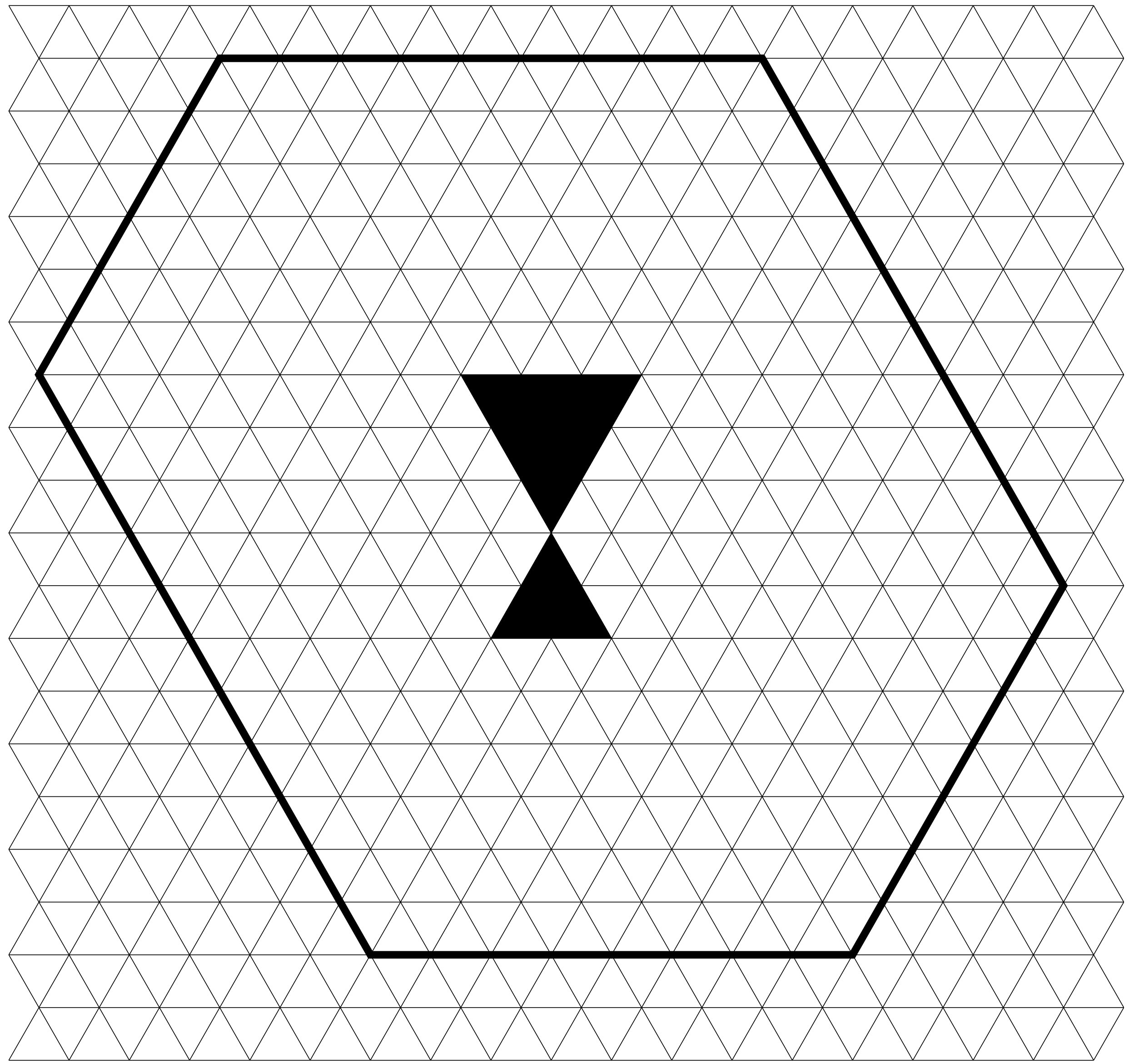
The hexagon $H_{6,8,4}$



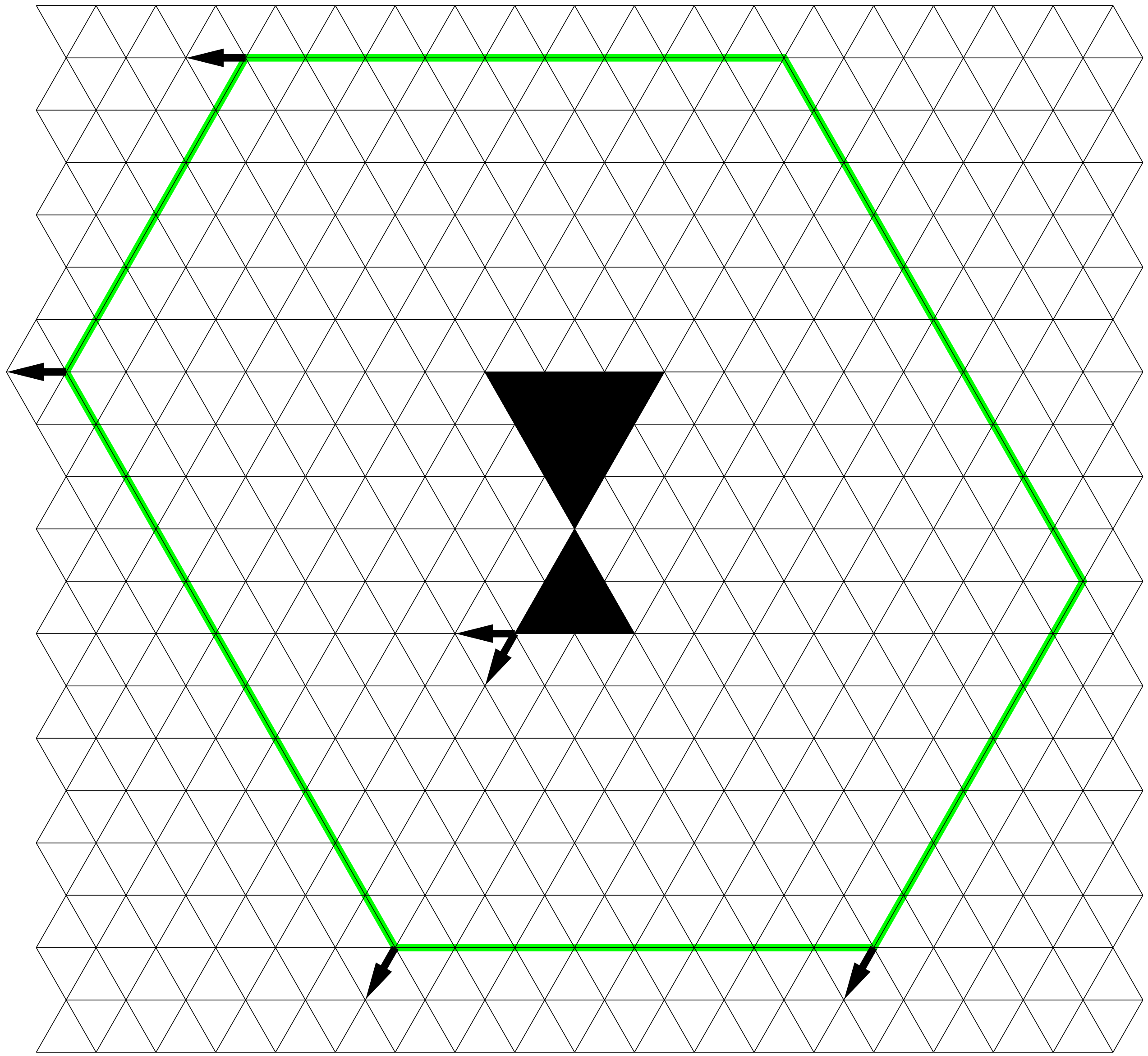


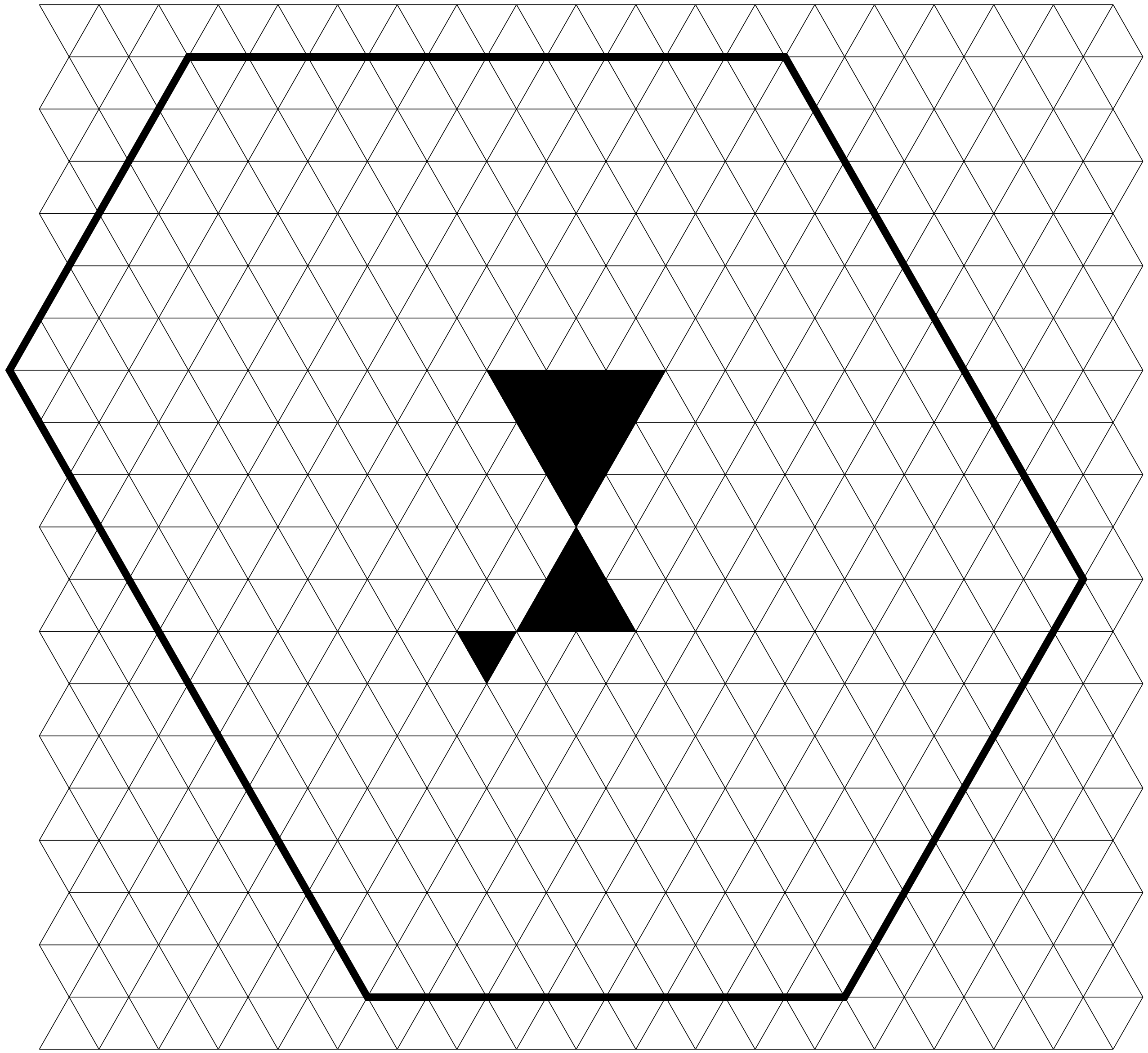
The cored hexagon $C_{6,8,4}(2)$



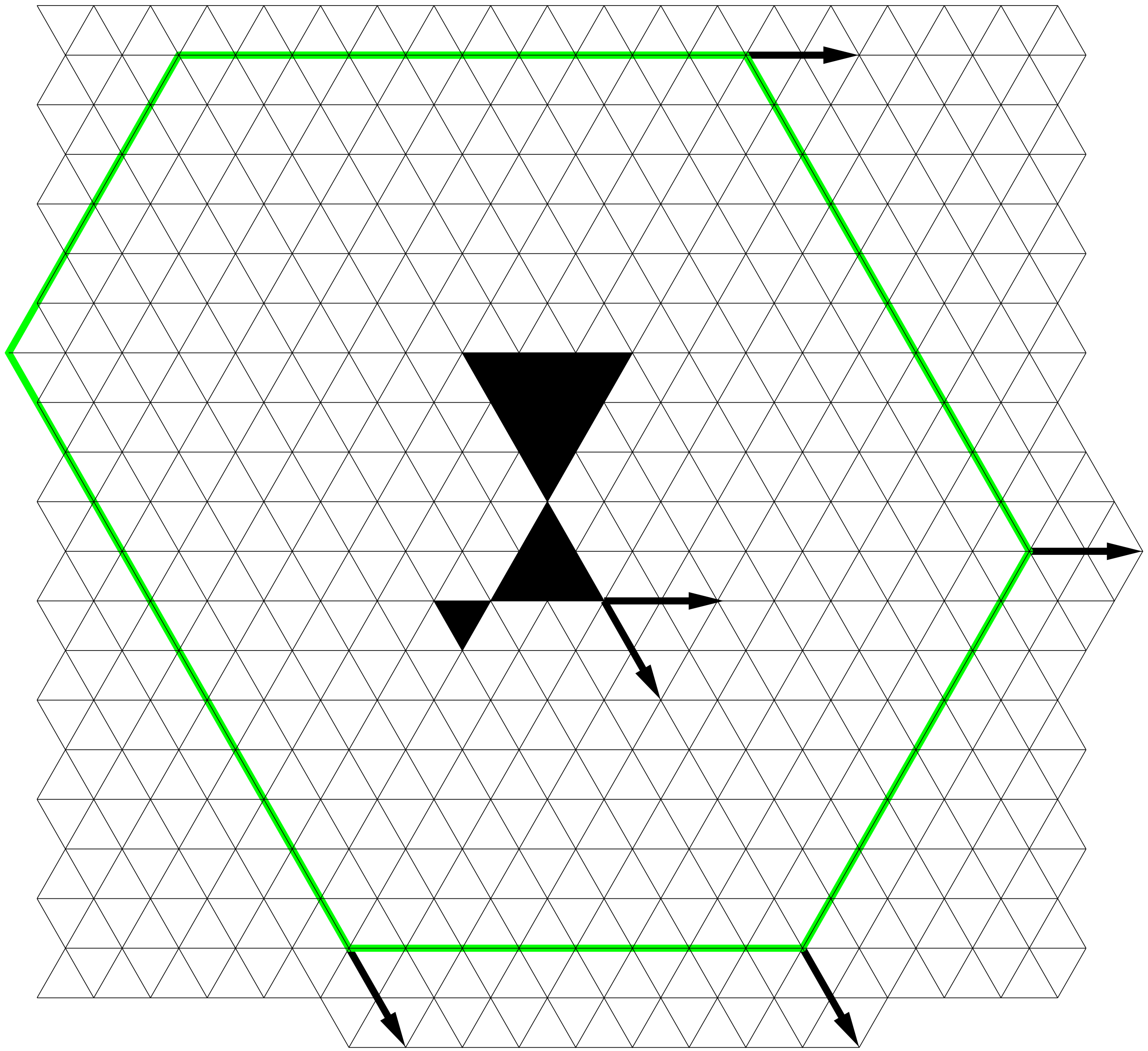


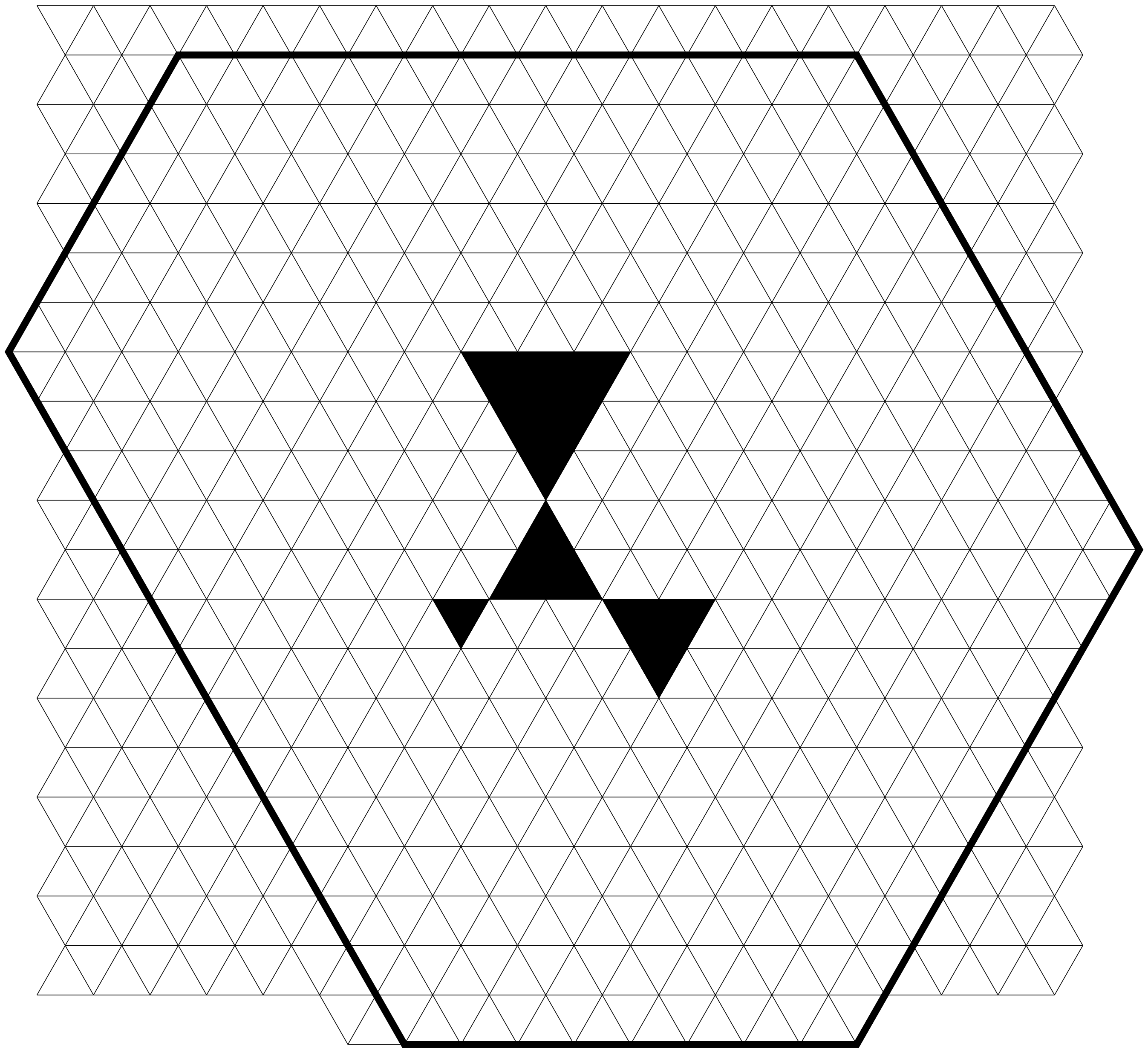
The first lobe





The first two lobes





