Length derivative of generating series for walks in the quarter plane

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collaborations with T. Dreyfus, J. Roques, MF. Singer

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A walk in the quarter plane with weight $\mathcal{W}$ is a sequence of points $(P_n)_{n \in \mathbb{Z}^+} \in (\mathbb{Z}^+)^2$ with

$$P_0 = (0,0), \text{ such that } P_{n+1} - P_n = (i,j) \in \mathcal{D} \text{ and } \mathbb{P}(P_n \to P_{n+1}) = d_{i,j}$$

Example

$\mathcal{D} = \{\leftarrow, \uparrow, \rightarrow, \searrow, \downarrow, \swarrow\}$

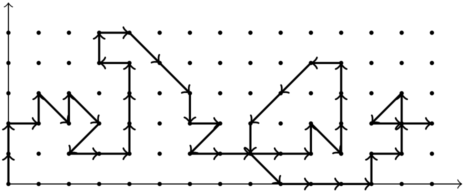

FIGURE – Walk with directions in $\mathcal{D}$ of Length 45 ending at $(15, 2)$

Probabilistic interpretation

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- Associated probabilities : $\mathbb{P}\left((0,0) \to^k (l,s)\right)$
- *Generating series*

$$Q_{\mathcal{W}}(x,y,t) = \sum_{l,s,k} \mathbb{P}\left((0,0) \to^k (l,s)\right) x^l y^s t^k$$

converges for $|x|, |y| \leq 1$ and $|t| < 1$.

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- algebraic over $\mathbb{Q}(x,y,t)$?
- holonomic over $\mathbb{Q}(x,y,t)$? (x, y, and t-holonomic)
- differentially algebraic in each of the variables ? $f(x,y,t)$ is *differentially algebraic* in x if for some n and polynomial $P(X_0,\dots,X_n) \in \mathbb{Q}(x,y,t)[X_0,\dots,X_n]^*$, we have

$$P(f, \frac{\partial}{\partial x}(f), \dots, \frac{\partial^n}{\partial x^n}(f)) = 0$$

Combinatorial classification for unweighted walks

Symmetries and classifying objects : Bousquet-Mélou, Mishna (2010)

- For *unweighted walks :* 256 possible choices for $\mathcal{D}$.
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For unweighted walks,
$|G_{\mathcal{W}}| < \infty$ if and only if $Q_{\mathcal{W}}(x, y, t)$ is holonomic.
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What are the differential algebraic properties of the series ?

Weighted walks and t-derivations

In this talk, we will see how, for genus 1 walks, one can use the theory of Mordell-Weil lattices of rational elliptic surfaces to produce an algorithm to find these differential algebraic relations.

The functional equation

Consider a set of weights $\mathcal{W} := \{d_{i,j}\}$.
The generating series $Q_{\mathcal{W}}(x,y,t) = \sum_{l,s,k} \mathbb{P}\left((0,0) \to^{k} (l,s)\right) x^l y^s t^k$ satisfies

$$K_{\mathcal{W}}(x,y,t) Q_{\mathcal{W}}(x,y,t) = \\ xy - K_{\mathcal{W}}(x,0,t) Q_{\mathcal{W}}(x,0,t) - K_{\mathcal{W}}(0,y,t) Q_{\mathcal{W}}(0,y,t) \\ + K_{\mathcal{W}}(0,0,t) Q_{\mathcal{W}}(0,0,t).$$

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$$K_{\mathcal{W}}(x, y, t) := xy \left(1 - t \sum_{(i,j)\in\{-1,0,1\}} d_{i,j} x^i y^j \right).$$

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The *Kernel curve* of the walk

Let $(C, |.|)$ be an algebraically closed complete field extension of $\mathbb{Q}(t)$ endowed with the t-valuation. The *Kernel curve* $E_{\mathcal{W}}$ of the walk is

$$E_{\mathcal{W}} = \{(x, y) \in \mathbb{P}^1(C) \times \mathbb{P}^1(C) | K_{\mathcal{W}}(x, y, t) = 0\}.$$

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Dreyfus-H.-Roques-Singer : Characterization of the direction sets of reducible and genus zero walks.

From now on, I will assume that $E_{\mathcal{W}}$ has genus 1.

Group of the walk

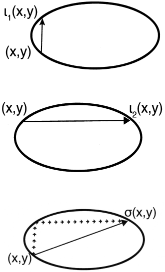
$$E_{\mathcal{W}} = \overline{\{(x,y) \mid K_{\mathcal{W}}(x,y,t)=0\}}^{Zariski} \subset \mathbb{P}^1(C) \times \mathbb{P}^1(C)$$

We define two involutions of $E_{\mathcal{W}}$ and an automorphism :

$$\iota_1(x,y) = \left(x, \frac{1}{y} \frac{\sum_{(i,-1)\in\mathcal{D}} d_{i,-1}x^i}{\sum_{(i,+1)\in\mathcal{D}} d_{i,1}x^i}\right)$$

$$\iota_2(x,y) = \left(\frac{1}{x} \frac{\sum_{(-1,j)\in\mathcal{D}} d_{-1,j}y^j}{\sum_{(+1,j)\in\mathcal{D}} d_{1,j}y^j}, y\right)$$

$$\sigma = \iota_2 \circ \iota_1$$



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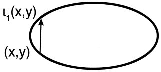