The S-cored hexagon $SC_{6,8,4}(3,1,2,2)$

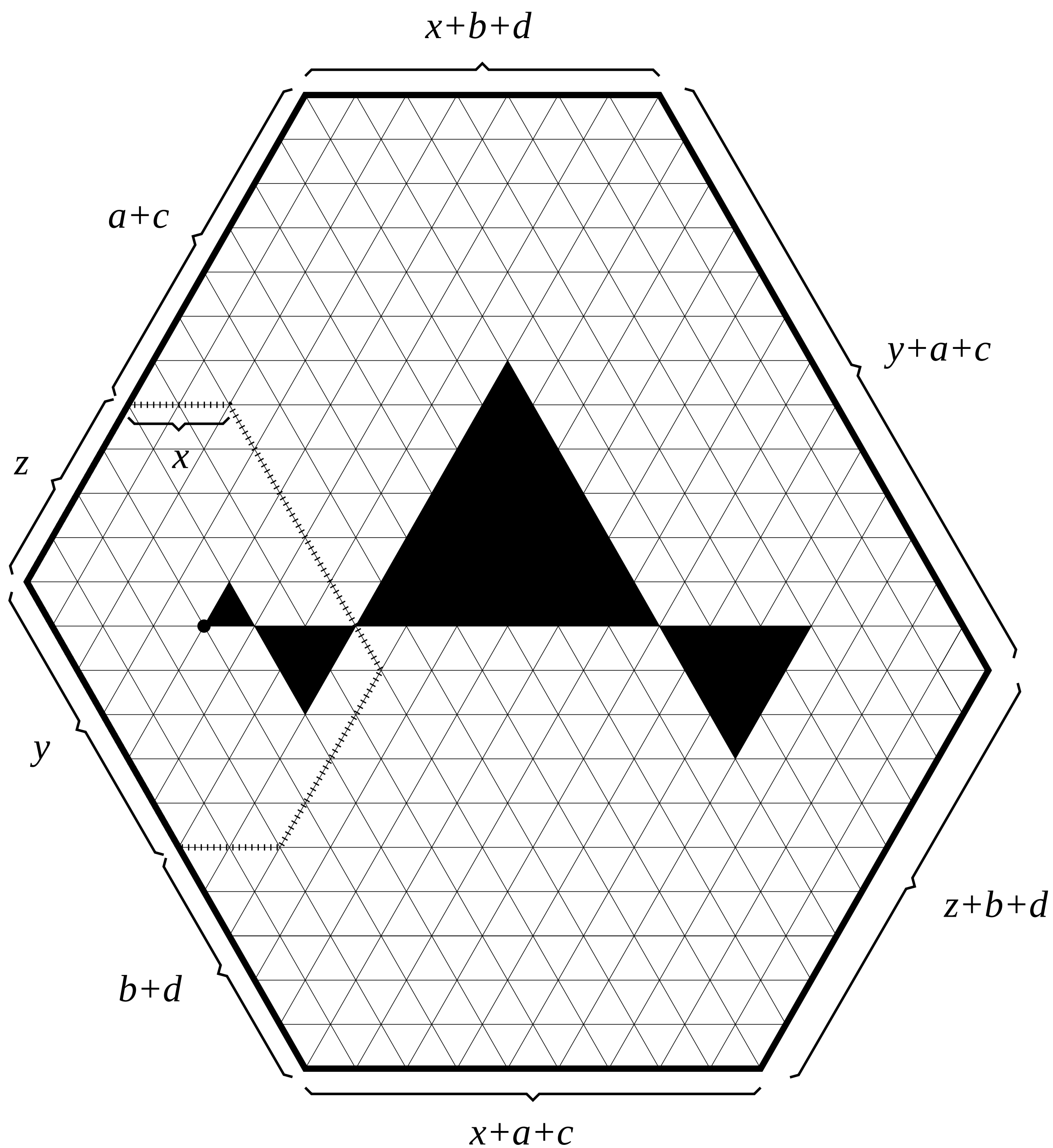
Define

$$\mathrm{H}(n) := \prod_{k=0}^{n} \Gamma(k),$$

$$\mathrm{H}(n + \frac{1}{2}) := \prod_{k=0}^{n} \Gamma(k + \frac{1}{2})$$

THEOREM 1 (C. AND KRATTENTHALER, 2013). *Let x, y, z, a, b, c and m be nonnegative integers.*
If y and z have the same parity, we have:

$$\mathrm{M}(SC_{x,y,z}(a,b,c,m)) = \frac{\mathrm{H}(m)^3\ \mathrm{H}(a)\ \mathrm{H}(b)\ \mathrm{H}(c)}{\mathrm{H}(m+a)\ \mathrm{H}(m+b)\ \mathrm{H}(m+c)}$$
$$\times \frac{\mathrm{H}(\lfloor \frac{x+y}{2} \rfloor + m + a + b)\ \mathrm{H}(\lceil \frac{x+z}{2} \rceil + m + a + c)\ \mathrm{H}(\frac{y+z}{2} + m + b + c)}{\mathrm{H}(\lceil \frac{x+y}{2} \rceil + m + c)\ \mathrm{H}(\lfloor \frac{x+z}{2} \rfloor + m + b)\ \mathrm{H}(\frac{y+z}{2} + m + a)}$$
$$\times \frac{\mathrm{H}(\lceil \frac{x+y}{2} \rceil + c)\ \mathrm{H}(\lfloor \frac{x+z}{2} \rfloor + b)\ \mathrm{H}(\frac{y+z}{2} + a)}{\mathrm{H}(\lfloor \frac{x+y}{2} \rfloor + a + b)\ \mathrm{H}(\lceil \frac{x+z}{2} \rceil + a + c)\ \mathrm{H}(\frac{y+z}{2} + b + c)}$$
$$\times \frac{\mathrm{H}(x+m+a+b+c)\,\mathrm{H}(y+m+a+b+c)}{\mathrm{H}(x+y+m+a+b+c)\,\mathrm{H}(x+z+m+a+b+c)}$$
$$\times \frac{\mathrm{H}(z+m+a+b+c)\,\mathrm{H}(x+y+z+m+a+b+c)}{\mathrm{H}(y+z+m+a+b+c)}$$
$$\times \frac{\mathrm{H}(\lceil \frac{x+y+z}{2} \rceil + m+a+b+c)\,\mathrm{H}(\lfloor \frac{x+y+z}{2} \rfloor + m+a+b+c)}{\mathrm{H}(\lfloor \frac{x+y}{2} \rfloor + m+a+b+c)\,\mathrm{H}(\lceil \frac{x+z}{2} \rceil + m+a+b+c)\,\mathrm{H}(\frac{y+z}{2} + m+a+b+c)}$$
$$\times \frac{\mathrm{H}(\lceil \frac{x}{2} \rceil)\,\mathrm{H}(\lfloor \frac{x}{2} \rfloor)\,\mathrm{H}(\lceil \frac{y}{2} \rceil)}{\mathrm{H}(\lceil \frac{x}{2} \rceil + \frac{m+a+b+c}{2})\,\mathrm{H}(\lfloor \frac{x}{2} \rfloor + \frac{m+a+b+c}{2})\,\mathrm{H}(\lceil \frac{y}{2} + \frac{m+a+b+c}{2})}$$
$$\times \frac{\mathrm{H}(\lfloor \frac{y}{2} \rfloor)\,\mathrm{H}(\lceil \frac{z}{2} \rceil)\,\mathrm{H}(\lfloor \frac{z}{2} \rfloor)}{\mathrm{H}(\lfloor \frac{y}{2} \rfloor + \frac{m+a+b+c}{2})\,\mathrm{H}(\lceil \frac{z}{2} \rceil + \frac{m+a+b+c}{2})\,\mathrm{H}(\lfloor \frac{z}{2} \rfloor + \frac{m+a+b+c}{2})}$$
$$\times \frac{\mathrm{H}(\frac{m+a+b+c}{2})^2\ \mathrm{H}(\lceil \frac{x+y}{2} \rceil + \frac{m+a+b+c}{2})\,\mathrm{H}(\lfloor \frac{x+y}{2} \rfloor + \frac{m+a+b+c}{2})}{\mathrm{H}(\lceil \frac{x+y+z}{2} \rceil + \frac{m+a+b+c}{2})\,\mathrm{H}(\lfloor \frac{x+y+z}{2} \rfloor + \frac{m+a+b+c}{2})\,\mathrm{H}(\lceil \frac{x+y}{2} \rceil)}$$
$$\times \frac{\mathrm{H}(\lceil \frac{x+z}{2} \rceil + \frac{m+a+b+c}{2})\,\mathrm{H}(\lfloor \frac{x+z}{2} \rfloor + \frac{m+a+b+c}{2})\,\mathrm{H}(\frac{y+z}{2} + \frac{m+a+b+c}{2})^2}{\mathrm{H}(\lfloor \frac{x+z}{2} \rfloor)\,\mathrm{H}(\frac{y+z}{2})}.$$



A hexagon with a *fern* removed from its center: $FC_{x,y,z}(a,b,c,d)$.

Set

$$s(b_1, b_2, \ldots, b_{2l}) = s(b_1, b_2, \ldots, b_{2l-1}) = \frac{\prod_{1\le i\le j\le 2l-1,\ j-i+1 \text{ odd}} \mathrm{H}(b_i + b_{i+1} + \cdots + b_j)}{\prod_{1\le i\le j\le 2l-1,\ j-i+1 \text{ even}} \mathrm{H}(b_i + b_{i+1} + \cdots + b_j)} \times \frac{1}{\mathrm{H}(b_1 + b_3 + \cdots + b_{2l-1})}.$$

THEOREM 2 (C., 2017). *Let x, y, z and $a_1, \dots, a_k$ be non-negative integers. Then the number of lozenge tilings of the F-cored hexagon $FC_{x,y,z}(a_1, \dots, a_k)$ is given by*

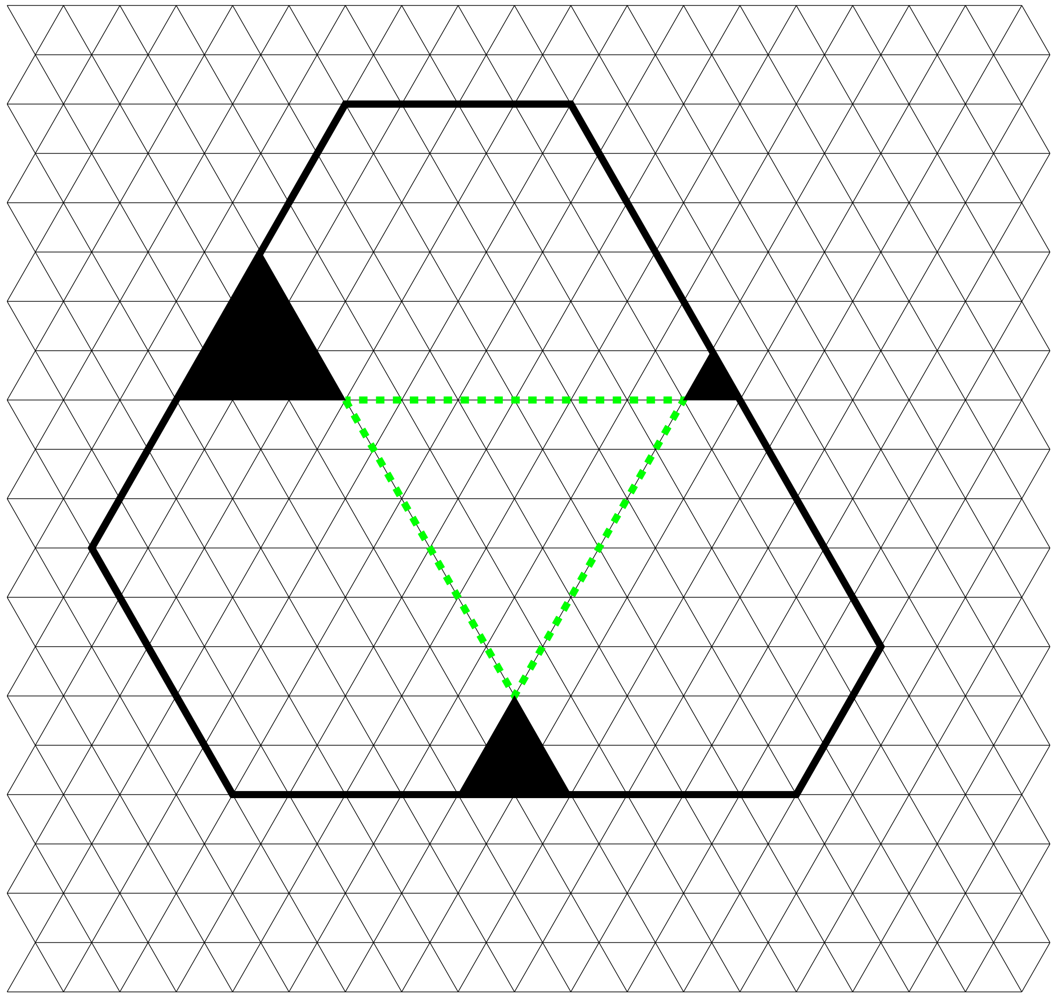
$$\frac{\mathrm{M}(FC_{x,y,z}(a_1,\dots,a_k))}{\mathrm{M}(FC_{x,y,z}(a_1+a_3+a_5+\cdots, a_2+a_4+a_6+\cdots))} = s(a_1,\dots,a_{k-1})s(a_2,\dots,a_k)$$

$$\times \frac{\mathrm{H}(\lfloor\frac{x+z}{2}\rfloor + a_1+a_3+\cdots)}{\mathrm{H}(\lfloor\frac{x+y}{2}\rfloor + a_1+a_3+\cdots)} \frac{\mathrm{H}(\lceil\frac{x+z}{2}\rceil + a_2+a_4+\cdots)}{\mathrm{H}(\lceil\frac{x+y}{2}\rceil + a_2+a_4+\cdots)}$$

$$\times \prod_{1\leq 2i+1\leq k} \frac{\mathrm{H}(\lfloor\frac{x+y}{2}\rfloor + a_1+\cdots+a_{2i+1})}{\mathrm{H}(\lfloor\frac{x+z}{2}\rfloor + a_1+\cdots+a_{2i+1})} \frac{\mathrm{H}(\lceil\frac{x+y}{2}\rceil + \overline{a_1+\cdots+a_{2i+1}})}{\mathrm{H}(\lceil\frac{x+z}{2}\rceil + \overline{a_1+\cdots+a_{2i+1}})}$$