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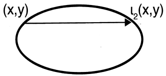

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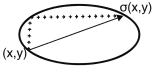

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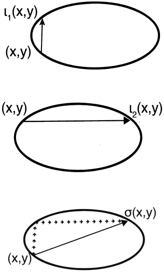

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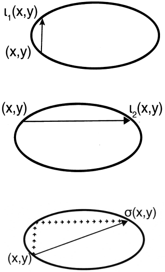

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Specializing

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for $(x,y) \in E_{\mathcal{W}}$
One obtain a difference equation of the form

$$\sigma F - F = \iota_1(xy) - xy,$$

where $F = K_{\mathcal{W}}(0,y,t)Q_{\mathcal{W}}(0,y,t)$.

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If $Q(0, y, t)$ is ∂_t-∂_y-algebraic over C then it is ∂_y-algebraic over $\mathbb{Q}$

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Theorem (Dreyfus-H.-Roques-Singer)

If $Q(0, y, t)$ is ∂_y-alg over $\mathbb{Q}$ then there exist $c_0, \dots, c_{n-1} \in C$ not all zero and $h \in C(E_{\mathcal{W}})$ such that

$$c_0 b + \cdots + c_{n-1}\delta^{n-1}(b) + \delta^n(b) = \sigma(h) - h, \tag{3.1}$$

with δ the invariant derivation of $C(E_{\mathcal{W}})$ and $b = \iota_1(xy) - xy \in C(E_{\mathcal{W}})$.

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We call (3.1) a *telescoper* for b.

Telescopers in $\mathbb{C}(E_{\mathcal{W}})$, $E_{\mathcal{W}}$ an elliptic curve

$E_{\mathcal{W}}$ elliptic curve, σ the addition by a non torsion point, $K = \mathbb{C}(E_{\mathcal{W}})$
Def.

- $\{u_Q \mid Q \in E_{\mathcal{W}}\}$ local param. are **coherent** if $u_{Q \ominus \Omega_{\mathcal{W}}} = \sigma(u_Q)$.
- For $g \in \mathbb{C}(E_{\mathcal{W}})$, $Q \in E_{\mathcal{W}}$, write

$$g = \frac{c_{Q,N}}{{u_Q}^N} + \cdots + \frac{c_{Q,i}}{u_Q^i} + \cdots + \frac{c_{Q,1}}{u_Q} + f$$

with f regular at Q. Then, the **i^th^ orbit residue** of g at Q is

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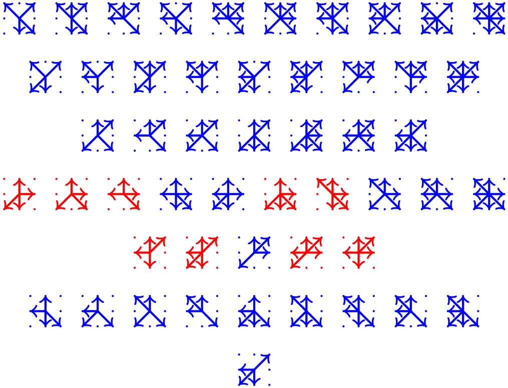
Existence of Telescopers. The following are equivalent :

- g satisfies a telescoper equation.
- For each $i \in \mathbb{N}_{>0}$, $Q \in E_{\mathcal{W}}$, $\mathrm{ores}_Q^i(g) = 0$.

Unweighted walks, genus$(E_{\mathcal{W}}) = 1$, $|G_{\mathcal{W}}| = \infty$



Theorem (Dreyfus-H.-Roques-Singer)

- 42 *cases :* $Q(0, y, t)$ *is* ∂_y*-diff trans. over* $\mathbb{Q}$
- 9 *cases :* $Q(0, y, t)$ *is* ∂_y*-diff. algebraic but not holonomic over* $\mathbb{Q}$.

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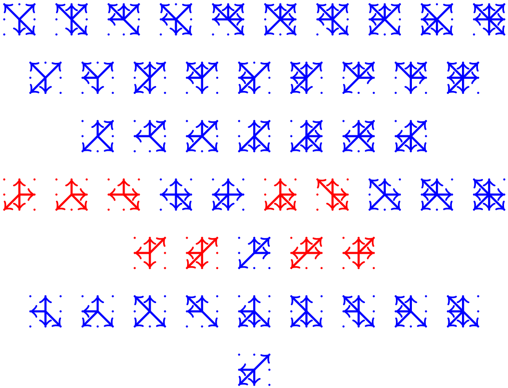

Theorem (Dreyfus-H.-Roques-Singer)

- 42 *cases* : $Q(0, y, t)$ *is* ∂_y*-diff trans. over* $\mathbb{Q}$
- 9 *cases* : $Q(0, y, t)$ *is* ∂_y*-diff. algebraic but not holonomic over* $\mathbb{Q}$.