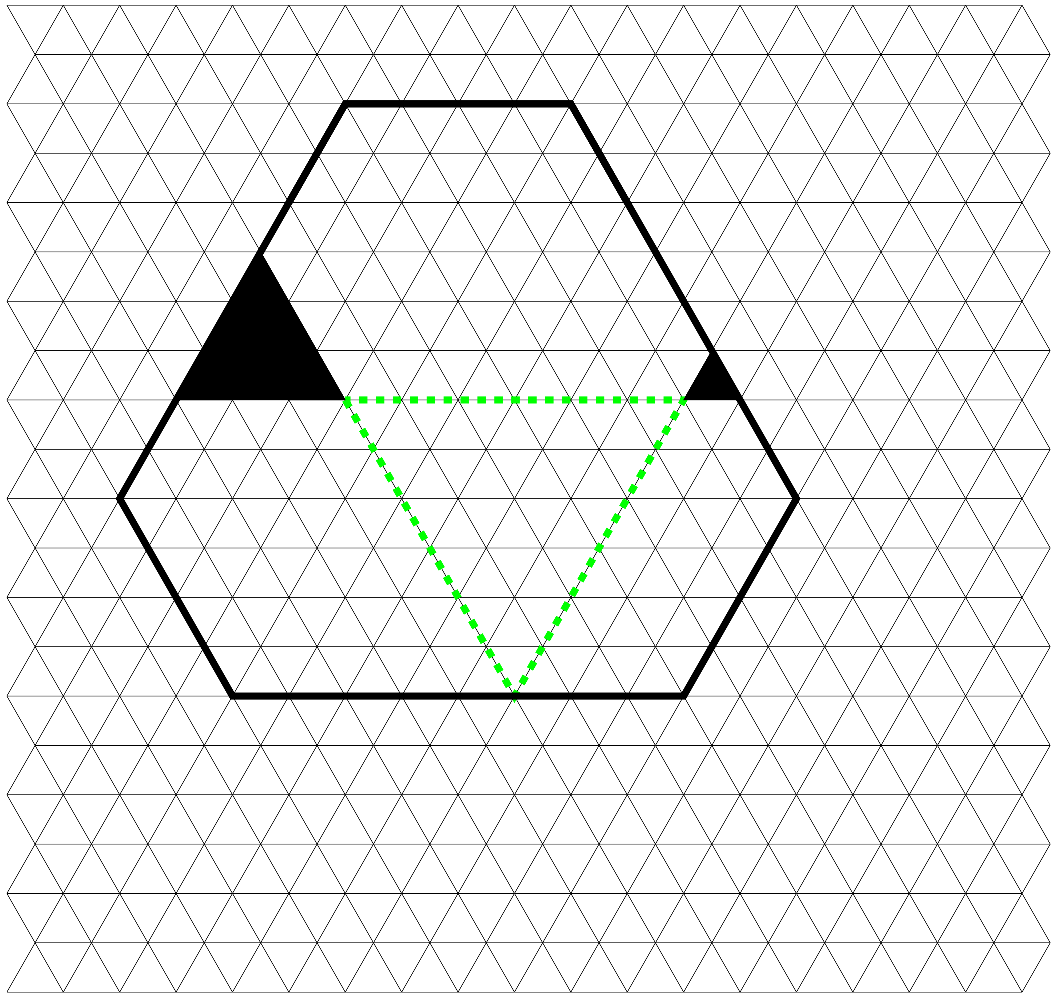
$$\times \prod_{1<2i<k} \frac{\mathrm{H}(\lfloor\frac{x+z}{2}\rfloor + a_1+\cdots+a_{2i})}{\mathrm{H}(\lfloor\frac{x+y}{2}\rfloor + a_1+\cdots+a_{2i})} \frac{\mathrm{H}(\lceil\frac{x+z}{2}\rceil + \overline{a_1+\cdots+a_{2i}})}{\mathrm{H}(\lceil\frac{x+y}{2}\rceil + \overline{a_1+\cdots+a_{2i}})},$$

where $\overline{a_1+\cdots+a_i}$ stands for $a_{i+1}+\cdots+a_k$.

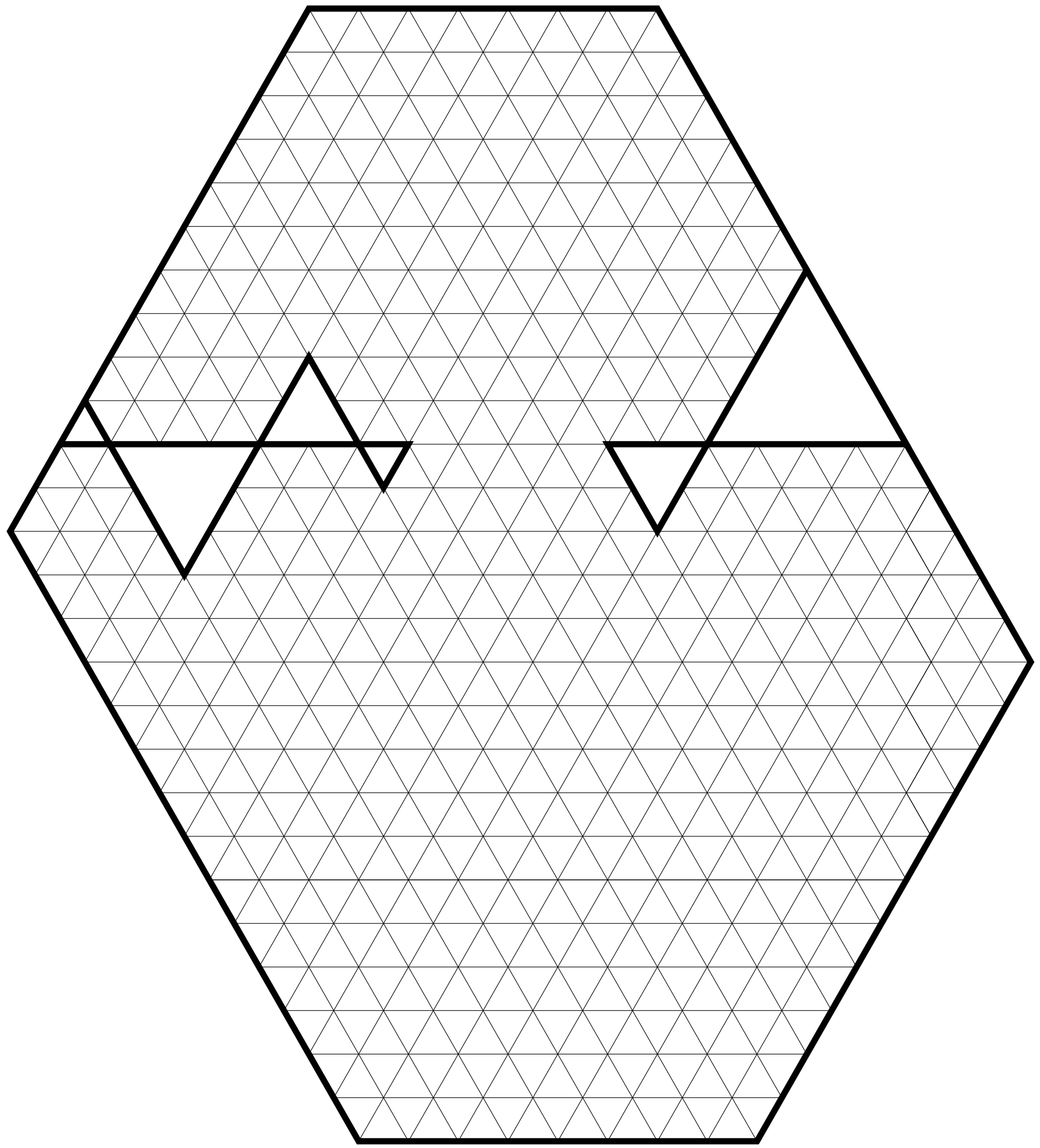


Hexagon with three dents

T. Lai (2015): Product formula for number of lozenge tilings of this family of regions.



A special case: Hexagon with two level dents



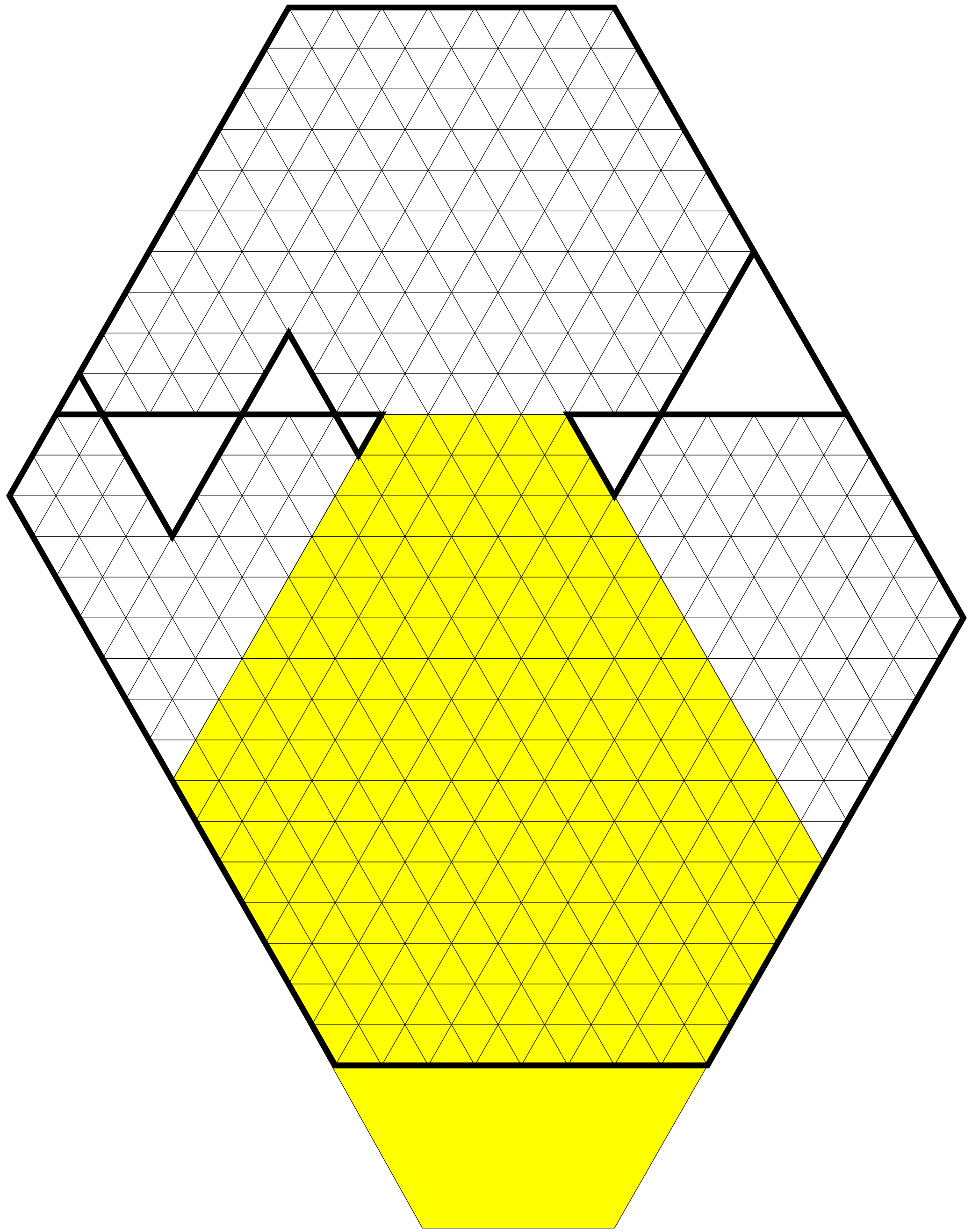
Generalization: Two level ferns removed

$$
\begin{aligned}
a &:= a_1 + a_2 + a_3 + \cdots \\
b &:= b_1 + b_2 + b_3 + \cdots \\
o_a &:= a_1 + a_3 + a_5 + \cdots \\
e_a &:= a_2 + a_4 + a_6 + \cdots \\
o_b &:= b_1 + b_3 + b_5 + \cdots \\
e_b &:= b_2 + b_4 + b_6 + \cdots
\end{aligned}
$$

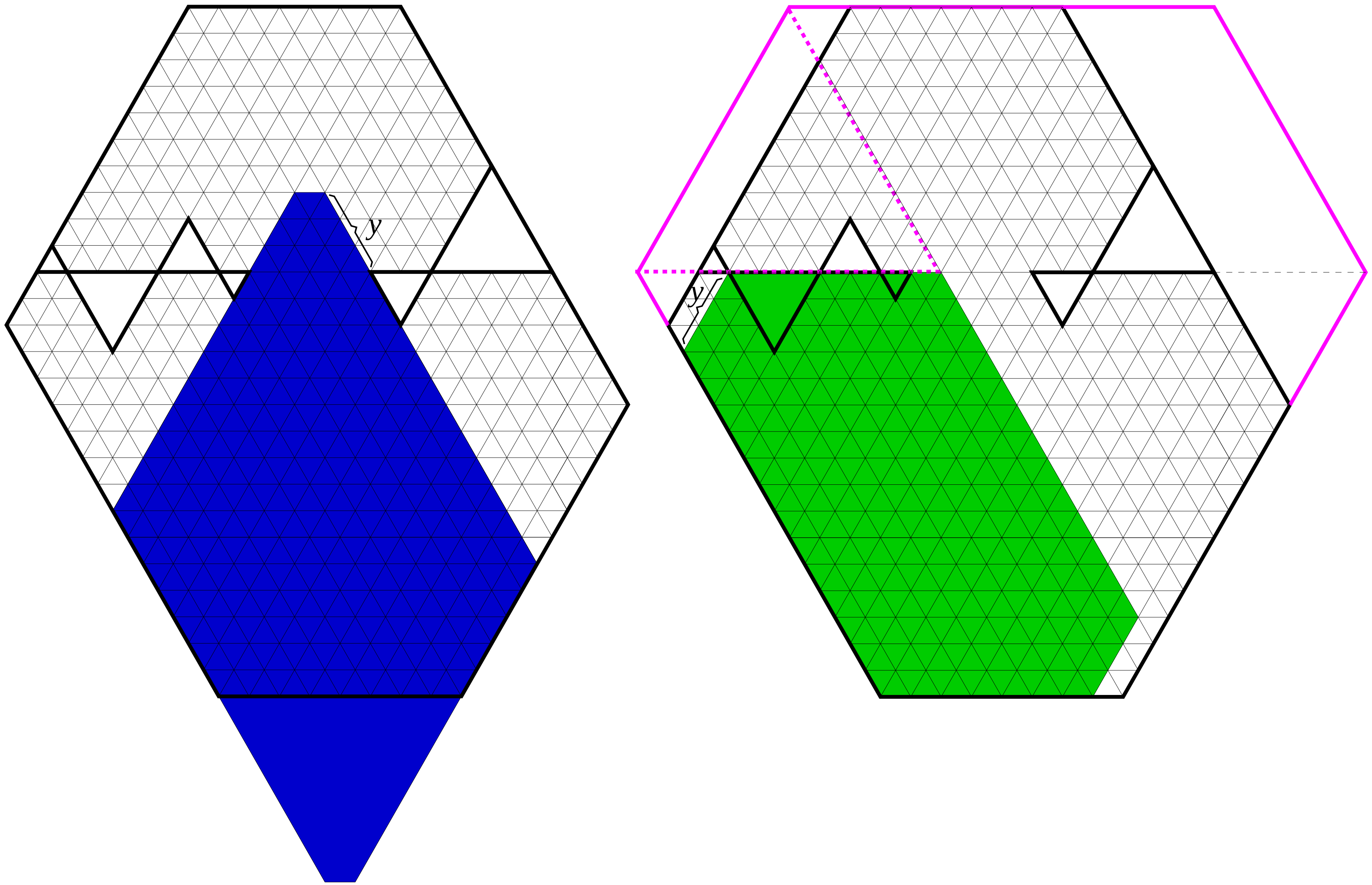
$$P(x, y, z) = \mathrm{M}(H_{x,y,z}).$$

THEOREM 3 (C. AND LAI, 2019).

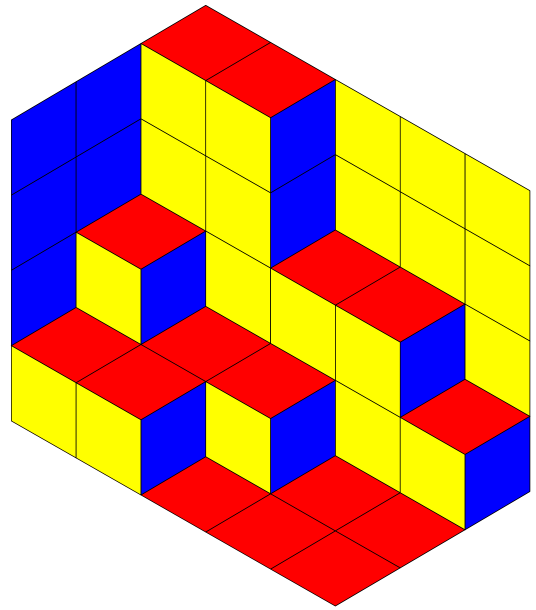
$$
\begin{aligned}
\mathrm{M}(P_{x,y,z,t}(a_1, \ldots, a_{2l}; b_1, b_2, \ldots, b_{2k})) = \frac{P(a + x + y, b + y + t, z)P(e_a + e_b + x + t, o_a + o_b, y)}{P(a + x, b + t, y + z)} \\
\times s(a_1, \ldots, a_{2l-1}, a_{2l} + y + z + b_{2k}, b_{2k-1}, \ldots, b_1) \\
\times s(x, a_1, \ldots, a_{2l}, y + z, b_{2k}, \ldots, b_1, t).
\end{aligned}
$$