Unweighted case : Decoupling functions

In parallell, Bernardi, Bousquet-Mélou and Raschel proved that

Theorem

- *For theses* 9 *cases, the function* xy *decouples, that is* , $xy = f(x) + g(y)$*on* $E_{\mathcal{W}}$ *for some* $f, g \in \mathbb{Q}(X)$.
- *If the function* xy *decouples then one can find an explicit differential algebraic equation for the* $Q(0, y, t)$ *in* y *and* t.

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If $E_{\mathcal{W}}$ is of genus 1 and $|G_{\mathcal{W}}| = \infty$ the following statements are equivalent

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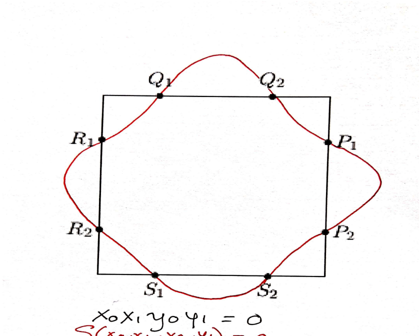
- the blow up of $\mathbb{P}^1 \times \mathbb{P}^1$ at the eight base points of the pencil of curves $\{E_\lambda := \{(x,y) \in \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C}) | K_{\mathcal{W}}(x,y,\lambda) = 0\}\}_{\lambda \in \mathbb{P}^1(\mathbb{C})}$



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where h is the Néron Tate height on the generic fiber $E_{\mathcal{W}}(\overline{\mathbb{Q}}(t))$.

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- Use Bernardi-Bousquet-Mélou-Raschel to find the differential algebraic equations satisfied by the generating series.

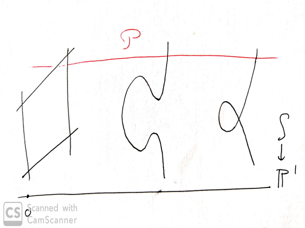
Direct criteria

When several base points merge, one can compute directly the heights

- via *intersection theory* and Shioda's formula

$$h(P) = 2 + 2(\mathcal{P}.\mathcal{O}) - \sum_{v \in R} \mathrm{contr}_v(P, P)$$

with $\mathcal{O}$ the zero section, $\mathcal{P}$ the section corresponding to P and $\mathrm{contr}_v(P, P)$ the contribution for the reducible fibres.



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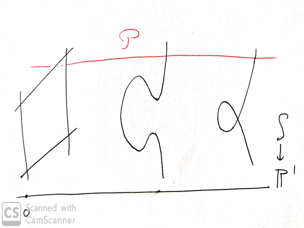
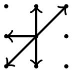


- and via *the classification of reducible fibres* for rational elliptic surfaces (See Shioda and Oguiso-Shioda)

For instance, If the directions set is



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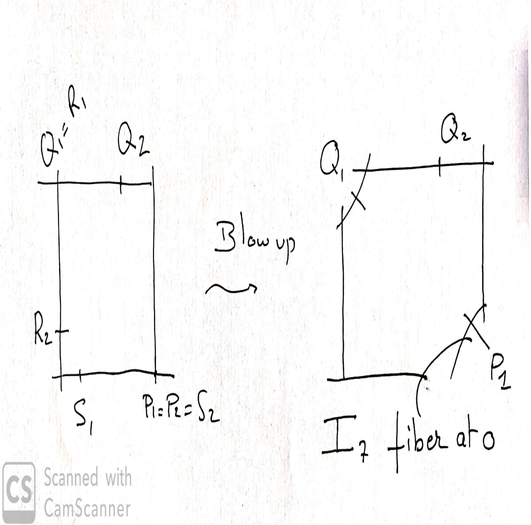

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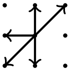

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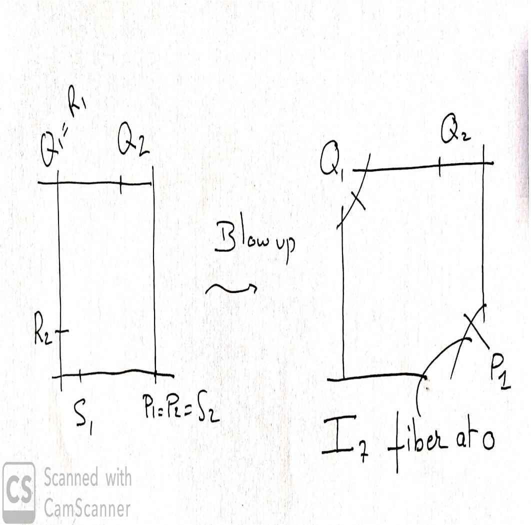

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Lemma (H.-Singer)

- *The group $G_{\mathcal{W}}$ is infinite*
- *The generating series is ∂_y-algebraic if and only iff*

$$d_{1,1}d_{-1,-1} - d_{0,-1}d_{0,1} = 0$$

.

Conclusion

Combinatorics of the base points and of the Dynkin diagrams of the reducible fibers

encode

the diff. alg. properties of the generating series

Thank you for your attention