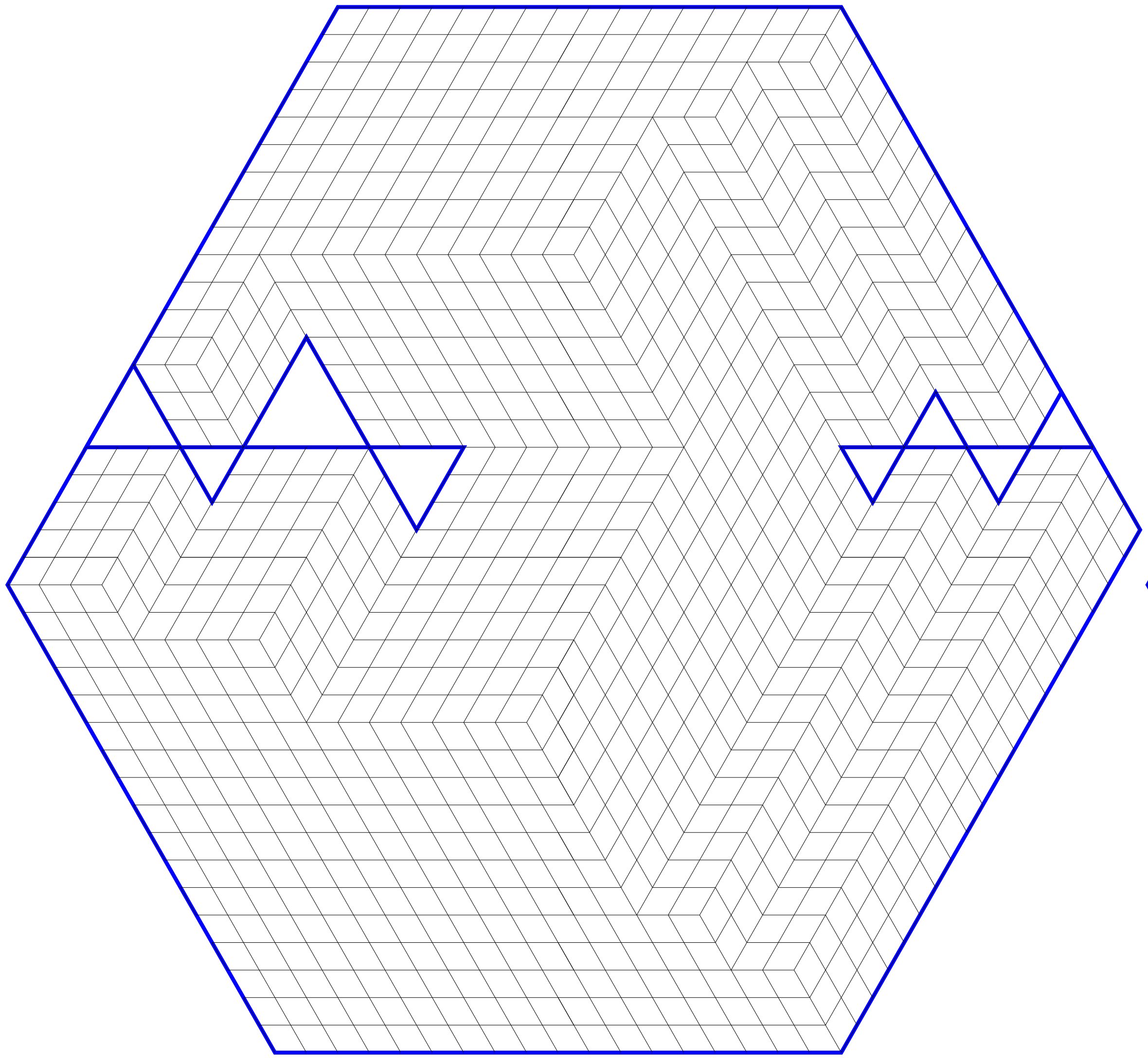
The hexagon in the denominator



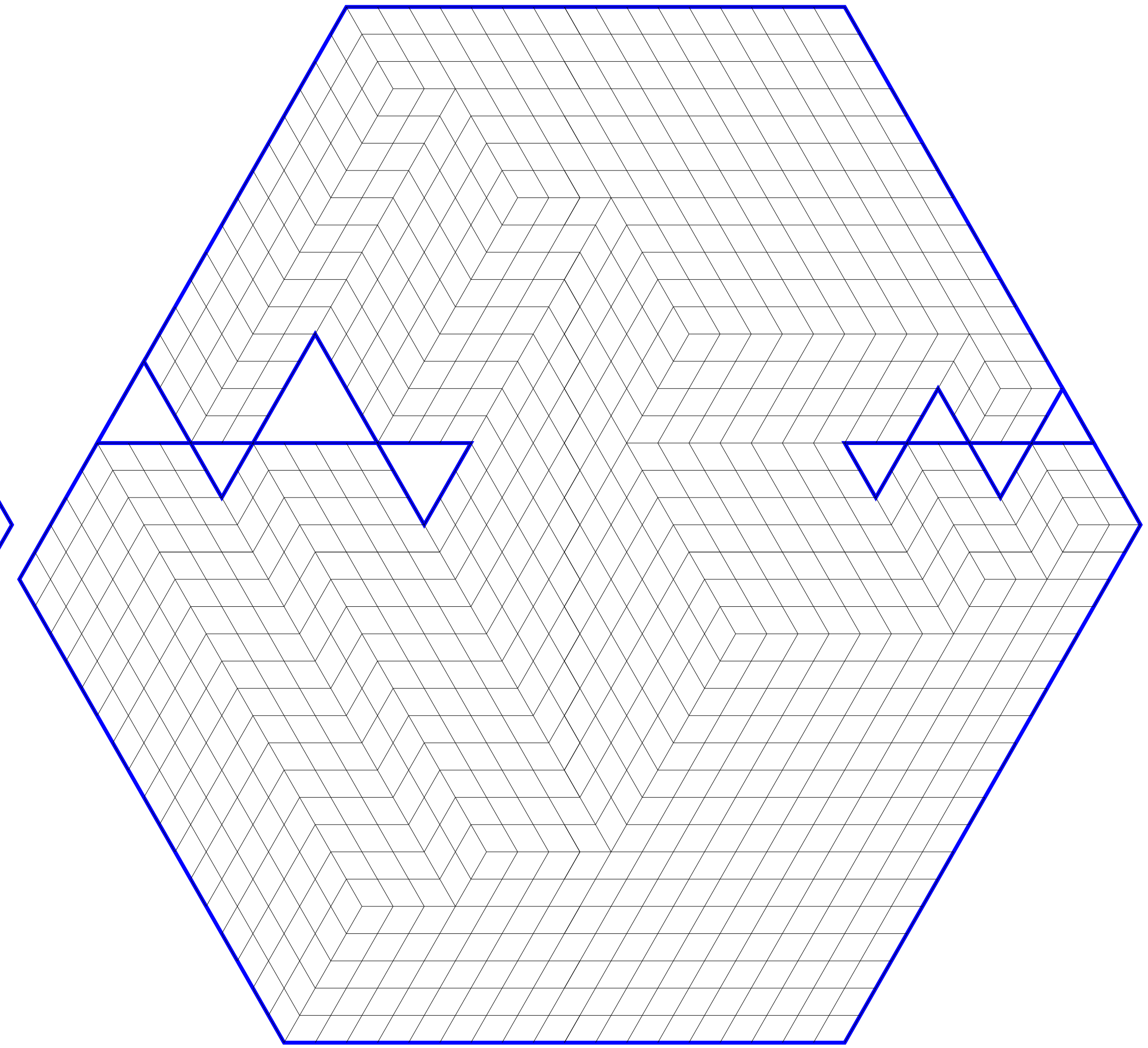
The two hexagons in the numerator



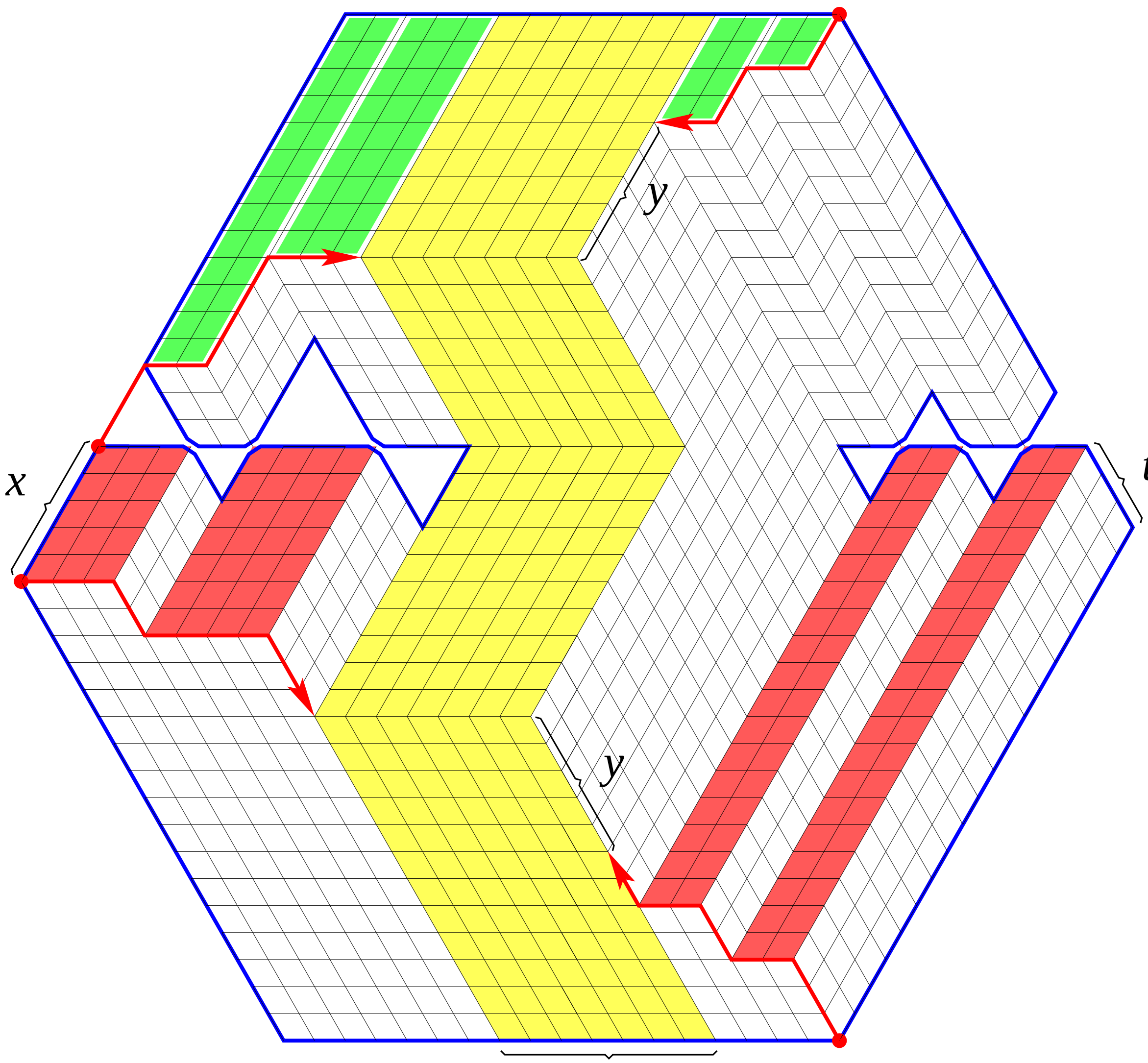
A lozenge tiling as a stack of unit cubes (called plane partition) — gets a natural volume



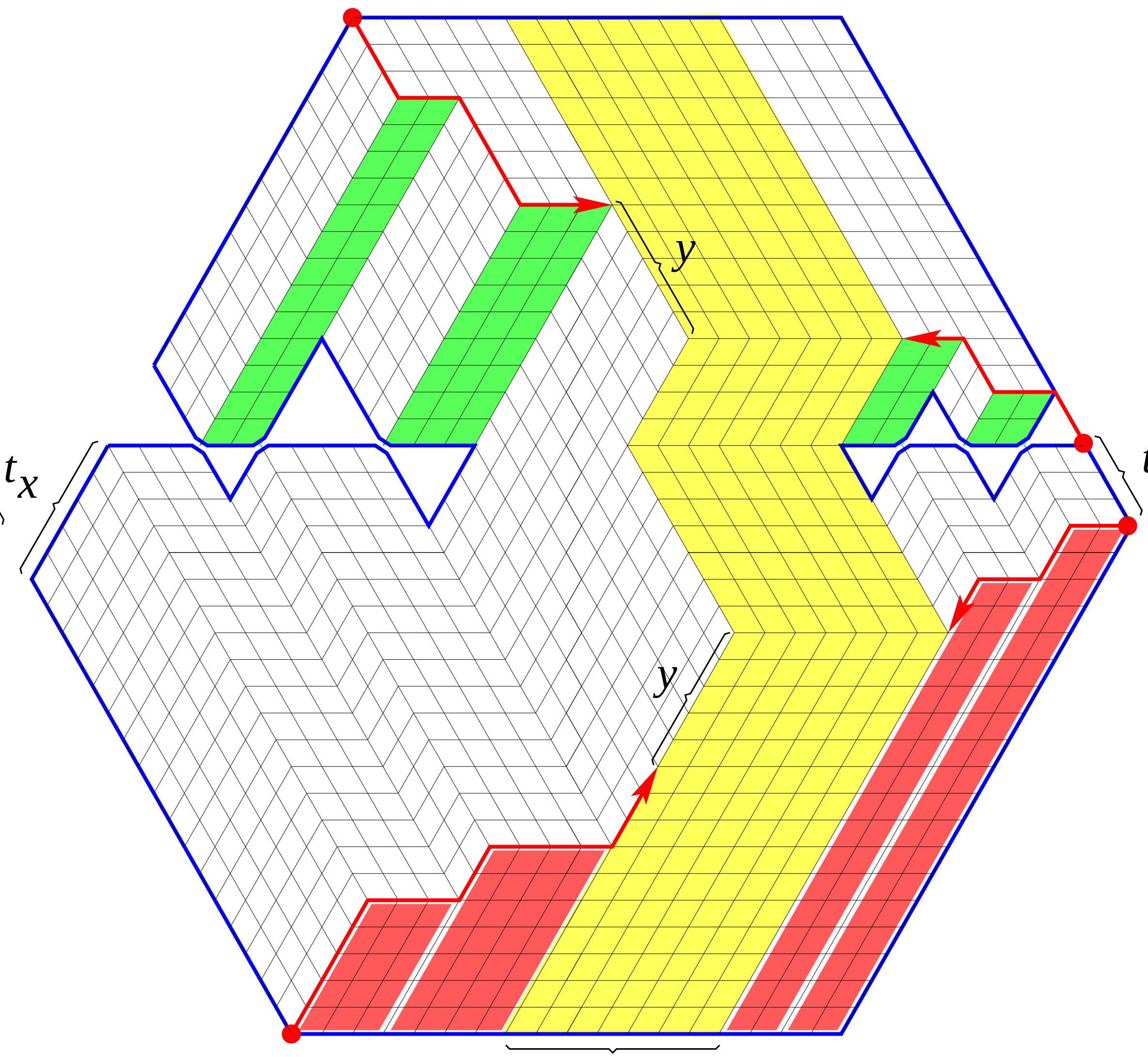
The highest tiling



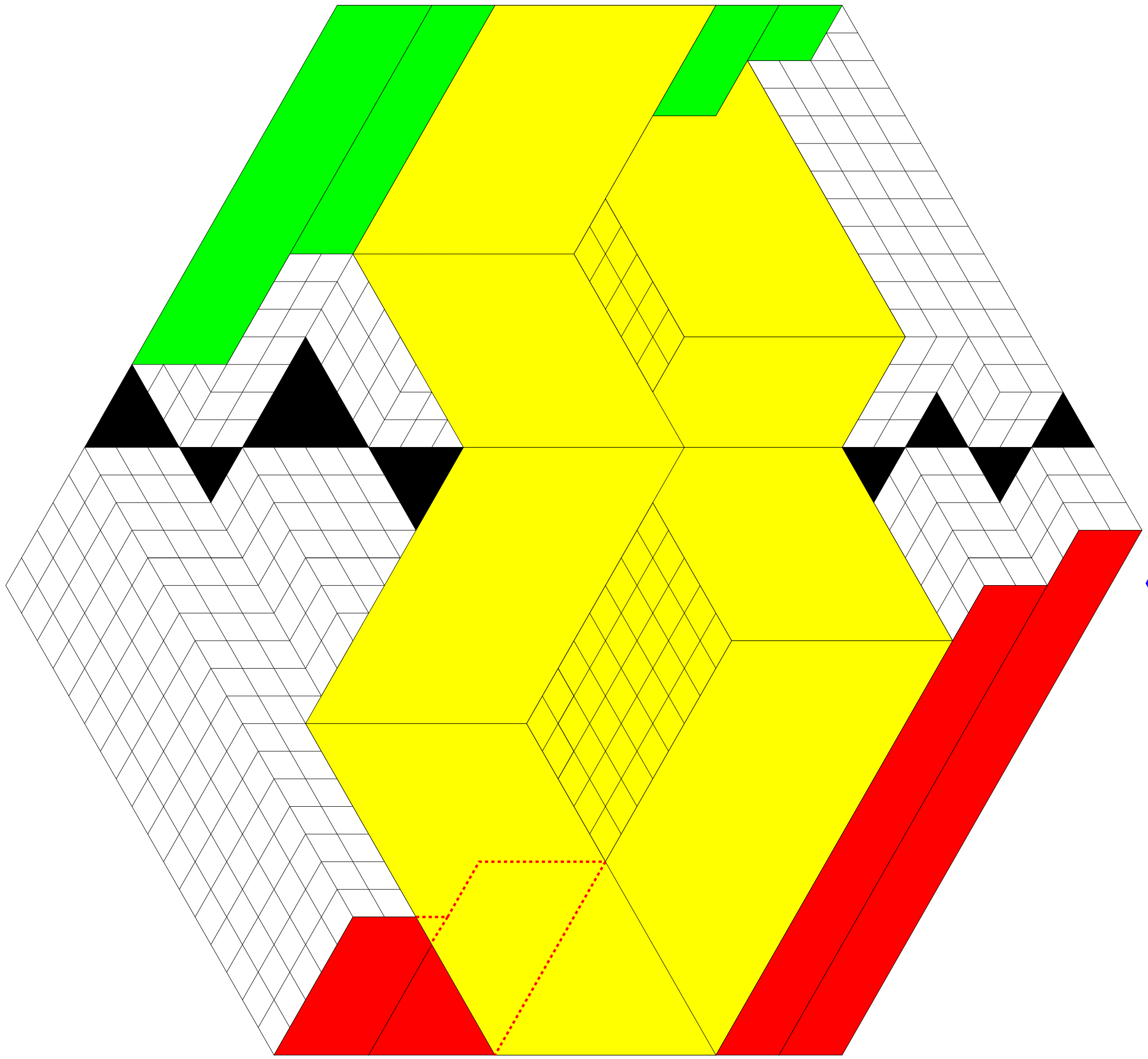
The lowest tiling



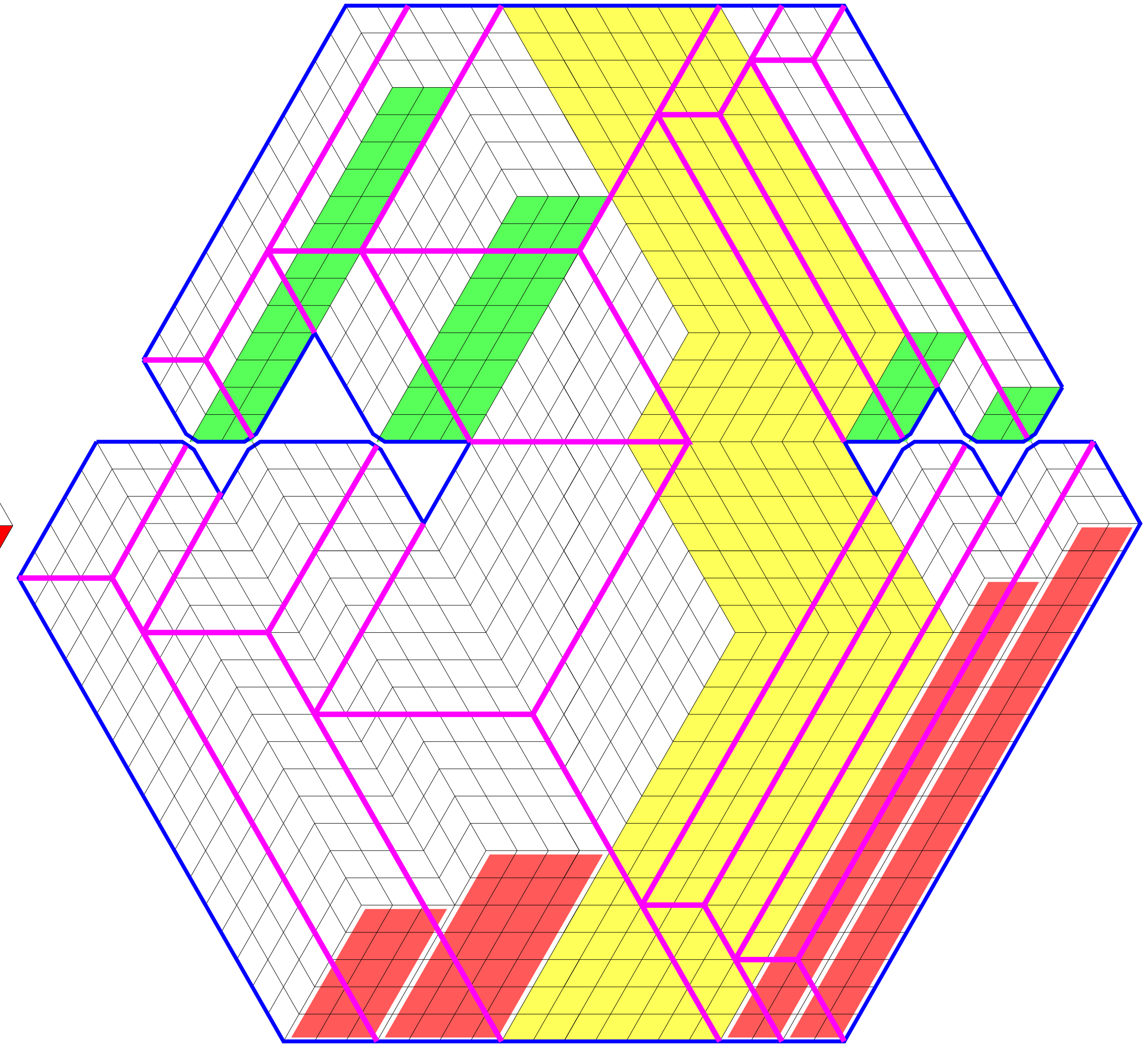
The upper envelope



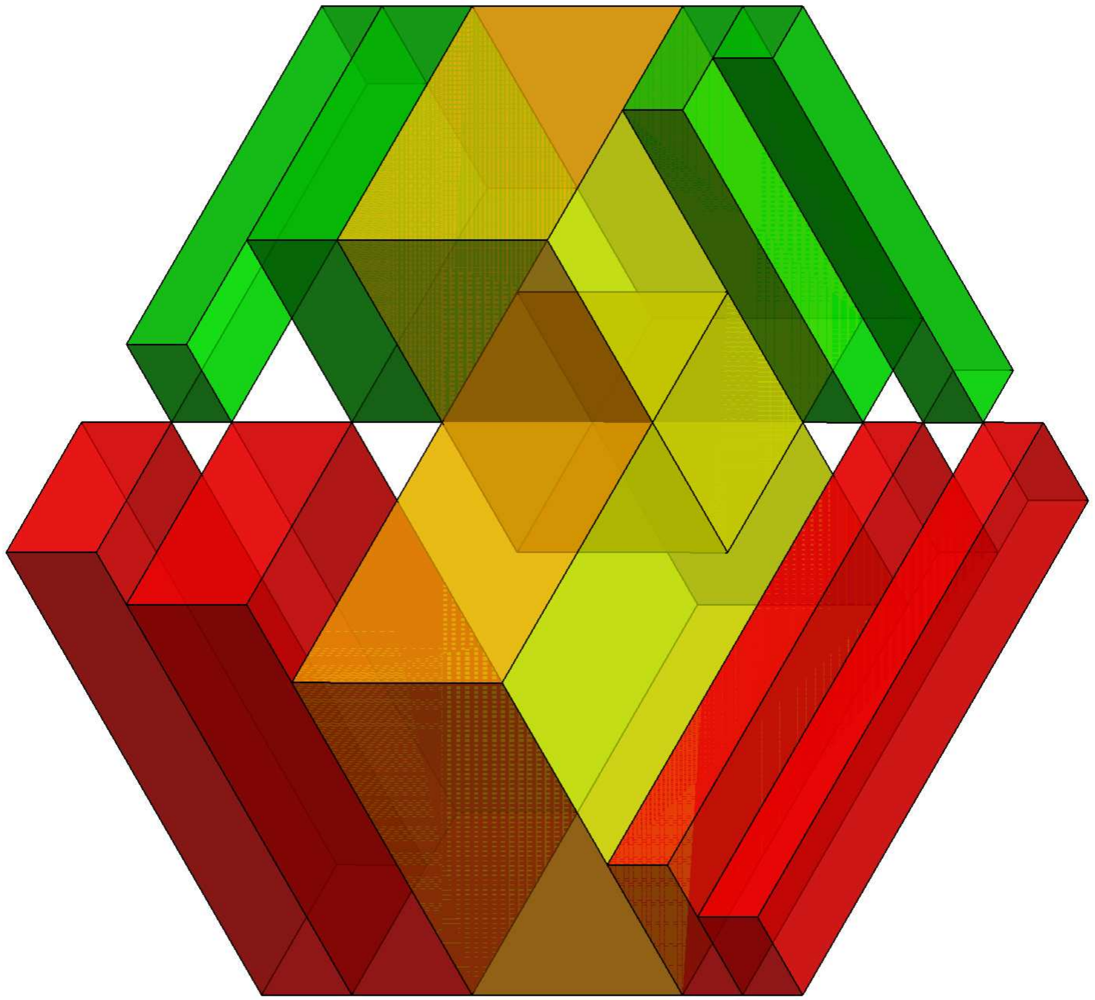
The lower envelope



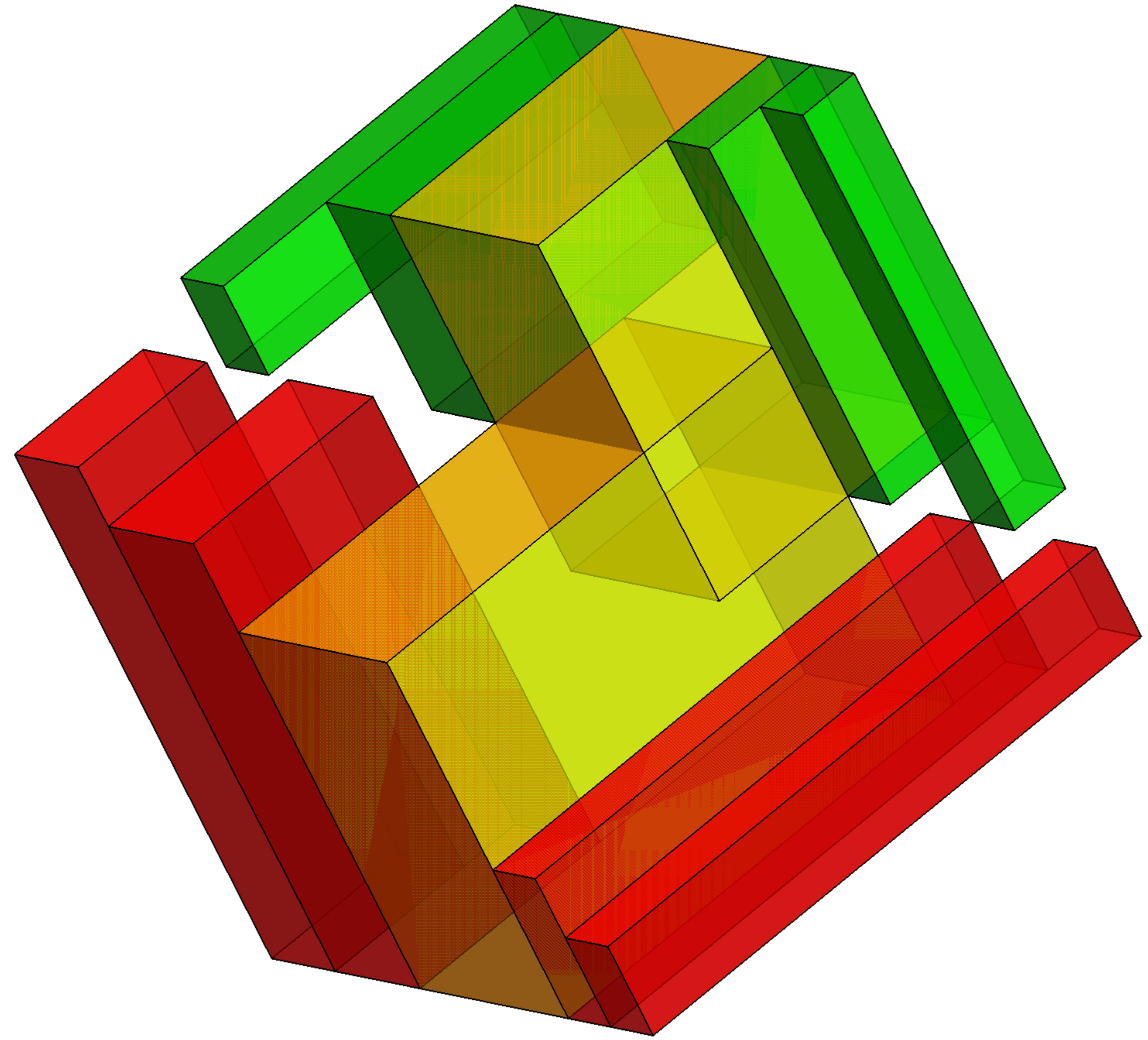
The house with slab and roof



The house with stairs and cupboards



View from the same viewpoint



View from a slightly different viewpoint

Weighted results

$$[n]_q := 1 + q + \cdots + q^{n-1}$$

$$[n]_q! := [1]_q \, [2]_q \, \cdots \, [n-1]_q$$

$$\mathrm{H}_q(n) := [1]_q! \, [2]_q! \, \cdots \, [n-1]_q!$$

MacMahon's q-enumeration of boxed plane partitions

$$\sum_{\pi \subset [x]\times[y]\times[z]} q^{Vol(\pi)} = \frac{\mathrm{H}_q(x)\,\mathrm{H}_q(y)\,\mathrm{H}_q(z)\,\mathrm{H}_q(x+y+z)}{\mathrm{H}_q(x+y)\,\mathrm{H}_q(x+z)\,\mathrm{H}_q(y+z)} =: P_q(x,y,x)$$

q-version of our result

THEOREM 4 (C. AND LAI, 2019).

$$\sum_{\pi\subset B_{x,y,z,t}(a_1,\dots,a_{2l};b_1,b_2,\dots,b_{2k})} q^{Vol(\pi)} =$$

$$\frac{P_q(a+x+y,b+y+t,z)P_q(e_a+e_b+x+t,o_a+o_b,y)}{P_q(a+x,b+t,y+z)}$$

$$\times\ s_q(a_1,\dots,a_{2l-1},a_{2l}+y+z+b_{2k},b_{2k-1},\dots,b_1)$$

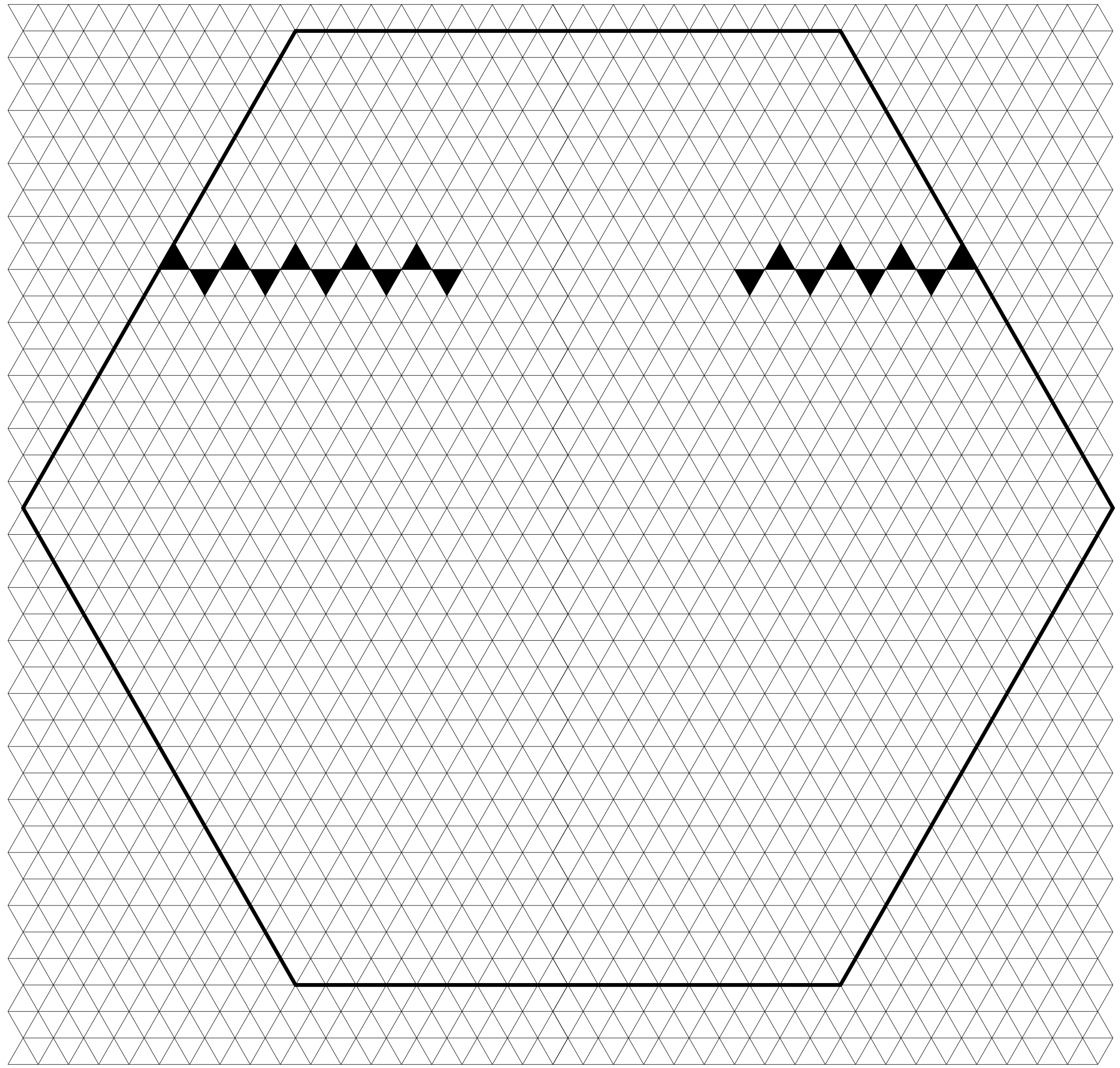
$$\times\ s_q(x,a_1,\dots,a_{2l},y+z,b_{2k},\dots,b_1,t),$$

where

$$s_q(b_1,b_2,\dots,b_{2l}) = s_q(b_1,b_2,\dots,b_{2l-1})$$

$$= \frac{1}{\mathrm{H}_q(b_1+b_3+\cdots+b_{2l-1})}\frac{\prod_{1\le i\le j\le 2l-1,\ j-i+1 \text{ odd}}\mathrm{H}_q(b_i+b_{i+1}+\cdots+b_j)}{\prod_{1\le i\le j\le 2l-1,\ j-i+1 \text{ even}}\mathrm{H}_q(b_i+b_{i+1}+\cdots+b_j)}.$$

- proof is inductive, based on Kuo's graphical condensation method
- natural weight difficult to work with (weight of a tile not well defined)
- work around this using two other weight functions



Open Problems: What ratio gives maximum in the scaling limit?

Limit shape?